Applicable Analysis, Vol. 32, pp. 183–202
Reprints available directly from the publisher
Photocopying permitted by license only

Printed in Great Britain

Homogenization of the Layered Medium Equation

Communicated by R. Carroll

MARIE-PASCALE BOSSE
Department of Mathematics, Iowa State University, Ames, Iowa 50010

R. E. SHOWALTER
Department of Mathematics, The University of Texas,
Austin, Texas 78712

AMS(MOS): 35K40

Abstract. A new partial differential equation to be called the layered medium equation is introduced, and it is proved that certain relevant initial or periodic boundary conditions give well-posed problems. Then, the homogenized limit of the layered medium equation is studied. It is shown to be preserved in limit in the physical problem in which the coefficients that arise from the dielectric layer are both proportional to thickness. Otherwise, a non-local problem is obtained as the limiting form.

KEY WORDS: Layered medium equation, homogenization

(*Received for Publication 22 June 1988*)

I. Introduction

From the integrated circuit and also the thin film technology there arise a new and important class of network elements: the distributed components. The theory of distributed networks is the natural form in which to study the behavior of a system in which wave lengths of interest are comparable to the physical dimensions of the system. Such networks have many advantages; for example, their use permits fewer components than comparable circuits using lumped elements. A distributed network is not easily described by the classical methods of network theory, and new techniques are being devised to treat them, either directly by methods of partial differential equations or indirectly by approximation with "nearly equivalent" lumped models to which traditional methods apply. See [8], [9], [11], [13].

Multilayered structures offer many advantages in integrated circuits

[11]. Here we consider a three-dimensional structure consisting of a finely layered arrangement of alternating conductive and dielectric materials. Such a structure is highly anisotropic and offers the opportunity for a variety of new phenomena. For example, the electrical response of such a structure depends not only on the component materials, the proportions and their arrangement but also on the shape of the final structure, the location and size of the contacts, and other geometric variables which are so numerous, partly because of the possibilities of a three dimensional structure. There appears to be no limit to how fine the system of layers can be constructed. However the extremely fine layering that is currently possible by intercalation [7] goes beyond our interest here, since the traditional laws no longer are valid on this submicron level.

The objective in the following is to develop by classical continuum methods a model of a distributed RC network in dimension 3 consisting of a sequence or a continuum of successive layers of (mostly) resistive and (mostly) dielectric materials. This results in a structure which is extremely anisotropic on the macroscopic level. This layered structure has electrical properties periodic in the vertical direction. Thus we consider successively the following mathematical models:

(1) -Discretely–layered... a parabolic system of partial differential equations. There are as many unknowns as layers and each is a function of the position $x = (x_1, x_2) \in R^2$ and time $t > 0$.

(2) -Continuum limit... a single new partial differential equation of implicit type which we shall call the *layered medium equation.* The unknown is a function of position $(x, z) = (x_1, x_2, z) \in R^3$ and time $t > 0$. Coefficients $G_H(x, z)$, $C(x, z)$ describe the distributed horizontal conductance and vertical capacitance of the limiting heterogeneous material.

(3) -Layered continuum... The case of the layered medium equation with coefficients rapidly changing (periodic) in the z-direction. This is the microstructure of the network, probably the *best model* of the real material. However this problem has many interfaces or singularities,

roughly the same number as there are layers. With such a large number of irregularities, it is effectively impossible to deal with them individually in any computational program to solve the problem.

(4) -Homogenized limit... This is a model of the structure with constant coefficients, the *effective* coefficients. The partial differential equation is obtained by studying the limit of (3). The numerical solution of this limit equation is essentially standard and it will serve as an approximation of the layered continuum.

II. The Layered Medium Equation.

We shall develop the network equations for distributed RC networks which are multi-layered structures consisting of alternating thin-films of resistive and dielectric materials. First we consider a structure consisting of a finite number of layers each with positive thickness. The resulting parabolic system of partial differential equations is too singular for conventional numerical treatment. Then we obtain the limiting case of a continuum of layers. This gives an implicit partial differential equation in three spatial variables which is used later as the fundamental equation for electrical conductivity in layered media, the layered medium equation. Then we show that various initial-boundary-value problems for equations of that type are well-posed; we also discuss corresponding results for periodic problems which intervene later. After presenting a model case which suffices for our later discussion of homogenization, we develop the theory to include many non-homogeneous terms which are useful models of input options in the application to layered RC circuits.

Model Problems.

Consider the voltage distribution $u_k(x_1, x_2, t)$ in the k-th conductive layer as a function of time $t > 0$ and position $x = (x_1, x_2)$ with respect to a planar reference region $\mathcal{O}$ in R^2. We shall consider the interaction of this voltage with vertical current flow into this layer. A voltage gradient (or electric field) in the k^{th} conductive layer induces a current $\overline{J}$ given by Ohm's law as $\overline{J} = -cG_H^k(x)\overline{\nabla}u_k$ where $G_H^k(x)$ is the spacially distributed conductance of this resistive layer, c is the thickness, and the gradient

$\overline{\nabla} = (\partial_{x_1}, \partial_{x_2})$ is taken in the horizontal direction. Similarly, a vertical electric field resulting from a voltage difference $u_{k+1} - u_k$ across the upper dielectric layer of thickness $d > 0$ gives a vertical current distribution into the k^{th} layer of magnitude $G_V^{k+1}(x)(u_{k+1} - u_k)/d$. This represents leakage or loss through the dielectric. A similar contribution of current arises from the lower dielectric layer and we assume $c \ll d$. Finally, the voltage differences across the dielectric layers induce on the k^{th} layer a charge of magnitude

$$Q = C^k(x)(u_k - u_{k-1})/d + C^{k+1}(x)(u_k - u_{k+1})/d$$

Let $\mathcal{O}_0$ be a part of $\mathcal{O}$. Since the sum of all currents coming into the k^{th} level over $\mathcal{O}_0$ through its boundary $\partial\mathcal{O}_0$ plus any outside sources of spatial density $F_k(x,t)$ and those as above distributed over $\mathcal{O}_0$ must equal the rate at which charge accumulates on the k^{th} layer over $\mathcal{O}_0$, the conservation of charge requires that

$$(d/dt)\int\int_{\mathcal{O}_0} Q\,dx = \int\int_{\mathcal{O}_0} \{cF_k + (G_V^{k+1}(u_{k+1} - u_k)/d$$
$$+ G_V^k(u_{k-1} - u_k)/d\}\,dx + \int_{\partial\mathcal{O}_0} cG_H^k(x)\overline{\nabla}u_k\,ds$$

Since $\mathcal{O}_0$ is arbitrary in $\mathcal{O}$ we obtain from the divergence theorem

$$\partial_t(C^{k+1}\delta u_{k+1} - C^k\delta u_k) + (G_V^{k+1}\delta u_{k+1} - G_V^k\delta u_k) + \nabla\cdot(cG_H^k\overline{\nabla}u) = -cF_k$$

as the partial differential equation for voltage distributions in the k^{th} layer. Here $\delta u_k = (u_k - u_{k-1})/d$ is the electric field in the dielectric. If we denote by $\delta^* v_k = (v_k - v_{k+1})/d$, the corresponding dual difference operator, this takes the form

$$d\frac{\partial}{\partial t}(\delta^* C^k\delta u_k) + d\delta^* G_V^k\delta u_k) - \nabla\cdot(cG_H^k\overline{\nabla}u) = cF_k(x,t)$$

of a parabolic system whose unknowns $u(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_N(x,t))$ are voltage distribution at horizontal layers; we may take $u_0(x,t) = 0$ as the reference level at the bottom. In summary, this system is a model of a discretely-layered medium consisting of conductive sheets of thickness $c(x)$ and spatial conductivity $G_H^k(x)$ alternating with dielectric sheets of thickness $d(x)$, capacitance $C^k(x)$, and a (small) vertical leakage conductance $G_V^k(x)$ at the position x on the k^{th} level. This is the model of a discretely-layered structure.

Consider now a three-dimensional structure consisting of very thin horizontal layers of conductor within a dielectric material. We shall describe it as a continuum model obtained as a limit of the system above of discretely-layered structures. Thus, at each level we assume the widths d, c are both scaled by the same factor $h > 0$, *i. e.* are replaced by hd and hc, respectively. Upon dividing our system equations by h and letting $h \to 0$ we obtain the partial differential equation

$$-d(x)\frac{\partial}{\partial t}(\partial_z C(x,z)\partial_z u) - d(x)(\partial_z G_V(x,z)\partial_z u) - \overline{\nabla} \cdot c(x)G_H(x,z)\overline{\nabla} u$$
$$= c(x)F(x,z,t)$$

for a continuously-layered medium. This equation is a model for an extremely anisotropic medium obtained as the limit by intercalation of layers by horizontally conducting sheets into dielectric material. The current is assumed to flow horizontally in the conductor and vertically in the dielectric. The numbers d and c give the ratio of the respective materials. We refer to it as the *layered medium equation.* Note that for a dielectric material with parameters C, G_V independent of (x,z), the corresponding terms that occur in the layered medium equation are dC and dG, respectively, so even if the dielectric thickness d varies with height z and position x, the *ratio* of these coefficients is constant. This case will take a prominent role in the development below. The unknown $u(x_1,x_2,z,t)$ is a voltage in the medium measured with respect to a reference level which we take arbitrarily along the bottom of the region of interest. The boundary conditions appropriate to specify for the layered medium equation are either the voltage level u or the current in the direction out of the sides of the region, $cG_H\vec{\nabla}u \cdot \vec{v_x} + dG_V\partial_z u v_z$, where $v = (\vec{v_x}, v_z)$ is the unit outward normal on the boundary of the region. The initial condition is to specify the charge distribution $d(x)Q_0(x) = -\partial_z C(x,z)\partial_z u(x,z,0)$ throughout the region.

We shall show that boundary and initial conditions as those above lead to a well posed problem for the layered medium equation. To begin, let us recall certain notations and basic material on Sobolev spaces and traces. Let G be a domain of R^N, $N > 1$. At each point of the boundary ∂G let $\nu = (\nu_1 \ldots, \nu_N)$ denote the unit outward normal. Let ∂G be partitioned into two measurable sets Γ_+ and Γ_- such that Γ_- includes all $s \in \partial G$ where $\nu_N(s) < 0$; *i.e.*, Γ_- includes the "bottom" of ∂G. Denote by $H^1(G)$ the Sobolev space $H^1(G) = \{v \in L^2(G) : \partial_i v \in L^2(G), 1 \leq j \leq N\}$ where ∂_j is the distributional derivative $\partial/\partial x_j$. $H^1(G)$ is a Hilbert space with the norm given by $\|v\|^2_{H^1} = \sum_{j=0}^n |\partial_j v|^2_{L^2}$, where ∂_0 is the identity. The trace operator is the continuous extension of the restriction to ∂G, $\gamma : H^1(G) \to H^{1/2}(\partial G)$, where the range $H^{1/2}(\partial G)$ is the indicated fractional-order space, dense and continuously imbedded in $L^2(\partial G)$. The kernel of γ is $H^1_0(G)$, the closure in $H^1(G)$ of $C_0^\infty(G)$. See [**14,22**] for these and related facts. Another Hilbert space that arises naturally in the following is

$$H^1_N(G) = \{v \in L^2(G) : \partial_N v \in L^2(G)\}$$

with the norm given by $\|v\|^2_{H^1_N} = |v|^2_{L^2} + |\partial_N v|^2_{L^2}$. The trace operator $\gamma_N = \nu_N \cdot \gamma$ extends continuously from $C^\infty(\bar{G})$ to $\gamma_N : H^1_N(G) \to H^{-1/2}(\partial G)$, the dual of $H^{1/2}(\partial G)$. See [**29**]. Finally, we mention the following inequality which permits us to drop the $L^2(G)$ part of the $H^1_N(G)$ norm and have an equivalent norm on the subspace

$$W \equiv \{v \in H^1_N(G) : \gamma_N v = 0 \quad \text{on } \Gamma_-\} .$$

Lemma (Poincaré). *Let G be bounded in the x_N-direction:*

$$\textit{assume } -K \leq x_N \leq 0 \quad \textit{for all } x = (x', x_N) \in G .$$
$$\textit{Then } |v|_{L^2(G)} \leq 2K |\partial_N v|_{L^2(G)} , \qquad v \in W .$$

Proof. Integrating the derivative $\partial_N(x_N v^2(x)) = v^2 + 2x_N v \partial_N v$ over G gives

$$|v|^2_{L^2} \leq \int_{\Gamma_+} (x_N v)(\nu_N v) + 2K |v|_{L^2} |\partial_N v|_{L^2}$$

and the boundary integral is non-positive since $x_n \le 0$ and $\nu_N \ge 0$ on Γ_+.

Let W denote the space above and define $V = \{v \in H^1(G) : \gamma v = 0$ a.e. on $\Gamma_-\}$. It follows as in **[29]** that V is dense in W. These spaces will be used to specify the boundary-value problems corresponding to the layered-medium equation.

Stationary Problems.

Let three functions C, G_1, and G_2 be given in $L^\infty(G)$ and assume $C(x) \ge \beta > 0$, $G_1(x) \ge \beta$ and $G_2(x) \ge 0$, a.e. $x \in G$, where $\beta > 0$. Define continuous linear operators $B : W \to W'$ and $A : V \to V'$ by

$$Bu(v) = \int_G C(x)\partial_N u \partial_N v \, dx \ , \qquad u, v \in W$$

$$Au(v) = \int_G (G_1(x)\overline{\nabla} u \cdot \overline{\nabla} v + G_2(x)\partial_N u \partial_N v) \, dx \ , \qquad u, v \in V$$

where $\overline{\nabla} = (\partial_1, \partial_2, \ldots, \partial_{N-1})$ is the gradient in the first $N-1$ coordinates. From the Lax-Milgram theorem and the Poincaré lemma above, it follows that B is an isomorphism of W onto W' and that, for each $\lambda > 0$, $\lambda B + A$ is an isomorphism of V onto V'. Let $F \in L^2(G)$ and $g \in H^{1/2}(\Gamma_+)$ be given and define $f \in W'$ by

$$f(v) = \int_G F(x)v(x)\, dx + \int_{\Gamma_+} g(s)\nu_N(s)v(s)\, ds \ , \qquad v \in W \ .$$

This is meaningful since $\gamma_N : W \to H^{-1/2}(\Gamma_+)$ is continuous. Consider the solution of the stationary problem

$$u \in V : \lambda B(u) + A(u) = f \ .$$

By standard methods **[22]** this solution is characterized by

(i) $-\lambda\partial_N(C(x)\partial_N u(x)) - \overline{\nabla}\cdot(G_1(x)\overline{\nabla} u) - \partial_N(G_2(x)\partial_N u(x))$
$= F(x)$, a.e. $x \in G$

(ii) $u\,|_{\Gamma_-} = 0$,

(iii) $\lambda\nu_N(s)C(s)\partial_N u(s) + (G_1(s)\overline{\nabla} u(s) + G_2(s)\partial_N u(s))\nu(s)$

$= \nu_N(s)g(s),\ s \in \Gamma^+$.

The partial differential equation (i) is the stationary layered medium equation, the boundary condition (ii) means the voltage reference level is taken at the lower part of the boundary, and (iii) is the prescribed current flux along the remaining boundary.

Initial-value Problem.

Certain initial-boundary-value problems for the layered-medium equation can be resolved as a special case of the following result from **[22]**.

Theorem 1. *Let W be a Hilbert space with Riesz map $B : W \to W'$. Let V be a Hilbert space dense and continuously embedded in W and let A be continuous and linear from V to V'. Assume that for some $\lambda > 0$ there is a $c > 0$ such that*

$$\lambda Bv(v) + Av(v) \geq c\|v\|_{V^2}\,, \qquad v \in V\,.$$

Then for each $u_0 \in W$ and each Hölder continuous $f \in C^\alpha([0,\infty), W')$, $0 < \alpha \leq 1$, there is a unique $u \in C((0,\infty), V)$ such that:

$$u \in C([0,\infty), W) \cap C^1((0,\infty), W)\,,$$
$$u(0) = u_0 \qquad \text{and}$$
$$Bu'(t) + Au(t) = f(t)\ , t > 0\,.$$

Proof. Define $D(M) = \{v \in V : Av \in W'\}$ and $M = B^{-1}A : D(M) \to W$. The scalar-product on W is $(u,v)_W = Bu(v)$ so it follows that M is accretive: $(Mu,u)_W = Au(u) \geq 0$. [We may assume $\lambda = 0$ above since a change of variable in the equation replaces A by $\lambda B + A$.] Also $I + M = B^{-1}(B + A)$ maps $D(M)$ onto W so M is m-accretive. Finally, note that A is sectorial, so then is M, and it follows **[10]** that $-M$ generates a holomorphic semigroup of contractions on W.

We can solve the layered-medium equation subject to mixed Dirichlet-Neumann boundary conditions as an application of Theorem 1. Let $F \in$

$C^\alpha([0,\infty), L^2(G))$ and $g \in C^\alpha([0,\infty), H^{1/2}(\Gamma_+))$ be given and define $f \in C^\alpha([0,\infty), W')$ by

$$f(t,v) = \int_G F(t,x)v(x)\,dx + \int_{\Gamma_+} g(t,s)\nu_N(s)v(s)\,ds\ , \quad v \in W\ .$$

There is a unique $u \in C^0((0,\infty), V)$ such that:

(i) $-\partial_t\partial_N(C(x)\partial_N u(x)) - \overline{\nabla}\cdot(G_1(x)\overline{\nabla}u) - \partial_N(G_2(x)\partial_N u(x))$
$= F(t,x)$, a.e. $x \in G$

(ii) $u\,|_{\Gamma_-} = 0$ for $t \in (0,\infty)$,

(iii) $\partial_t\nu_N(s)C(s)\partial_N u(t,s) + (G_1(s)\overline{\nabla}u(t,s) + G_2(s)\partial_N u(t,s))\nu(s)$
$= \nu_N(s)g(t,s,t), s \in \Gamma_+$

(iv) $u(0) = u_0$

Periodic problems.

We shall need to resolve the layered-medium equation subject to periodic boundary conditions. Such problems are not covered by Theorem 1 since B gives only a semi-norm on the correspondng space.

Theorem 2. *Let W be a semi-normed space whose semi-norm is obtained from a non-negative symmetric bilinear form $(u,v) \to Bu(v)$ associated with the linear $B : W \to W'$. Let V be a Hilbert space dense and continuously embedded in W and let $A : V \to V'$ be continuous, linear and symmetric. Assume that for some λ there is a $c > 0$ such that $(\lambda B + A)v(v) \geq c \inf\{\|v+\xi\|_V : \xi \in \mathrm{Ker}(A)\}^2$. Then for each $h \in W'$ and each Hölder continuous $f \in C^\alpha([0,\infty), W')$, $0 < \alpha \leq 1$, there is a $u \in C^0((0,\infty), V)$ which is a solution of*

$$\begin{aligned} &Bu \in C^0([0,\infty), W') \cap C^1((0,\infty), W'),\\ &Bu(0) = h \quad \text{and}\\ &(Bu)'(t) + Au(t) = f(t), t > 0\ . \end{aligned}$$

Proof. Note first that u is a solution if and only if $v(t) = \exp(-\lambda t)u(t)$ is a solution of the corresponding problem with A replaced by $\lambda B + A$. Thus we may take $\lambda = 0$ above and assume $\mathrm{Ker}(A) \subset \mathrm{Ker}(B)$. Let K be the kernel of B, let W/K be the corresponding quotient space, and denote by H the completion of W/K. Regard the quotient map $q : W \to W/K$ as a semi-norm-preserving injection of W into H and denote the corresponding dual map by $q^* : H' \to W'$. Note that q^* is an isomorphism. If $B_0 : H \to H'$ is the Riesz map associated with the scalar-product on H inherited from W, then we easily check that B factors according to

$$Bu = q^* B_0 q(u) \ , \qquad u \in W \ .$$

In order to simultaneously factor A we consider the subspace $D = \{v \in V : Av \in W'\}$ where we identify $W' \subset V'$. Then for each $u \in D$ we have

$$|Au(v)| \leq \text{const. } (Bv(v))^{1/2}$$

If $u \in K \cap D$ then setting $v = u$ above shows $u \in \mathrm{Ker}(A)$. Thus, $K \cap D = \mathrm{Ker}(A)$, there is a unique linear $A_0 : q[D] \to H'$ for which

$$Au = q^* A_0 q(u) \ , \qquad u \in D \ ,$$

and we have $D/\mathrm{Ker}(A) = D/K \equiv q[D]$. Define $M : q[D] \to H$ by $B_0^{-1} A_0$; it follows that M is sectorial in H so it suffices to show that $Rg(\lambda B_0 + A_0) = H'$. To this end, consider $\lambda B + A : V \to V'$. From the "semi-coercive" estimate, it follows that $\lambda B + A$ induces an isomorphism of $V/\mathrm{Ker}(A)$ onto $\mathrm{Ker}(A)^\perp$, the annihilator of $\mathrm{Ker}(A)$ in V'. Also we note $W' \subset \mathrm{Ker}(B)^\perp \subset \mathrm{Ker}(A)^\perp \subset V'$ and $q[D] = D/\mathrm{Ker}(A) \subset V/\mathrm{Ker}(A)$, so this isomorphism extends $q^*(\lambda B_0 + A_0)$, which necessarily maps $q[D]$ onto W'. Since q^* is an isomorphism, this gives the desired range condition and finishes the proof.

Consider the special case in which G is a cylinder, $G = \Omega \times T$ where Ω is a smoothly bounded domain in R^{N-1} and $T = (0, 1) \subset R^1$. Then we can identify $H^1_N(G) = H^1(T, L^2(\Omega))$, a Sobolev space of $L^2(\Omega)$-valued functions of $x_N \in T$. Such functions are absolutely continuous so the trace operators γ_- and γ_+ defined by $\gamma_-(v) = v\,|_{x_N=0}$ and $\gamma_+(v) = v\,|_{x_N=1}$ are continuous from $H_N^1(G)$ onto $L^2(\Omega)$. In order to resolve problems periodic in x_N we shall use the closed subspace of H_N^1 given by

$$W_p(G) = \{v \in H_N^1(G) : \gamma_-(v) = \gamma_+(v)\} .$$

We choose $W = W_p(G)$, $V = W_p(G) \cap H^1(G)$ and define B and A exactly as above. Since $\lambda B + A$ is coercive over the space $\{v \in H^1(G) : \int_G v(x)\,dx = 0\}$ when $\lambda > 0$, the desired estimate in Theorem 2 is obtained. Note that $\mathrm{Ker}(B) \cong L^2(\Omega)$, the functions in $L^2(G)$ independent of $x_N \in T$, and $\mathrm{Ker}(B + A)$ consists of constant functions. In order to characterize W_p', note that $\partial_N : W_p \hookrightarrow L^2(G)$ has closed range $Rg(\partial_N) = L^2(\Omega)^\perp$ in $L^2(G)$ on which its surjective dual $\partial_N^* : L^2(G) \to \mathrm{Ker}(B)^\perp \cong W_p'$ is one-to-one. Thus, $f \in W_p'$ if and only if there is an $F \in L^2(G)$ (with $\int_0^1 F(x', x_N)\,dx_N = 0$ for a.e. $x' \in \Omega$ if desired) for which

$$f(w) = \int_G F(x)\partial_N w(x)\,dx, \qquad w \in W_p .$$

It follows that $f = \partial_N^* F \in H^{-1}(T, L^2(\Omega))$, so we have $W_p'(G) = H^{-1}(T, L^2(\Omega))$. Note that $f = \varphi \in L^2(T, L^2(\Omega)) = L^2(G)$ if and only if $-\partial_N F = \varphi$ in $D'(G)$ and $\int_0^1 \varphi(x', s)\,ds = 0$ for a.e. $x' \in \Omega$. A solution of the stationary equation

$$u \in V : \lambda B u + A u = f \text{ in } W_p'$$

is characterized as before by

$$\begin{aligned}
u \in H^1(G) : \quad & \\
& -\partial_N(\lambda C + G_2)\partial_N u - \overline{\nabla} \cdot G_1 \overline{\nabla} u = f \qquad \text{in } G , \\
& \gamma_- u = \gamma_+ u \\
& (\lambda C + G_2)\partial_N u \,|_{XN=0} = (\lambda C + G_2)\partial_N u \,|_{x_N=1} \quad \text{in } \Omega , \\
& G_1 \partial u / \partial \nu = 0 \qquad \text{on } \partial\Omega \times T .
\end{aligned}$$

That is, u satisfies the layered-medium equation, the Neumann boundary conditions on the boundary of the cylinder, and it is periodic in the vertical direction.

The corresponding evolution problem is resolved by Theorem 2. Thus, suppose we are given $\psi \in L^2(G)$ with $\int_0^1 \psi(x', s)\,ds = 0$, a.e. $x' \in \Omega$, and $\varphi \in C^\alpha((0,1), L^2(G))$ with $\int_0^1 \varphi(x', s, t)\,ds = 0$ for a.e. $x' \in \Omega$ and all $t > 0$. Then there is a unique $u \in C^0([0,1), V)$ satisfying:

$$\begin{aligned}
&Bu \in C^0([0,1), W_p') \cap C^1((0,1), W_p'),\\
&Bu(0) = \psi\\
&-(\partial/\partial t)\partial_N(C\partial_N u(t)) - \overline{\nabla}\cdot G_1\overline{\nabla}u(t) - \partial_N(G_2\partial_N u(t)) = \varphi(\cdot, t) \quad \text{in } G\\
&\gamma_- u = \gamma_+ u\\
&((d/dt)C\partial_N u + G_2\partial_N u)\,|_{x_N=0} = ((d/dt)C\partial_N u + G_2\partial_N u\,|_{x_N=1} \quad \text{in } \Omega\,,\\
&\quad G_1(\partial u/\partial\nu)(t) = 0 \qquad \text{on } \partial\Omega \times T\,.\\
&\int_G u(x,t)\,dx = 0 \qquad \text{for all } t > 0\,.
\end{aligned}$$

This last constraint serves only to achieve uniqueness by picking a specific representative from the equivalence classes modulo constants.

III. The Homogenized Equation.

Homogenization Methods.

We consider the method of homogenization developed by J. L. Lions, L. Tartar, F. Murat, E. Sanchez-Palencia, S. Spagnolo (as in **[4-6]**, **[15-18]**, **[21]**, **[23-28]** and references therein). The objective of this method is to average the partial differential equations of physics in heterogeneous materials with a periodic structure when the period goes to zero. Heuristically, the method is based on the consideration of two length scales associated with the microscopic and macroscopic phenomena. Here we shall apply the method to equations of the form

$$\partial_t\partial_z C\partial_z u + \partial_z G_2\partial_z u + \overline{\nabla}_x(G_1\overline{\nabla}_x u) = f$$

where we are considering functions of $(x,z) \in R^N$, $x \in R^{N-1}$, $z \in R$. The slot where z is will be called the *second* slot for simplicity, and we will make the following assumptions throughout this section:

$$C, G_1, G_2 \in L^\infty(G) \,,$$
$$C \geq \beta > 0 \,, G_1 \geq \beta > 0 \,, G_2 \geq \beta > 0 \quad \text{a.e. in } G \,,$$
$$C, G_1, G_2 \quad Z\text{-periodic in the “second” variable} \,, Z \in R^+ \,,$$
$$C^\varepsilon(x,z) \equiv C(x, z/\varepsilon) \,, G_1^\varepsilon(x,z) \equiv G_1(x, z/\varepsilon) \,, G_2^\varepsilon(x,z) = G_2(x, z/\varepsilon) \,.$$

Asymptotic Expansion.

We are going to look for the formal limit of the initial-value problem.

$$\begin{aligned} \partial_t \partial_z C^\varepsilon \partial_z u^\varepsilon + \partial_z G_2^\varepsilon \partial_z u^\varepsilon + \overline{\nabla}_x G_1^\varepsilon \overline{\nabla}_x u^\varepsilon &= f \\ u^\varepsilon(x, z, 0) = v \quad \text{independent of } \varepsilon \,. & \end{aligned} \tag{1}$$

We use the formal expansion of u^ε as a sum of powers of ε with pseudo-periodic coefficient functions. We have to gear the coefficients so as to take into account the fact that the periodicity is only in the z direction, so we set

$$u^\varepsilon(x,z,t) = u^0(x,z,y,t) + \varepsilon u^1(x,z,y,t) + \varepsilon^2 u^2(x,z,y,t) + \ldots$$

with $y = z/\varepsilon$ and the $u^i(x,z,y,t)$ smooth enough and Z-periodic in y. The differentiation ∂_z gives $\partial_z + (1/\varepsilon)\partial_y$ and with any function K, $\partial_z K \partial_z u^\varepsilon$ is given by

$$\begin{aligned} &\varepsilon^{-2} \partial_y K \partial_y u^0 + \varepsilon^{-1}\{\partial_y K \partial_y u^1 + \partial_z K \partial_y u^0 + \partial_y K \partial_z u^0\} \\ &\qquad + \partial_z K \partial_z u^0 + \partial_z K \partial_y u^1 + \partial_y K \partial_z u^1 + \partial_y K \partial_y u^2 + \varepsilon \ldots \,. \end{aligned}$$

Let us define the stationary operators

$$\begin{aligned} A_1 &\equiv \partial_y C \partial_t \partial_y + \partial_y G_2 \partial_y \\ A_2 &\equiv \partial_y C \partial_t \partial_z + \partial_y G_2 \partial_z + \partial_z C \partial_t \partial_y + \partial_z G_2 \partial_y \\ A_3 &\equiv \partial_z C \partial_t \partial_z + \partial_z G_2 \partial_z + \overline{\nabla}_x G_1 \overline{\nabla}_x \end{aligned}$$

as in Section 2. From (1) we obtain

$$\varepsilon^{-2} A_1 u^0 + \varepsilon^{-1}\{A_1 u^1 + A_2 u^0\} + A_1 u^2 + A_3 u^0 + A_2 u^1 + \varepsilon \ldots = f \,. \tag{2}$$

We have discussed already the solvability of the periodic problem

$$A_1\Phi = F$$
$$\Phi \in W(0,Z) \equiv \{\Phi \in H^1((0,Z)\times R_+ : \Phi(0,t) = \Phi(Z,t)\ \forall\, t \in R_+\}$$

at the end of Section II, and this will be used in following. For F to be orthogonal to the kernel of A_1 is a necessary and sufficient condition for existence and unicity up to any function of time only. This orthogonality is expressed by

$$\int_0^z F(y,t)\,dy = 0 \qquad \forall\, t \in R_+ \,.$$

We shall consider successively the terms in (2).

Terms Factor of ε^{-2}:

The first term satisfies

$$\begin{aligned} &A_1 u^0 = 0\,, \\ &u^0(x,z,y,t) \quad Z\text{-periodic in } y\,. \end{aligned} \tag{3}$$

The solutions of that equation are independent of y and therefore of $\varepsilon(y = z/\varepsilon)$. It follows that $u^0 = u^0(x,z,t)$.

Terms Factor of ε^{-1}:

The second term in (2) satisfies

$$A_1 u^1 + A_2 u^0 = 0\,. \tag{4}$$

Let us suppose u^0 known and find u^1 in terms of u^0. Note that $u^1(x,z,y,0) = 0$ because the initial condition is independent of ε. We need to solve

$$\begin{aligned} \partial_y C\partial_t\partial_y u^1 + \partial_y G_2\partial_y u^1 + \partial_y C\partial_t\partial_z u^0 + \partial_y G_2\partial_z u^0 \\ + \partial_z C\partial_t\partial_y u^0 + \partial_z G_2\partial_y u^0 = 0\,. \end{aligned}$$

Since u^0 is independent of y this simplifies to

$$\partial_y C\partial_t(\partial_y u^1) + \partial_y G_2(\partial_y u^1) + (\partial_y C)(\partial_t\partial_z u^0) + (\partial_y G_2)(\partial_z u^0) = 0\,.$$

**Case* (G_2/C) *Independent of* y*:* We multiply through by $\exp\{(G_2/C)t\}$ to make a change of variable.

$$\begin{aligned} e^{t(G_2/C)}[\partial_y C\partial_t\partial_y u^1 + \partial_y G_2\partial_y u^1] + e^{t(G_2/C)}[\partial_y C\partial_t\partial_z u^0 + \partial_y G_2\partial_z u^0] &= 0 \\ \partial_y C\partial_t[e^{t(G_2/C)}\partial_y u^1] + \partial_y C\partial_t[e^{t(G_2/C)}\partial_z u^0] &= 0 \\ \partial_y C\partial_y\partial_t[e^{t(G_2/C)}u^1] + (\partial_y C)\partial_t[e^{t(G_2/C)}\partial_z u^0] &= 0 \end{aligned} \tag{5}$$

In order to represent the solution we first solve

$$\partial_y C\partial_y\chi + \partial_y C = 0$$
$$\chi(0) = \chi(Z)\ ,\chi\in H^1(0,Z)$$

to obtain

$$\chi = -y + K\Big(\int_0^y (1/C)\,dy\Big) + \chi(0)$$

To find K we use the periodic condition $\chi(0) = \chi(Z)$ to obtain $K = Z(\int_0^Z(1/C)\,dy)^{-1}$ and therefore $\chi(y) = -y + Z(\int_0^Z(1/C)\,dy)^{-1}(\int_0^y(1/C)\,dy)$ $+\chi(0)$. With this expression for χ we can write (5) in the form

$$\partial_t[e^{t(G_2)/C)}u^1] = \partial_t[e^{t(G_2/C)}u^0]\chi(y) + k(x,z,t)\ ,$$
$$e^{t(G_2/C)}u^1 = \{(e^{t(G_2/C)}u^0 - u^0(x,z,0)\}\chi(y) + \int_0^t k(x,z,s)\,ds\ ,$$

and, hence, obtain the solution of (4) in the form

$$u^1 = \{u^0 - e^{-t(G_2/C)}u^0(x,z,0)\}\chi(y) + e^{-t(G_2/C)}\int_0^t k(x,z,s)\,ds\ ,$$

Terms Factor of ε^0*:*

$$A_1u^2 + A_2u^1 + A_3u_0 = f\ . \tag{6}$$

A necessary condition is that $\int_0^Z(A_2u^1 + A_3u_0 - f)\,dy = 0$, that is,

$$\begin{aligned} &\int_0^Z \partial_y(C\partial_t\partial_z u^1 + G_2\partial_z u^1)\,dy \\ &\quad + \int_0^Z (\partial_z(C\partial_t\partial_y u^1) + G_2\partial_y u^1) + \partial_z(C\partial_t\partial_z u^0 + G_2\partial_z u^0)\,dy \\ &\quad + \overline{\nabla}_x\Big(\int_0^Z G_1\,dy\Big)\overline{\nabla}_x u^0 = Z\,f\ . \end{aligned}$$

Since the first integral equals zero by periodicity, this gives

$$\partial_z \int_0^Z \{C\partial_t\partial_y u^1 + G_2\partial_y u^1 + C\partial_t\partial_z u^0 + G_2\partial_z u^0\}\,dy + \overline{\nabla}_x\Big(\int_0^Z G_1\,dy\Big)\overline{\nabla}_x u^0 = Z\,f \tag{7}$$

This is the homogenized equation in the general case. It is a non-local equation.

**Case* (G_2/C) *Independent of* y*:*

We multiply and divide by $\exp(\{tG_2/C)\}$ to make a change of variable.

$$\partial_z \int_0^Z \{Ce^{-t(G_2/C)}\partial_t(e^{t(G_2/C)}\partial_y u^1) + C\partial_t\partial_z u^0 + G_2\partial_z u^0\}\,dy + \overline{\nabla}_x\Big(\int_0^Z G_1\,dy\Big)\overline{\nabla}_x u^0 = Z\,f$$

$$\partial_z \int_0^Z \{Ce^{-t(G_2/C)}(\partial_y\chi)(\partial_t e^{t(G_2/C)}\partial_z u^0) + (C\partial_t\partial_z u^0 + G_2\partial_z u^0)\}\,dy + \overline{\nabla}_x\Big(\int_0^Z G_1\,dy\Big)\overline{\nabla}_x u^0 = Z\,f$$

$$\partial_z \int_0^Z \{(\partial_y\chi)(C\partial_t\partial_z u^0 + G_2\partial_z u^0) + (C\partial_t\partial_z u^0 + G_2\partial_z u^0)\}\,dy + \overline{\nabla}_x\Big(\int_0^Z G_1\,dy)\overline{\nabla}_x u^0 = Z\,f$$

Now recall that

$$\partial_y\chi = -1 + (Z/C)\Big(\int_0^Z 1/C\,dy\Big)^{-1}$$

Therefore we get successively

$$\partial_z \int_0^Z \{(-1 + Z/C\Big(\int_0^Z 1/C\,dy)^{-1} + 1\Big)(C\partial_t\partial_z u^0 + G_2\partial_z u^0)\}\,dy$$

$$+ \nabla_x\Big(\int_0^Z G_1\,dy)\nabla_x u^0 = Z\,f\,,$$

$$\partial_z\Big(\int_0^Z Z\Big(\int_0^Z 1/C\,dy\Big)^{-1} dy\Big)\partial_t\partial_z u^0$$

$$+ \Big(\int_0^Z G_1/C\,dy\Big)Z\Big(\int_0^Z 1/C\,dy\Big)^{-1}(\partial_z)^2 u^0\Big)$$

$$+ \nabla_x\Big(\int_0^Z G_1\,dy\Big)\nabla_x u^0 = Z\,f\,,$$

$$Z\Big(\int_0^Z 1/C\,dy\Big)^{-1}\partial_z\partial_t\partial_z u^0 + \Big(\int_0^Z 1/C\,dy\Big)^{-1}\Big(\int_0^Z G_2/C\,dy\Big)(\partial_z)^2 u^0$$

$$+ \nabla_x 1/Z\Big(\int_0^Z G_1\,dy\Big)\nabla_x u^0 = f\,,$$

and finally

$$Z\Big(\int_0^Z 1/C\,dy\Big)^{-1}\partial_t(\partial_z)^2 u^0 + (ZG_2/C)\Big(\int_0^Z 1/C\,dy\Big)^{-1}(\partial_z)^2 u^0$$
$$+ 1/Z\Big(\int_0^Z G_1\,dy\Big)(\nabla_x)^2 \overline{u}^0 = f\,. \tag{8}$$

This is the homogenized equation.

Convergence Theorem.

We want to study the limit of the solutions u^ε to equation (1) when ε goes to 0; the usual hypotheses on the coefficients will be assumed. Let us check that the sequence u^ε is weakly convergent. Let us take the L^2

product of (1) and u^ε.

$$\partial_t(\partial_z C^\varepsilon \partial_z u^\varepsilon, u^\varepsilon) + (\partial_z G_2^\varepsilon \partial_z u^\varepsilon, u^\varepsilon) + (\overline{\nabla}_x G_1^\varepsilon \overline{\nabla}_x u^\varepsilon, u^\varepsilon) = (f, u^\varepsilon)$$

$$\partial_t(C^\varepsilon \partial_z u^\varepsilon, \partial_z u^\varepsilon) + (G_2^\varepsilon \partial_z u^\varepsilon, \partial_z u^\varepsilon) + (\overline{G}_1^\varepsilon \overline{\nabla}_x u^\varepsilon, \nabla_x u^\varepsilon) = (f, u^\varepsilon)$$

$$\frac{1}{2}\frac{d}{dt}\{(C^\varepsilon \partial_z u^\varepsilon, \partial_z u^\varepsilon)\} + (G_2^\varepsilon \partial_z u^\varepsilon, \partial_z u^\varepsilon) + (\overline{G}_1^\varepsilon \overline{\nabla}_x u^\varepsilon, \nabla_x^\varepsilon) = (f, u^\varepsilon)$$

$$\frac{1}{2}(C^\varepsilon \partial_z u^\varepsilon(t), \partial_z u^\varepsilon(t)) + \int_0^t (G_2^\varepsilon \partial_z u^\varepsilon, \partial_z u^\varepsilon)\, ds$$

$$+ \int_0^t (G_1^\varepsilon \overline{\nabla}_x u^\varepsilon, \nabla_x u^\varepsilon)\, ds = \frac{1}{2}(C^\varepsilon \partial_z u^0, \partial_z u^0) + \int_0^t (f, u^\varepsilon)\, ds$$

$$\frac{1}{2}\beta |u^\varepsilon(t)|_{H^1 n} + \beta \int_0^t \{|u^\varepsilon|_{H_N^1} + \|\nabla_x u^\varepsilon\|_{L^2}\}\, ds$$

$$\leq \frac{1}{2}(C^\varepsilon \partial_z u^0, \partial_z u^0) + \|f\|_{L^2} \|u^\varepsilon\|_{L^2}$$

G bounded. Gronwall Lemma gives then

$$|u^\varepsilon(t)|_{L_\infty(0,T;H^1 N)} < C$$

This then gives

$$|\partial_t u^\varepsilon(t)|, \infty(0, T; H_N^1) < C\ .$$

Then unicity of Laplace transform gives convergence to u.

Theorem. *Let us assume that (2) holds. The sequence of solutions u^ε of (1) converges towards the solution u^0 of (19) in $W^{1,\infty}(0,T;H_N^1)$.*

In conclusion, the model of this periodic layered structure leads us to the layered medium equation which has properties different from those of already known equations. The correct relevant physical problem is well-posed, therefore the equation can be used effectively as a model. The computational problem posed by the original problem is non-solvable, therefore the methods of homogenization are useful as we can see that a standard computational problem gives a good approximation of the solution.

We have not shown that the discretely-layered model in I converges to the layered medium equation; rather, we used it as an efficient introduction to this continuum model. Note that another model of discrete layers is obtained in the layered medium equation when the coefficients are chosen to be step functions which characterize the layers. Here the finite-difference coupling is replaced by natural or variational flux conditions on the interfaces. Of course this latter model has been shown above to converge to the continuum model. We have not addressed the question of which discretely-layered model is better.

References

1. I. Babuska, Numerical Solutions of PDE III Synspade 1975, Hubbard B., ed., Academic Press.
2. I. Babuska, SIAM J. Math. Anal., **7**, No. 5, 1976.
3. I. Babuska, SIAM J. Math. Anal., **9**, No. 6, 1977.
4. A. Bensoussan, J. L. Lions, G. Papanicolaou, Functional analysis and numerical analysis. Japan France Seminar. Tokyo & Kyoto 1976, H. Fujita, ed. Japan Society for the promotion of science 78.
5. A. Bensoussan, J. L. Lions, G. Papanicolaou, Asymptotic analysis for periodic structures, North Holland, 1978.
6. D. Cioranescu, F. Murat, Séminaire du Collège de France Nonlinear PDE & their applications. Brézis & Lions, ed., Research Notes in Math. 60870 Pitman London (1982), pp.98–138 and 154–178.
7. M. S. Dresselhaus, Physics Today, **37**, March 1984, pp.60–68.
8. L. P. Huelsman, IEEE Spectrum, **6**, pp.51–58, August 1969.
9. L. P. Huelsman, ed., Active Filters: Lumped, Distributed, Integrated, Digital, and Parametric, Inter-University Electronics Series, Vol. 11, McGraw-Hill, 1970.
10. T. Kato, Perturbation theory for linear operators, Springer Verlag, 1966.
11. J. J. Kelly, M. S. Ghausi, Introduction to distributed parameter networks, Holt, Rinehart and Winston, New York, 1968.
12. R. Landauer, AIP Conf. Proc., **4**, 1978.
13. F. A. Lindholm, Happ W. W., Microelectron Rel., **3**, December 1964, pp.209–226.
14. J. L. Lions, E. Magenès, Problèmes aux limites non-homogènes et applications, Dunod, Paris, 1968.
15. J. L. Lions, Remarks on some numerical aspects of the homogenization methods in composite materials, Novosibirrsk, 1976.
16. J. L. Lions, Rocky Mountain Journal of Mathematics, **10**, No. 1, Winter 1980.
17. J. L. Lions, Some methods in mathematical analysis of systems and their control, Science Press, Beijing, China, 1981. Gordon and Breach Science Publishers, Inc., New York.

18. F. Murat, Ann. Sc. Norm. Sup. Pisa **5**, 3, 1978, pp.489–507.
19. R. W. Newcomb, IEEE Trans. Circuits and Systems, **CAS-28**, Jan. 1981, pp.62–71.
20. D. A. Pinnow, R. L. Abrams, J. F. Lopsteich, D. M. Henderson T. K. Plant, R. R. Stephens, C. M. Walker, Appl. Phy. Lett., **34**, 1979, 391.
21. E. Sanchez-Palencia, Non-homogeneous media and vibration theory, Springer Verlag, Lecture Notes in Physics #127.
22. R. E. Showalter, Hilbert space methods in partial differential equations, Pitman, 1977.
23. S. Spagnolo, Numerical Solutions of PDE III, Synspade 1975, ed. Bert Hubbard, Academic Press, pp.469–498.
24. S. Spagnolo, Bolletino U. M. I. **5** 14.B, 1977, pp.547–568.
25. L. Tartar, Computing Methods in Ap. Sciences & Eng., pp.364–373; Lecture Notes in Mathematics 704, Springer Verlag.
26. L. Tartar, Functional analysis and numerical analysis. Japan France Seminar. Tokyo & Kyoto 1976 H. Fujita, ed. Japan Society for the promotion of science 78.
27. L. Tartar, Non Linear Analysis and Mechanics: Heriot-Watt Symposium. R. J Knops, ed. **IV** Pitman Research Notes in Math. *39, Londres, 1979, pp.136–212.
28. L. Tartar, Non Linear PDE & Applications. Séminaire du Collège de France, **IV**, Research Notes in mathematics, 1984. Pittman. Londres.
29. R. Temam, Annali di Matematica pura ed applicata, (IV) **LXXIX** 1968, pp.191–380.