1. Give an example of a power series with radius of convergence equal to 4.

**Sol’n:** Of course, there are many correct answers; \( \sum_{n=0}^{\infty} 4^{-n} z^n \) is one.

2. Find the radius of convergence of the following power series: \( \sum_{n=0}^{\infty} \frac{3^n}{(2n)!} z^n \).

**Sol’n:** When the ratio test is applied, one considers

\[
\lim_{n \to \infty} \frac{\frac{3^{n+1}|z|^{n+1}}{(2(n+1))!}}{\frac{3^n|z|^n}{(2n)!}} = \lim_{n \to \infty} \frac{3|z|}{(n+2)(n+1)} = 0.
\]

Since the limit is 0 regardless of the choice of \( z \) (unequal to 0, of course), we see that the series converges for every \( z \), so the radius of convergence is \( \infty \).

3. Find the power series expansion centered at \( z_0 = \pi/3 \) for \( \sin z \).

**Sol’n:** Set \( w = z - \pi/3 \). Then we have

\[
\sin z = \sin(w + \pi/3) = \sin \frac{\pi}{3} \cos w + \sin w \cos \frac{\pi}{3} = \frac{\sqrt{3}}{2} \cos w + \frac{1}{2} \sin w
\]

\[
= \frac{\sqrt{3}}{2} \sum_{n=0}^{\infty} (-1)^n \frac{w^{2n}}{(2n)!} + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{w^{2n+1}}{(2n+1)!}
\]

\[
= \frac{\sqrt{3}}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(z - \pi/3)^{2n}}{(2n)!} + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(z - \pi/3)^{2n+1}}{(2n+1)!}
\]

4. Let \( f \) be an entire function (i.e., analytic in all of \( \mathbb{C} \)). Suppose that there exist finite, positive constants \( C_0 \) and \( C_1 \) such that \( |f(z)| \leq C_0 \left( 1 + |z| \right)^{C_1} \) holds for all \( z \in \mathbb{C} \). Show that \( f \) is a polynomial.

**Sol’n:** The Cauchy estimate applied to \( D(0, R) \) tells us that

\[
|f^{(n)}(0)| \leq C_0 \left( 1 + r \right)^{C_1} \frac{n!}{r^n}.
\]

If \( n > C_1 \), then the limit of the right-hand side, as \( r \to \infty \), is 0. So we have \( f^{(n)}(0) = 0 \) for \( n > C_1 \). Letting \( N \) denote the largest integer that is less than or equal to \( C_1 \) and using the power series expansion for \( f \), we obtain

\[
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} z^n
\]
showing that $f$ is a polynomial.

5. Determine the type of singularity at $z = 0$. If the singularity is a pole, find the order of the pole.

$$\frac{z^2}{\cos(z) - 1 + z^2/2}.$$ 

**Sol’n:** Using the power series for $\cos z$, we see that

$$\cos(z) - 1 + z^2/2 = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} - 1 + z^2/2$$

$$= \sum_{n=2}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$$= \sum_{m=0}^{\infty} (-1)^{(m+2)} \frac{z^{2m+4}}{(2m+4)!}$$

$$= z^4 \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{(2m)!}$$

and we observe that $\sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{(2m)!}$ defines an entire function $h(z)$ that does not vanish at 0. Then we have

$$\frac{z^2}{\cos(z) - 1 + z^2/2} = z^2 \frac{1}{h(z)},$$

so 0 is a pole of order 2.

6. Give an example of a function with an essential singularity at $z_0 = 3$.

**Sol’n:** Again there are many correct solutions; one is $e^{1/(z-3)}$.

7. Suppose that $P(\cos \theta, \sin \theta)$ is a polynomial in $\cos \theta$ and $\sin \theta$. Show that if $P(\cos \theta, \sin \theta) = 0$ holds for $\theta = \pi/n$, for all positive integers $n$, then $P(\cos \theta, \sin \theta) = 0$ for all $0 \leq \theta \leq 2\pi$.

[Hint: An example of a polynomial in $\cos \theta$ and $\sin \theta$ is $P(\cos \theta, \sin \theta) = 1 + \cos \theta + \cos^2 \theta + 2 \cos \theta \sin \theta + 3 \sin^2 \theta$. This example does not equal 0 for all $\theta = \pi/n$; it is given here solely to illustrate the notion of a polynomial in $\cos \theta$ and $\sin \theta$.]
Sol’n: We consider the entire function
\[ f(z) = P(\cos z, \sin z). \]
Since \( f(\pi/n) = 0 \) for \( n = 1, 2, \ldots \), we see that \( z = 0 \) is a non-isolated zero of \( f \). But with the sole exception the identically zero function, the zeros of an analytic function are isolated in any connected open set. Thus \( f \equiv 0 \). In particular \( f(\theta) = 0 \) for \( \theta \in [0, 2\pi] \).

8. Find the Laurent series for \( \frac{1 + 3z + 3z^2}{z^2(z + 1)} \) at 0.

Sol’n: One can write
\[ \frac{1 + 3z + 3z^2}{z^2(z + 1)} = \frac{1}{z^2} + \frac{2}{z} + \frac{1}{1 + z}. \]
Using
\[ \frac{1}{1 + z} = \frac{1}{1 - (-z)} = \sum_{n=0}^{\infty} (-1)^n z^n, \]
we conclude that the Laurent series is
\[ \frac{1 + 3z + 3z^2}{z^2(z + 1)} = \frac{1}{z^2} + \frac{2}{z} + \sum_{n=0}^{\infty} (-1)^n z^n. \]

9. Use the residue calculus to compute \( \int_{0}^{\infty} \frac{x}{1 + x^4} \, dx. \)

[Hint: Show that the integral of \( z/(1+z^4) \) along the imaginary axis starting at 0 and going through \( i \) and continuing on to \( \infty \) equals \( -\int_{0}^{\infty} \frac{x}{1+x^4} \, dx \). Use this fact to help you choose the contour (i.e., curve) over which to integrate.]

Sol’n: First, we observe that if we parametrize the upper imaginary axis by \( z = it, \, 0 \leq t < \infty \), and call that path \( \gamma \), then we have
\[ \int_{\gamma} \frac{z}{1 + z^4} \, dz = \int_{0}^{\infty} \frac{it}{1 + t^4} \, i \, dt = -\int_{0}^{\infty} \frac{t}{1 + t^4} \, dt. \]
Next, if we consider the positively oriented quarter circle \( c \) of radius \( R \) shown in the figure, we see that
\[ \int_{c} \frac{z}{1 + z^4} \, dz \sim R^{-2} \]
for large $R$.

So the positively oriented line integral around the path $a$, followed by $c$, followed by $b$ converges, as $R \to \infty$, to 2 times the integral we wish to evaluate.

The roots of $1 + z^4$ are

\[
\begin{align*}
  z_1 &= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \\
  z_2 &= -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \\
  z_3 &= -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \\
  z_4 &= \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}.
\end{align*}
\]

The only pole inside the quarter circle (for $R > 1$) is at $z_1$ and the residue there is

\[
\text{Res} = \frac{z_1}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} = 2 \frac{1 + i}{2(2 + 2i)2i} = (1/4) \frac{1}{i} = \frac{-i}{4}.
\]

We have

\[
\int_{a+c+b} \frac{z}{1 + z^4} \, dz = 2\pi i \text{Res} = \frac{\pi}{2}.
\]
We conclude that

\[ \int_0^\infty \frac{x}{1 + x^4} \, dx = \frac{\pi}{2}. \]