MTH 614: Exercise set #2

Problems are due (electronically or in my mailbox in Kidder) on Dec 6 (Thursday) at 4pm = 1600.

1. In the classical theory of Fourier series, the orthonormal basis for \( L^2(0, \ell) \) consists of cosine and sine functions. Denote these by \( \{v_j(x) : j = 1, 2, \ldots \} \). Each \( f \in L^2(0, \ell) \) is represented by
\[
f = \sum_{j=1}^{\infty} (f, v_j)_{L^2} v_j.
\]
The \( n \)-th partial sum is
\[
f_n = \sum_{j=1}^{n} (f, v_j)_{L^2} v_j,
\]
and one can actually sum the series to obtain the representation
\[
f_n(x) = \int_0^\ell f(y) D_n(y - x) \, dy.
\]
The integrand is the Dirichlet kernel \( D_n(x) \).

Denote the space of continuous \( \ell \)-periodic functions by \( C_p[0, \ell] \). For \( f \in C_p[0, \ell] \) define
\[
T_n f = f_n(0) = \int_0^\ell f(y) D_n(y) \, dy
\]
to get a sequence \( \{T_n\} \) in the dual space \( C_p'[0, \ell] \). From an explicit computation one finds that \( \|T_n\| \to \infty \).
Show there exists a continuous periodic function whose Fourier series is not convergent at \( x = 0 \).

2. Let \( H \) be a hyperplane in a normed linear space \( X \). That is, \( H = \{x \in X : f(x) = \alpha\} \) where \( f \in X^{\ast} \) and \( \alpha \in \mathbb{R} \). Show that \( H \) is either closed or dense in \( X \).

3. Let \( B \in \mathcal{L}(X, Y') \) where \( X \) and \( Y \) are Banach spaces. Show the following are equivalent:
   - \( B : X \to Y' \) has closed range,
   - \( B(X) = (\text{Ker} \, B')^o \),
   - there is a \( \beta > 0 \) for which \( \sup_{y \in Y} \frac{\|Bx\|}{\|y\|} \geq \beta \inf_{z \in \text{Ker} \, B} \|x + z\|_X \), \( x \in X \).

4. a) Let \( V = H^1(0, \ell) \) and \( a(\cdot) \in L^\infty(0, \ell) \) with \( a(x) \geq c > 0 \). Show that
\[
u \in V : \int_0^\ell a(x) \partial u(x) \partial v(x) \, dx = \int_0^\ell F(x)v(x) \, dx + \alpha v(\ell)
\]
characterizes a boundary-value problem.
   b) Let \( A : V \to V' \) denote the corresponding operator. What is the kernel of \( A \)? For the functional
\[
f(v) = \int_G F(x)v(x) \, dx + \alpha v(\ell),
\]
when is \( f \) in \( \text{Rg} \, A \)?