Closed Range Theorem

Let $V$ be a Hilbert space and denote by $\mathcal{R}_V : V \to V'$ the Riesz isomorphism, $\mathcal{R}_V v(w) = (v, w)_V$, $\forall v, w \in V$. For any subset $W \subset V$ we have $\mathcal{R}_V(W^\perp) = W^a \subset V'$. That is, orthogonal complement corresponds to annihilator. We identify $V = V''$.

Let $S$ be another Hilbert space and $\mathcal{B} \in \mathcal{L}(V, S')$, that is, it is continuous and linear. Its dual $\mathcal{B}' \in \mathcal{L}(S, V')$, and a direct computation gives

$$\text{Ker}(\mathcal{B}) = \{v \in V : \mathcal{B}v(s) = 0 \, \forall s \in S\} = \{v \in V : \mathcal{B}'s(v) = 0 \, \forall s \in S\} = (\text{Rg} \mathcal{B}')^a.$$

**Lemma 0.1.** If $\mathcal{B} \in \mathcal{L}(V, S')$, the following are equivalent:

1. $\text{Rg} \mathcal{B}$ is closed in $S'$.
2. $\text{Rg} \mathcal{B} = (\text{Ker} \mathcal{B}')^a$.
3. There is a $\mathcal{B}^R \in \mathcal{L}(\text{Rg} \mathcal{B}, (\text{Ker} \mathcal{B})^\perp)$ and a constant $c_b > 0$ such that $\mathcal{B} \mathcal{B}^R = I$ on $\text{Rg} \mathcal{B}$ and $c_b \|\mathcal{B}^R(g)\|_V \leq \|g\|_{S'}$ for all $g \in \text{Rg} \mathcal{B}$.
4. For some constant $c_b > 0$,

$$\inf_{v \in V} \sup_{s \in S} \frac{\mathcal{B}v(s)}{\|v\|_V \|s\|_S} \geq c_b.$$

**Proof.** From above we have $\text{Ker}(\mathcal{B}') = (\text{Rg} \mathcal{B})^a$ and so $\text{Ker}(\mathcal{B}')^a = (\text{Rg} \mathcal{B})^{aa} = \overline{\text{Rg} \mathcal{B}}$. This shows (1) is equivalent to (2).

From (1) we have $\mathcal{B} : (\text{Ker} \mathcal{B})^\perp \to \overline{\text{Rg} \mathcal{B}}$ is continuous and injective, so it is necessarily an isomorphism, and this implies (3); (1) follows directly from (3) since $\mathcal{B}$ is continuous.

Finally, we note the equivalence of (3) and (4) follows from that of

$$c_b \inf_{w \in \text{Ker} \mathcal{B}} \|v + w\|_V \leq \|\mathcal{B}v\|_{S'} \text{ and } c_b \|v\|_{V/\text{Ker} \mathcal{B}} \leq \sup_{s \in S} \frac{\mathcal{B}v(s)}{\|s\|_S}.$$
**Corollary 0.2.** \( \text{Rg } \mathcal{B}' \) is closed in \( V' \).

**Proof.** If \( s \in (\text{Ker } \mathcal{B}')^\perp \) then \( R_S(s) \in (\text{Ker } \mathcal{B}')^\perp = \text{Rg } \mathcal{B} \) and so \( R_S(s) = \mathcal{B} v \) where we define \( v \equiv \mathcal{B}^R(g) \). Thus we have \( c_b \|v\|_V \leq \|R_S(s)\|_{S'} = \|s\|_S \) from which there follows

\[
\|s\|^2_S = R_Ss(s) = \mathcal{B}v(s) = \mathcal{B}'s(v) \leq \|\mathcal{B}'s\|_{V'}\|v\|_V \leq \|\mathcal{B}'s\|_{V'} \frac{1}{c_b} \|s\|_S.
\]

This implies \( c_b \|s\|_S \leq \|\mathcal{B}'s\|_{V'} \) (with the same constant \( c_b \)) and hence that \( \text{Rg } \mathcal{B}' \) is closed. \( \Box \)

The Corollary shows \( \text{Rg } \mathcal{B} \) is closed if and only if \( \text{Rg } \mathcal{B}' \) is closed, and we have obtained the **Closed Range Theorem**.

**Theorem 0.3.** If \( \mathcal{B} \in \mathcal{L}(V, S') \), the following are equivalent:

1. \( \text{Rg } \mathcal{B} \) is closed in \( S' \).
2. \( \text{Rg } \mathcal{B} = (\text{Ker } \mathcal{B})^\perp \).
3. There is a right-inverse \( \mathcal{B}^R \in \mathcal{L}(\text{Rg } \mathcal{B}, (\text{Ker } \mathcal{B})^\perp) \) and a constant \( c_b > 0 \) such that \( \mathcal{B} \mathcal{B}^R = I \) on \( \text{Rg } \mathcal{B} \) and \( c_b \|\mathcal{B}^R(g)\|_V \leq \|g\|_{S'} \) for all \( g \in \text{Rg } \mathcal{B} \).
4. For some constant \( c_b > 0 \),

\[
\inf_{v \in V} \sup_{s \in S} \frac{\mathcal{B}v(s)}{\|v\|_{V/\text{Ker } \mathcal{B}} \|s\|_S} \geq c_b.
\]

5. \( \text{Rg } \mathcal{B}' \) is closed in \( V' \).
6. \( \text{Rg } \mathcal{B}' = (\text{Ker } \mathcal{B}')^\perp \).
7. There is a right-inverse \( \mathcal{B}'^R \in \mathcal{L}(\text{Rg } \mathcal{B}', (\text{Ker } \mathcal{B}')^\perp) \) and a constant \( c_b > 0 \) such that \( \mathcal{B}' \mathcal{B}'^R = I \) on \( \text{Rg } \mathcal{B}' \) and \( c_b \|\mathcal{B}'^R(f)\|_S \leq \|f\|_{V'} \) for all \( f \in \text{Rg } \mathcal{B}' \).
8. For some constant \( c_b > 0 \),

\[
\inf_{s \in S} \sup_{v \in V} \frac{\mathcal{B}'s(v)}{\|s\|_{S/\text{Ker } \mathcal{B}'} \|v\|_V} \geq c_b.
\]