

**A TAXONOMY OF QUANTITATIVE METHODS
FOR ASSESSING RISK**

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What is risk?

1. A potential negative impact to an asset or characteristic of value that may arise from some present process or future event.

Components of risk:

- The list of potential hazards

P_r (hazard occurs)

- The list of consequences resulting from a hazard occurring

P_r (consequence | hazard occurred)

- The loss resulting from the consequence

E (loss | consequence occurred from a hazard)

2. Risk is the expected loss if a problem occurs.

3. Risk assessment is the set of tools for determining potential risks and the strategies for managing them.

- a. Prioritize the likelihood of hazards
- b. Perform cost benefit analysis for managing risks
- c. Analyze how a system was built and is operated
- d. Determine the probabilities (frequencies) of events leading to exposure of hazards
- e. Determine the magnitude of consequences for each scenario and its risk (expected loss)

Comment: The concern is not the bottom line BUT identifying the major components contributing to risk.

- f. Evaluate effective strategies to reduce risk
 - i. Available analytical techniques
 - ii. Knowledge of systems and its limitations
 - iii. Identify conditions that can lead to problems and determine the potential consequences

- iv. Express the analysis as a fault tree, which is
- (1) Inverted tree structure with an undesirable outcome as the mode event
 - (2) Branches spread downward representing failure logic from the intermediate system event failure down to component event failure
 - (3) Consists of two types of symbols
 - (a) Events: failure logic
 - (b) Gates: Boolean expressions
 - (4) Cutset: set of component failure modes, which if they occur together will cause the system to fail.
 - (5) Minimal cutset: necessary and sufficient combination of component failures which, if they occur together, will cause the system to fail.
 - (6) Strategy:
 - (a) Determine minimal cut sets (find smallest combinations of basic failure events that will prevent the system from performing)
 - (b) Ignore insignificant cut sets
 - (c) Use simulations and sensitivity studies to interpret the analysis

The study of risk

1. Risk has never evolved into its own language and methodologies.
2. Risk analysis is a cross-cutting topic that combines such diverse topics as:
 - Engineering
 - Medicine
 - Finance theory
 - Public policy
 - Marketing
 - Environmental sciences
 - Etc.
3. The study of risk has developed in a variety of ways:
 - a. Building upon statistical theory subsumed in probabilistic risk assessment
 - b. Developing strategies that are robust against specific kinds of uncertainty
 - c. Constructing strategies in dynamically changing action spaces such as in an economic environment or in a military setting
4. Much of the relevant literature is scattered in professional journals and books. Wiley & Sons will be publishing in July 2008 The Encyclopedia of Qualitative Risk Analysis and Assessment with the aim of drawing together varied intellectual threads so that risk analysts in one area can gain from the experience of researchers in other areas.

This talk will focus on quantitative models that have played important roles in risk analysis.

Preliminary

1. Axiomatic models of perceived risk (Pollatsek & Tversky)
 - a. Risk is a property of options.
 - b. Options can be meaningfully ordered with respect to their riskiness.
 - c. Risk is related to dispersion (variance) of its outcomes.
 - d. Comments:
 - i. Rotar & Sholomitsky generalized the mean variance model of Pollatsek & Tversky.
 - ii. Based on experimentation, some authors have proposed asymmetrical measures for situations when considering losses versus gains, i.e., people tend to take a higher risk position when facing a loss and become risk averse when facing a gain.
 - iii. Jia, Dyer and Butler show relationships between financial measures of risk and psychological measures of risk.

2. Bayesian statistics is a form of statistical inference that combines qualitative and quantitative information. The process begins with a numerical estimate of the degree of belief in a hypothesis and updates the belief as new information becomes available.

Components of Bayesian statistics:

- a. Prior probability (subjective probability) is the degree of belief about a hypothesis without numerical data (Ramsey and de Finetti).
- b. Posterior probability is the updated degree of belief conditioned on available information.
- c. Markov Chain Monte Carlo algorithms are used to sample from posterior densities and to numerically calculate multi-dimensional integrals. The algorithms have allowed for extending the range of single-parameter sampling methods to multivariate situations where the parameters have different densities (Smith and Gelfand).
- d. Credibility intervals (vs. confidence intervals) that cover the true parameter with 95% probability.

e. Special applications

- i. Allows for modeling hierarchically or spatio-temporarily correlated effects by conditioning on priors.

Friessen modeled job exposures in historical epidemiological studies by modeling 3 stages:

- Stage 1: Specify likelihood given unknown randomly distributed cluster effects.
- Stage 2: Specify the density of the population of cluster effects.
- Stage 3: State the priors on the population parameters

- ii. Exceedance analysis

Lye proposed methods for building on a flood plain and Van Gelder determined the necessary size required for building dams.

3. Decision Theory is a methodology for making optimal decisions involving situations of uncertainty that can occur when a particular action is taken.

a. Based on subjective and objective information

b. Analytical approach involving the modeling of:

- i. Judgment of uncertainty (subjective probability)
- ii. Preferences (utility function)

c. Utility function (von Neumann and Morgenstern)

- i. Basic axioms of utility: set of axioms that justify decision making based on expected utility
- ii. Basic steps:
 - (1) Choose options whose outcomes may be uncertain at the time of decision making
 - (2) Convert options within a project to utilities (e.g., monetary payoff)
 - (3) Compute the expected utility for each project
 - (4) Select the option with the largest expected utility
- iii. Problems
 - (1) Assessing utility functions
 - (2) Analyzing behavioral properties – individuals often do not follow axioms (Kahneman and Tversky)
 - (3) Example: individuals are risk seekers for losses (not want a sure loss) but risk averters for gains (want a sure gain)

d. Analysis is connected with Bayesian statistics. Extensions include:

i. Temporal relationship (decision tree)

ii. Value of information:

$$\begin{array}{c} \text{maximum expected utility with data} \\ \text{minus} \\ \text{maximum expected utility without data} \end{array}$$

Some problems require inverting utility functions to obtain the financial value of information.

Note: The literature of decision theory and risk are almost identical. The major difference is:

Decision Theory: **Uncertainty** and **value** are equally important.

Risk: Greater emphasis is on the modeling of uncertainty.

Important Statistical Measures in Risk Analysis

1. Extreme Value Theory is the study of events that occur with small probability.
 - a. Distribution of the largest order statistic (Fisher–Tippet Theorem 1928)
 - i. Distribution of the extreme value of observations selected from blocked data, i.e., joint distribution of the largest order statistics selected from a random sample of observations that have been blocked.
 - ii. Peaks over Threshold (POT) is the positive difference between sample values and a threshold.
 - (1) Preferable when estimating quantities
 - (2) Can be extended to dependent data
 - (3) Distribution of exceedance is the generalized Pareto Distribution.
 - iii. Extreme value distributions have 3 parameters: location, scale, and shape
 - Type I: Gumbel distribution which is for data from a distribution whose tail falls off exponentially such as the normal. The scale parameter approaches zero.
 - Type II: Fréchet distribution that includes the Pareto family, which is for data from distributions that fall off as a polynomial (fat-tailed distributions) such as the t-distribution.
 - Type III: Weibull distribution, which is for data from distributions with a finite tail such as the beta distribution.
 - b. Extreme value distributions play a major role in ruin theory of finance and insurance. It is used for determining surplus or reserve requirements needed for insurance portfolios and for borrowing money.

2. Value at Risk (VaR) is a measure of risk based on a confidence interval that covers the worst expected loss over a given time interval under normal marked conditions.

Example: the VaR (worst loss) of a \$100 million equity portfolio with a 15% measure of variability per annum over 10 days at the 99% confidence level is \$7 million.

$$\$100M(15% * \sqrt{10/252})2.33 = \$7M$$

Comments

- a. There is a 1% confidence level that the portfolio will decrease by more than \$7M per 10 days.
- b. The probability that this event occurs cannot be determined.
- c. VaR gives no information about the severity of a loss.
- d. If the distribution of the returns is unknown, VaR can be determined by simulating the distribution of returns and determining percentiles.
- e. Not sub-additive, i.e., it is possible to construct 2 portfolios A and B such that

$$\text{VaR}(A + B) > \text{VaR}(A) + \text{VaR}(B)$$

Counterintuitive since the portfolio (diversification) should reduce risk. Artzner et al. provides axioms (coherent risk properties) that a risk measure should possess to be coherent. An example of a coherent risk measure is Conditional Value at Risk (CVaR that is expected tail loss).

3. Probability that a system will perform and maintain its function during a specified time interval (o, t).

a. Reliability (survival) function: $R(t) = P(T > t)$

Time to failure before time t: $F(t) = 1 - R(t)$

- i. Usually positively skewed
- ii. Often reflects censored observations meaning the end points have not been reached. With censoring, the actual survival time is larger than the censored survival time.

The probability of failure in the infinitesimal interval (t, t + dt): $f(t) dt$

Hazard function is the rate of failure among items that have survived to time t:

$$h(t) = \frac{f(t)}{R(t)} = \frac{-d \ln R(t)}{dt}$$

Cumulative hazard rate: $H(t) = \int_0^t h(x) dx$

b. Relationships

- i. $R(t) = \exp(-H[t])$
- ii. If $h(t)$ increases with age, $H(t)$ is the increasing failure rate.
Example: object wearing out, aging

If $h(t)$ decreases with age, $H(t)$ is the decreasing failure rate.
Example: Infant mortality, burn-in period

If $h(t)$ is constant with age, $H(t)$ is constant.
Example: Failure time does not depend on age
- iii. $f(t)$ is the proportion of the initial number of items that fail per unit time interval.

$h(t)$ is the proportion of items in service that fail per unit time interval, i.e., represents the risk of failure that changes with age or time.

iv. Distributions for failure times

- (1) Exponential = constant hazard function
- (2) Lognormal = Hazard rate increases at first, then decreases if $\sigma \leq 1$, or has its maximum value at $t=0$ when $\sigma > 1$ (more useful for length of time to repair than for modeling times to failure).
- (3) Weibull has a shape parameter m

$m < 1 \equiv$ hazard rate is decreasing

$m > 1 \equiv$ hazard rate is increasing

$m = 1 \equiv$ hazard rate is constant

- (a) The Weibull is especially useful for failure of structural components in a system that fails when the weakest components fail.
- (b) A bathtub curve that has the 3 stages (Infant mortality region/Useful life region/Wearout region) can be described by defining changing the value of m over the regions.

c. Cox proportional hazard model

$$h(t) = \lambda_0(t) \exp[\gamma + \underline{X}\underline{\beta}]$$

$\lambda_0(t) \equiv$ base hazard function of unspecified shape

$\underline{X} \equiv$ vector of risk factors measured on each individual

$\underline{\beta} \equiv$ vector of parameters describing the relative risk associated with risk factors

Example (non-parametric)

$$X = \begin{cases} 1 & \text{high risk} \\ 0 & \text{low risk} \end{cases}$$

Thus,

$$h(t|x=1) = \lambda_0(t) \exp[\gamma + \beta]$$

$$h(t|x=0) = \lambda_0(t) \exp[\gamma]$$

$$\frac{h(t|x=1)}{h(t|x=0)} = \exp(\beta) \quad \begin{cases} \text{the instantaneous relative risk} \\ \text{conditioned upon survival at time } t \end{cases}$$

Comments:

- i. $\lambda_0(t)$ is usually unknown and cannot be estimated from the data.
- ii. Example: credit risk modeling for corporate bonds based on interest rates and market conditions

d. Kaplan-Meier estimator of the survival function from life-time data

i. Examples

- (1) Measures the fraction of patients living for a certain amount of time after treatment.
- (2) Measures the length of time people remain unemployed after a job loss
- (3) Measures the time until failure of a machine

ii. Advantages

- (1) Nonparametric (empirical distribution)

$$s(t) = \prod_{t_{(i)} \leq t} \left(1 - \frac{d_i}{n_i} \right)$$

where

$t_{(1)} \leq t_{(2)} \leq \dots \leq t_{(n)}$ $t_{(i)}$ is the observed time of death,
 d_i is the number of deaths at $t_{(i)}$

n_i a stochastic process indicating the number of individuals at risk at time $t_{(i)}$

- (2) Can be used with censored data

e. Frailty models are extensions of the Cox model (assume a homogeneous population) that arise in populations with a mixture of hazards.

i. It is a random effects survival model that describes unexplained heterogeneity, which influences unobserved risk factors.

ii. Form

$$h(t) = Zh_0(t)\exp(\gamma + \underline{X}^T\underline{\beta})$$

where Z (frailty) varies from individual to individual and is not observed.

(1) It is univariate if the characteristic varies from individual to individual

(2) It is multivariate if the characteristic is shared with individuals in a group.

iii. Examples

(1) Time between the first fibrillation and the first stroke. The frailty variable is often modeled as being generated from a gamma distribution, compound Poisson, or log-normal.

(2) Right censoring

4. Distributions for loss-modeling

a. Parametric families

i. Generalized beta

$$F(x) = \beta \left(\tau, \alpha, \frac{(x/\theta)^\gamma}{1 + (x/\theta)^\gamma} \right) \quad x > 0$$

ii. Generalized gamma

$$F(x) = \Gamma \left(\alpha; (x/\theta)^\tau \right) \quad x > 0$$

iii. Inverse generalized gamma distribution

$$F(x) = 1 - \Gamma \left(\alpha; (\theta/x)^\tau \right) \quad x > 0$$

b. The following tables (Panjer) show the relationships within the families generated by changing parameters.

Models are determined by:

- i. experimenting by changing parameter values,
- ii. determining the best descriptor of tail distributions,
- iii. comparing hazard rate functions.

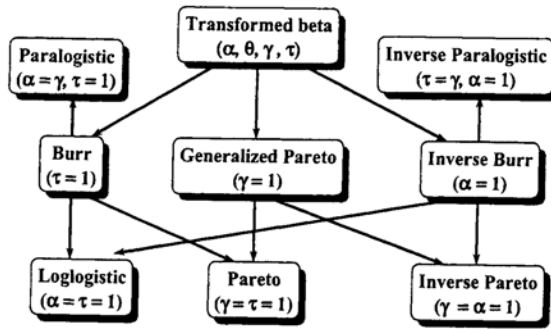


Fig. 1 Transformed beta family

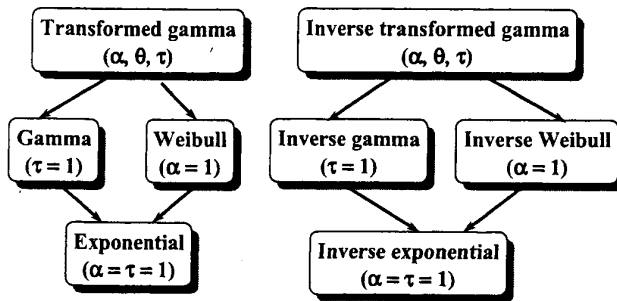


Fig. 2 Transformed/inverse transformed gamma family

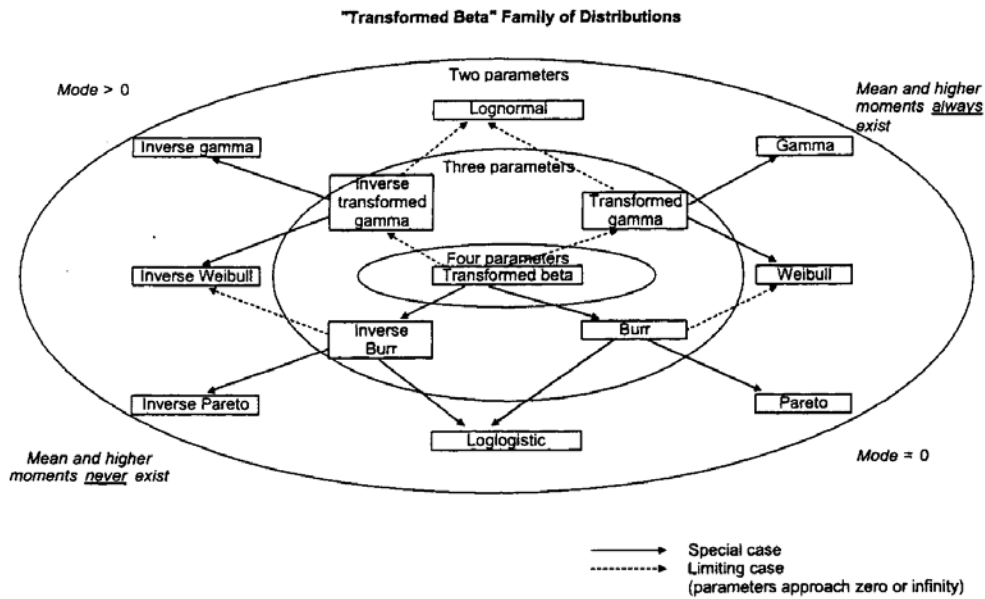


Fig. 3 Distributional relationships and characteristics

c. Counting distributions for describing the, N , number of losses (Johnson)

- i. $(a, b, 0)$ class $P(N = k) = p_k = (a + b/k)p_{k-1}$ $k = 1, 2, \dots$
 ii. $(a, b, 1)$ class $P(N = k) = p_k = (a + b/k)p_{k-1}$ $k = 2, 3, \dots$

Table 1: The $(a, b, 0)$ class

Distribution	a	b	p_0
Poisson	0	λ	$e^{-\lambda}$
Binomial	$-\frac{q}{1-q}$	$(m+1)\frac{q}{1-q}$	$(1-q)^m$
Negative binomial	$\frac{\beta}{1+\beta}$	$(r-1)\frac{\beta}{1+\beta}$	$(1+\beta)^{-r}$
Geometric	$\frac{\beta}{1+\beta}$	0	$(1+\beta)^{-1}$

Table 2: The $(a, b, 1)$ class

Distribution	p_0	a	b	Parameter Space
Poisson	$e^{-\lambda}$	0	λ	$\lambda > 0$
ZT Poisson	0	0	λ	$\lambda > 0$
ZM Poisson	Arbitrary	0	λ	$\lambda > 0$
Binomial	$(1-q)^m$	$-\frac{q}{1-q}$	$(m+1)\frac{q}{1-q}$	$0 < q < 1$
ZT binomial	0	$-\frac{q}{1-q}$	$(m+1)\frac{q}{1-q}$	$0 < q < 1$
ZM binomial	Arbitrary	$-\frac{q}{1-q}$	$(m+1)\frac{q}{1-q}$	$0 < q < 1$
Negative binomial	$(1+\beta)^{-r}$	$\frac{\beta}{1+\beta}$	$(r-1)\frac{\beta}{1+\beta}$	$r > 0, \beta > 0$
ETNB	0	$\frac{\beta}{1+\beta}$	$(r-1)\frac{\beta}{1+\beta}$	$r > -1, \beta > 0$
ZM ETNB	Arbitrary	$\frac{\beta}{1+\beta}$	$(r-1)\frac{\beta}{1+\beta}$	$r > -1, \beta > 0$
Geometric	$(1+\beta)^{-1}$	$\frac{\beta}{1+\beta}$	0	$\beta > 0$
ZT geometric	0	$\frac{\beta}{1+\beta}$	0	$\beta > 0$
ZM geometric	Arbitrary	$\frac{\beta}{1+\beta}$	0	$\beta > 0$
Logarithmic	0	$\frac{\beta}{1+\beta}$	$-\frac{\beta}{1+\beta}$	$\beta > 0$
ZM logarithmic	Arbitrary	$\frac{\beta}{1+\beta}$	$-\frac{\beta}{1+\beta}$	$\beta > 0$

5. Multivariate distributions and copulas (Sklar)

a. Copula

- i. Invariant transformation to combine marginal probability functions to form multivariate distributions
- ii. Measures of dependent structure to form multivariate distributions

b. Procedure for generating multivariate distributions

- i. Determine the marginal distributions
$$F_i(x_i) \quad i = 1, \dots, n$$
- ii. Introduce the probability integral transformations
$$u_i = F_i(x_i) \quad \text{where } u_i \text{ uniform on } [0, 1]$$
- iii. Copula: Is the multivariate distribution
$$c(u_1, \dots, u_n) \text{ on } [0, 1]^n$$
$$= p(U_1 \leq u_1, \dots, U_n \leq u_n)$$
$$= H(F^{-1}(u_1), \dots, F^{-1}(u_n))$$
$$= H(x_1, \dots, x_n) \quad \text{the multivariate CDF of } x_1, \dots, x_n.$$
$$= C(F_1(x_1), \dots, F_n(x_n)) \quad \text{for all } x_1, \dots, x_n.$$
- iv. Comments:
 - (1) $H(x_1, \dots, x_n)$ is the multivariate CDF with marginals $F_i(x_i) \quad i = 1, \dots, n$
 - (2) $c(u_1, \dots, u_n)$ is unique if all $F_i(x_i)$ are continuous.
 - (3) If $F_i(x_i)$ are not continuous, the copulas are unique on the range of positive values for the marginal distributions.

- i. Example: suppose X_1 and X_2 are statistically independent, then

$$\begin{aligned}c(u_1, u_2) &= P(U_1 \leq u_1 \quad U_2 \leq u_2) = u_1 u_2 \\ &= H(F_1^{-1}(u_1), F_2^{-1}(u_2))\end{aligned}$$

$$H(x_1, x_2) = F_1(x_1) F_2(x_2).$$

If x_1 and x_2 are not statistically independent, then

$$c(F_1(x_1), F_2(x_2)).$$

b. Issue: How to form copulas

- i. Generalize concept of ρ
- (1) Linear correlations often too restrictive
 - (2) Only useful in the elliptical family
 - (3) Kendall (τ) and Spearman (ρ) are used because moments not need to exist AND invariant under monotonic transformations
- ii. Listing of copulas in Nelsen
- iii. Archimedean copulas are most popular: additive, continuous, decreasing convex functions

e. Examples

- i. Extreme value theory: asymmetrical tail dependence
- ii. Economics: modeling correlated risks such as groups of individuals exposed to similar economic and physical environments
- iii. Finance: modeling joint default probabilities in credit portfolios
- iv. Actuarial science: modeling joint mortality patterns

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