1. Linear Algebra Basics

We begin with a discussion of the solvability of a system of \( n \) equations in \( n \) unknowns:

\[
\sum_{i=1}^{n} a_{ij} x_j = f_i, \quad j = 1, \ldots, n.
\]

The corresponding homogeneous system is

\[
\sum_{i=1}^{n} a_{ij} x_j = 0, \quad j = 1, \ldots, n,
\]

and it plays an important role. The coefficients, unknown and right side of the system (1) are given as the \( n \times n \) matrix and column vectors

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix},
\]

and the system (1) is written as \( Ax = f \).

**Theorem 1.** The matrix \( A \) is non-singular if and only if for every vector \( f = (f_1, f_2, \ldots, f_n)^T \) the system (1) has exactly one solution, and this holds if and only if the only solution of the homogeneous system (2) is the null vector \( x = 0 = (0, 0, \ldots, 0)^T \).

In this case, the solution of (1) is given by \( x = A^{-1}f \). Use the row reduction method to find \( x \), or to find \( A^{-1} \).

**Definition 1.** If \( \sum_{i=1}^{n} c_i x^i = 0 \) holds for a non-zero vector \( c = (c_1, c_2, \ldots, c_n)^T \), we say that the vectors \( x^i \) are linearly dependent.

This implies that one of the vectors is a linear combination of the others. On the other hand, if the equation \( \sum_{i=1}^{n} c_i x^i = 0 \) implies that all \( c_i = 0 \), then the vectors \( x^i \) are linearly independent.
2. The Initial-Value Problem

Our objective is to describe the set of solutions of the initial-value problem

\( x'(t) + A(t)x(t) = f(t), \quad a < t < b, \)  
\( x(t_0) = x_0, \)  

in which the matrix of coefficients and right side are given by

\[
A(t) = \begin{pmatrix}
  a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\
  a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t)
\end{pmatrix}, \quad f(t) = \begin{pmatrix}
  f_1(t) \\
  f_2(t) \\
  \vdots \\
  f_n(t)
\end{pmatrix}.
\]

The system (3a) is written in components as

\[
x_1'(t) + a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) = f_1(t), \\
x_2'(t) + a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) = f_2(t), \\
\vdots \\
x_n'(t) + a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) = f_n(t),
\]

for the solution \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T. \) This is a general linear first-order system of differential equations, and (3b) is the initial condition. The discussion depends on the following existence-uniqueness theorem:

**Theorem 2.** Assume that the functions \( a_{ij}(t), f_i(t) \) are continuous on the interval \( a < t < b \) and that \( t_0 \) is a point in this interval. Then for any initial vector, \( x_0, \) there is exactly one solution \( x(t) \) of the initial-value problem (3), and the solution exists on the entire interval \((a, b)\).

Note that the set of vector-valued functions on the interval \((a, b)\) is a linear space, so the notion of linearly independent applies as well to them.

2.1. Homogeneous equation. We begin by describing the general solution of the corresponding homogeneous equation

\( x'(t) + A(t)x(t) = 0. \)

Note that if \( x_j(t) \) are solutions to (4), then for any set of constants, \( c_j, \) the linear combination \( \sum_{j=1}^{n} c_j x_j(t) \) is also a solution of (4).
Definition 2. Let the \( n \) functions \( x_1(t), x_2(t), \ldots, x_n(t) \) be solutions of (4). The Wronskian of these solutions is the function

\[
W(t) = |x_1(t) x_2(t) \ldots x_n(t)| = \begin{vmatrix}
x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\
x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\
\vdots & \vdots & \ddots & \vdots \\
x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t)
\end{vmatrix},
\]

the indicated determinant of the row matrix of column vectors.

Note that for any \( t \), the matrix \( (x_1(t), x_2(t), \ldots, x_n(t)) \) is singular if and only if \( W(t) = 0 \). We shall apply the linear algebra results above.

Suppose that there is a \( t_0 \) in \((a, b)\) for which \( W(t_0) = 0 \). Then there is a set of constants \([c_1, c_2, \ldots, c_n]\), not all zero, such that

\[
\sum_{j=1}^{n} c_j x_j(t_0) = 0.
\]

This implies that the function \( \sum_{j=1}^{n} c_j x_j(t) = x(t) \) is a solution of the initial-value problem (3) with \( f(t) = 0, \ x_0 = 0 \). But the zero function is a solution of this same initial-value problem, so by the uniqueness part of Theorem 2, we must have \( x(t) = 0 \). That is, we have shown there is a set of constants, \([c_1, c_2, \ldots, c_n]\), not all zero, such that

\[
\sum_{j=1}^{n} c_j x_j(t) = 0, \quad a < t < b.
\]

This says that the solutions \( x_1(t), x_2(t), \ldots, x_n(t) \) are linearly dependent. On the other hand, if the solutions \( x_1(t), x_2(t), \ldots, x_n(t) \) are linearly dependent, then there are constants \([c_1, c_2, \ldots, c_n]\), not all zero, such that (7) holds, and this implies that \( W(t) = 0 \) for every \( t \) in the interval \((a, b)\). Thus we have shown that \( W(t_0) = 0 \) for some \( t_0 \) in the interval \((a, b)\) if and only if \( W(t) = 0 \) for all \( t \) in the interval \((a, b)\), and this holds if and only if the solutions \( x_1(t), x_2(t), \ldots, x_n(t) \) are linearly dependent.

Now suppose that \( t_0 \) is a point in \((a, b)\) and \( W(t_0) \neq 0 \). Let \( x(t) \) be any solution of (4). Since the matrix of coefficients is non-singular, we can
solve the system of algebraic equations

\[ \sum_{j=1}^{n} c_j x_j(t_0) = x(t_0) \]

for a unique set of constants \([c_1, c_2, \ldots, c_n]\). The system (8) implies that the functions \(\sum_{j=1}^{n} c_j x_j(t)\) and \(x(t)\) are both solutions of the same initial-value problem (3) with \(f(t) = 0\). But then we must have (by uniqueness of solutions)

\[ x(t) = \sum_{j=1}^{n} c_j x_j(t), \quad a < t < b. \]

Thus, every solution of the homogeneous equation (4) is given by (9) for a corresponding set of constants, \([c_1, c_2, \ldots, c_n]\).

**Definition 3.** If all solutions of the homogeneous equation (4) are given by linear combinations of the solutions \(x_1(t), x_2(t), \ldots, x_n(t)\), we say that these solutions are a fundamental set or basis of solutions of (4).

We summarize this in the following.

**Theorem 3.** Let the functions \(x_1(t), x_2(t), \ldots, x_n(t)\) be solutions of (4). The Wronskian \(W(t)\) is either identically zero or never zero on the interval \((a, b)\).

In the first case, the solutions \(x_1(t), x_2(t), \ldots, x_n(t)\) are linearly dependent.

In the second case, the functions \(x_1(t), x_2(t), \ldots, x_n(t)\) are a fundamental set of solutions of (4). In particular, for any choice of initial values, \(x_0\), there is a unique set of constants, \([c_1, c_2, \ldots, c_n]\) for which (9) is the solution of the initial-value problem consisting of the homogeneous equation (4) together with the initial conditions (3b).

An algorithm for representing all solutions to (4) follows immediately. Find a fundamental set of solutions to (4), and then the general solution is represented as linear combinations of these \(n\) solutions. The Wronskian provides a means to test these to determine whether they are a fundamental set or they are linearly dependent.
2.2. Non-homogeneous equation. Now we can describe the solutions of the non-homogeneous equation (3a). Suppose that we have a solution $x_p(t)$ of the non-homogeneous equation (3a). If $x(t)$ is any solution of (3a), then the difference satisfies $(x(t) - x_p(t))' + A(x(t) - x_p(t)) = 0$, so $x(t) - x_p(t)$ is a solution of the homogeneous equation (4). As such, it must be a linear combination of the fundamental set $x_1(t), x_2(t), \ldots, x_n(t)$, so we have

\begin{equation}
  x(t) = \sum_{j=1}^{n} c_j x_j(t) + x_p(t), \quad a < t < b.
\end{equation}

**Theorem 4.** Let the functions $x_1(t), x_2(t), \ldots, x_n(t)$ be a fundamental set of solutions of the homogeneous equation (4), and let $x_p(t)$ be a solution of the non-homogeneous equation (3a). Then the general solution of (3a) is given by (10) for an appropriate set of constants. For any choice of initial values, $x_0$, there is a unique set of constants, $[c_1, c_2, \ldots, c_n]$ for which (10) is the solution of the initial-value problem (3a).

Thus, in order to find all solutions of (3a), or to find the solution of the initial-value problem (3), it suffices to find a fundamental set of solutions to the homogeneous equation (4) and one solution to the non-homogeneous equation (3a).