Final Exercise

We showed that \( V = \{ u \in L^2(a, b) : \partial u \in L^2(a, b) \text{ and } u(a) = 0 \} \) is a Hilbert space with the scalar product

\[
(u, v)_V = \int_a^b (uv + \partial u \partial v) \, dx.
\]

**Exercise 1.** Show that for each \( f \in L^2(a, b) \) there is a unique

\[
u \in V : \int_a^b (uv + \partial u \partial v) \, dx = \int_a^b f v \, dx \quad \forall v \in V.
\]

**Exercise 2.** Denote (1) by \( u = G(f) \). Show this is equivalent to a boundary-value problem.

**Exercise 3.** Show that \( G \in \mathcal{L}(L^2(a, b), V) \) and that \( G \) is one-to-one.

**Exercise 4.** Show that \( G \in \mathcal{L}(L^2(a, b)) \) is self-adjoint and compact. Hint: Let \( u = G(f), \ v = G(g) \), compute \( (f, G(g))_{L^2} = (f, v)_{L^2} \) and \( (g, G(f))_{L^2} = (g, u)_{L^2} \). Recall that the identity \( H^1(a, b) \to L^2(a, b) \) is compact.

**Exercise 5.** Compute the eigenvalues and eigenfunctions for \( G \).

Hint: Note \( G(u) = \mu u \) is equivalent to \( G(\lambda u) = u \) with \( \lambda = \mu^{-1} \), and use Exercise 2.

**Exercise 6.** Find the range \( Rg(G) \) and show it is a Hilbert space with the scalar-product

\[
(u, v)_{H^2} = \int_a^b (uv + \partial u \partial v + \partial^2 u \partial^2 v) \, dx.
\]

Show that \( G \in \mathcal{L}(L^2(a, b), H^2(a, b)) \) Hint: Show that the graph of \( G \) is closed.