Notation. \( \mathbb{R}^3 \) is the space of vectors \( \mathbf{x} = < x_1, x_2, x_3 > \) with each \( x_j \in \mathbb{R} \). Note that vectors are given with bold face notation \( \mathbf{x} \) and their components by \( x_j \) for \( j = 1, 2, 3 \). The standard basis on \( \mathbb{R}^3 \) is

\[
\mathbf{i} = < 1, 0, 0 >, \quad \mathbf{j} = < 0, 1, 0 >, \quad \mathbf{k} = < 0, 0, 1 >.
\]

Each vector in \( \mathbb{R}^3 \) is \( \mathbf{x} = < x_1, x_2, x_3 > = x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k} \).

Vectors represent *position*. That is, the vector \( \mathbf{x} \) determines the point in space with coordinates \( (x_1, x_2, x_3) \). Vectors represent *displacement*. That is, the displacement \( \mathbf{x} \) moves each point \( \mathbf{y} \) in space to the point \( \mathbf{x} + \mathbf{y} \). Displacement is independent of position, since it acts on *all* positions, but position is the displacement from the origin, \( \mathbf{0} = < 0, 0, 0 > \). Which notion is appropriate should be clear from the context. Vectors are also appropriate to represent velocity, acceleration, force, current, .... In general, each vector is equivalent to a *length* and a *direction*.

The usefulness of vectors depends on the correspondence between their *algebraic* and *geometric* notions.

<table>
<thead>
<tr>
<th>Algebra</th>
<th>Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{x} = &lt; x_1, x_2, x_3 &gt; )</td>
<td>length &amp; direction</td>
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<tr>
<td>addition</td>
<td>parallelogram law</td>
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<tr>
<td>scalar multiplication</td>
<td>changes length</td>
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<tr>
<td>norm (</td>
<td></td>
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<tr>
<td>unit vector ( \frac{\mathbf{x}}{</td>
<td></td>
</tr>
<tr>
<td>dot product ( \mathbf{x} \cdot \mathbf{y} )</td>
<td>angle between vectors</td>
</tr>
<tr>
<td>cross product ( \mathbf{x} \times \mathbf{y} )</td>
<td>perpendicular to both</td>
</tr>
</tbody>
</table>

**Angle and Dot Product.** Radian measure of an *angle* \( \theta \) is the distance traveled along the *unit circle* from the x-axis to the point that determines the angle. The coordinates of that point are \( (\cos \theta, \sin \theta) \). This defines the trigonometric functions, *cosine* and *sine*. 

1
Let’s compute the angle between two unit vectors in the plane, \( \mathbb{R}^2 \). They correspond to two points on the unit circle. These are determined by their respective angles measured counterclockwise from the x-axis, \( \alpha \) and \( \beta \), and the coordinates of these points are \((\cos \alpha, \sin \alpha)\) and \((\cos \beta, \sin \beta)\). We assume \( \beta \) is the larger angle, and we want to compute the angle \( \theta = \beta - \alpha \) between these unit vectors.

Rotate the sector from \( \alpha \) to \( \beta \) to align \( \alpha \) with the x-axis. Then \( \beta \) is aligned with the point \((\cos \theta, \sin \theta)\). The distance from \((\cos \alpha, \sin \alpha)\) to \((\cos \beta, \sin \beta)\) is equal to the distance from \((1, 0)\) to \((\cos \theta, \sin \theta)\), so we have

\[
(\cos \beta - \cos \alpha)^2 + (\sin \beta - \sin \alpha)^2 = (\cos \theta - 1)^2 + (\sin \theta - 0)^2
\]

and simplifying this gives the result

\[
(0.1) \quad \cos \theta = \cos \beta \cos \alpha + \sin \beta \sin \alpha
\]

for the angle between the unit vectors.

Now let \( \mathbf{u} \) and \( \mathbf{v} \) be any two non-zero vectors. We use our previous result to compute the angle between them. Orient the pair of vectors so they lie in the plane, \( \mathbb{R}^2 \). The unit vectors \( \frac{\mathbf{u}}{||\mathbf{u}||} = (\frac{u_1}{||\mathbf{u}||}, \frac{u_2}{||\mathbf{u}||}) = (\cos \beta, \sin \beta) \) and \( \frac{\mathbf{v}}{||\mathbf{v}||} = (\frac{v_1}{||\mathbf{v}||}, \frac{v_2}{||\mathbf{v}||}) = (\cos \alpha, \sin \alpha) \) lie on the unit circle, and from (0.1) the angle between them is \( \theta \), where

\[
\cos \theta = \frac{u_1 v_1}{||\mathbf{u}|| ||\mathbf{v}||} + \frac{u_2 v_2}{||\mathbf{u}|| ||\mathbf{v}||}.
\]

This shows that we have

\[
||\mathbf{u}|| ||\mathbf{v}|| \cos \theta = u_1 v_1 + u_2 v_2.
\]

This quantity is the dot product between the vectors, and more generally for \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{R}^3 \) we define

\[
(0.2) \quad \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.
\]

This provides a means to calculate the angle \( \theta \) between vectors \( \mathbf{u} \) and \( \mathbf{v} \) by

\[
\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| ||\mathbf{v}|| \cos \theta.
\]

**Projection.**