A Regularization-Stabilization Technique for Nonlinear Conservation Equation Computations

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A regularization procedure for nonlinear conservation equations is introduced and demonstrated to have a stabilizing effect on the numerical solution of the associated approximate problem. Representative results for a least-squares finite-element method are given, and the numerical performance of the stabilization procedure explored. The effect of the regularization term is similar to a local numerical dissipation dependent on the numerical itegration time step.

I. INTRODUCTION

Numerical experiments conducted on nonlinear conservation equations, such as the inviscid Burgers' equation and the Euler equations for compressible flow, indicate that standard computational techniques may become nonlinearly unstable as steep solution gradients associated with shocks develop. For example, as the local solution gradients become large, there is frequently an oscillatory overshoot that grows catastrophically. This difficulty has led to the use of artificial dissipation techniques as a means to stabilize the calculation (e.g., see Löhner et al [1], Hughes and Mallet [2], Selmin and Quartapelle [3], and Jiang and Carey [4]). In the present note, we consider an alternative regularization involving a time derivative and then examine the performance of a least-squares finite-element approximation of this regularized problem. This regularized model problem may be related to a model problem arising in the conservation relations for gas absorption described in Tychonov and Samarski [5], and has been the subject of related mathematical analysis (Showalter [6]).

A. Formulation and Approximate Method

Consider the model nonlinear conservation equation

$$u_t + \{a(u)\}_x = 0 \tag{1}$$

with initial data

$$u(x, 0) = g(x) \tag{2}$$

and where $a(u)$ is a specified relation; e.g., $a(u) = \frac{1}{2}u^2$ for the familiar inviscid Burgers' equation.

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Usually, Eq. (1) is regularized by adding an artifical viscous dissipation term of the form εu_{xx}. But here we introduce instead the regularization term εu_{xt} so that Eq. (1) becomes

$$u_t + \varepsilon u_{xt} + \{a(u)\}_x = 0 \tag{3}$$

Writing u_{xt} as $(u_x)_t$ and discretizing Eq. (3) with respect to time yields a semidiscrete system to be solved in each time step Δt. Differencing Eq. (3) from t_n to $t_{n+1} = t_n + \Delta t$, we obtain

$$\frac{u^{n+1} - u^n}{\Delta t} + \varepsilon \frac{(u_x^{n+1} - u_x^n)}{\Delta t} + \Theta\{a(u)\}_x^{n+1} + (1 - \Theta)\{a(u)\}_x^n = 0 \tag{4}$$

where $u^n = u(x, t_n)$ is assumed known and $0 \leqq \Theta \leqq 1$. The implicit schemes $\Theta = 1$ (backward time differencing) and $\Theta = ½$ (central or Crank–Nicolson time differencing) are of particular interest in the calculations presented later.

Next, we set $\{a(u)\}_x = a_u u_x = A(u)u_x$ and iteratively linearize within time step Δt so that $\{A(u)u_x\}^{n+1} \approx A(u^{n+1,k})u_x^{n+1}$ for iterate $k + 1$ with $u^{n+1,0} = u^n$. The approximation to Eq. (4), then, is

$$u^{n+1} - u^n + \varepsilon(u_x^{n+1} - u_x^n) + \Delta t\Theta A u_x^{n+1} + \Delta t(1 - \Theta)\{a(u)\}_x^n = 0 \tag{5}$$

where we let A denote $A(u^{n+1,k})$ for notational convenience.

For admissible u^{n+1}, Eq. (5) defines the residual R at iterate k and in the present approach we seek a solution that minimizes the L^2 norm of this residual. That is, we define on domain Ω

$$I = \int_\Omega R^2\, dx \tag{6}$$

so that $\delta I = 0$ implies

$$\int_\Omega [(u^{n+1} - u^n) + \varepsilon(u_x^{n+1} - u_x^n) + \Delta t\Theta A u_x^{n+1} + \Delta t(1 - \Theta)\{a(u)\}_x^n] \times [v + \varepsilon v_x + \Delta t\Theta A v_x]\, dx = 0 \tag{7}$$

That is

$$\int_\Omega [u^{n+1} + (\varepsilon + \Delta t\Theta A)u_x^{n+1}][v + (\varepsilon + \Delta t\Theta A)v_x]\, dx = \int_\Omega [u^n + \varepsilon u_x^n - \Delta t(1 - \Theta)\{a(u)\}_x^n][v + (\varepsilon + \Delta t\Theta A)v_x]\, dx \tag{8}$$

yields the weak form of the problem for time step Δt.

We introduce a finite-element discretization with piecewise polynomial basis $\{\phi_j\}$, $j = 1, 2, \ldots, N$; the corresponding approximation and test functions are given by

$$u_h^{n+1}(x) = \sum_{j=1}^{N} u_j^{n+1}\phi_j(x), \qquad v_h(x) = \phi_i(x) \tag{9}$$

Substituting u_h, v_h for u, v in Eq. (8), we have

$$\sum_{j=1}^{N}\left[\int_{\Omega}[\phi_j + (\varepsilon + \Delta t\Theta A)\phi_{jx}][\phi_i + (\varepsilon + \Delta t\Theta A)\phi_{ix}]\,dx\right]u_j^{n+1}$$
$$= \sum_{j=1}^{N}\left[\int_{\Omega}[\phi_j + \varepsilon\phi_{jx} - \Delta t(1-\Theta)\{a(u_h)\}_x^n][\phi_i + (\varepsilon + \Delta t\Theta A)\phi_{ix}]\,dx\right]u_j^n \tag{10}$$

That is, in each iterate for given time step Δt we solve a system of the form

$$\mathbf{M}\mathbf{u}^{n+1} = \mathbf{b} \tag{11}$$

where $\mathbf{M}$ and $\mathbf{b}$ depend upon known values $\mathbf{u}^n$ and $\mathbf{u}^{n+1,k-1}$.

To explore briefly the effect of this regularization, denote by ∂ the derivative $\partial/\partial x$ as an operator on an appropriate function space. Now, Eq. (3) can be written in the form $(I + \varepsilon\partial)u_t + \partial\{a(u)\} = 0$ and we obtain

$$u_t + (I + \varepsilon\partial)^{-1}\partial a(u) = 0 \tag{12}$$

an ordinary differential equation in function space. Formally, we have $(I + \varepsilon\partial)^{-1} \cong I - \varepsilon\partial$, so Eq. (3) is close to

$$u_t + a(u)_x = \varepsilon a(u)_{xx} \tag{13}$$

a regularization of Eq. (1) obtained by the addition of a nonlinear viscosity term. In the linear case, $a(u) = u$, it is known that the solution u^ε of Eq. (13) converges to the solution of Eq. (1) as $\varepsilon \to 0$; such a result is not known for the nonlinear case. Finally, note that the norm of $(I + \varepsilon\partial)^{-1}\partial$ is given by $1/\varepsilon$. Thus, for the difference scheme (4), in the linear case, we have

$$u^{n+1} = [I + \Delta t\Theta(1 + \varepsilon\partial)^{-1}\partial]^{-1}[I - \Delta t(1-\Theta)(I + \varepsilon\partial)^{-1}\partial]u^n \tag{14}$$

and stability will be determined by the norm of this operator—hence, by $\Delta t/\varepsilon$.

B. Test Problem

As a test case, we consider the familiar Burgers' equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \qquad 0 \leqq x < \infty \tag{15}$$

with a "slant step" as initial data (Figure 1) and boundary data $u(0,t) = 1$. We solve the approximate problem for $0 < t < T$ on the truncated domain $0 < x < 1$ and for our least-squares approach also specify $u(1,t) = 0$, $0 < t < T$. [Recall that the least-squares scheme is equivalent to a Galerkin method for a (dissipative) second-order equation (Jiang and Carey [4]).] All computations were made using a uniform mesh of 50 linear elements. The time step Δt was varied with ε to examine their dependence and relative influence on the nonlinear stability of the calculation. Both the backward ($\Theta = 1$) and central $\Theta = ½$) time difference schemes were investigated.

In Figure 1, we show an initial slant step that steepens to a shock and compare numerical solutions at $T = 0.4$ for $\varepsilon = 0.0$ and $\varepsilon = 0.0075$ with $\Delta t = 0.01$ and $\Theta = 1.0$. Note the oscillatory overshoot for the case $\varepsilon = 0.0$. The solution

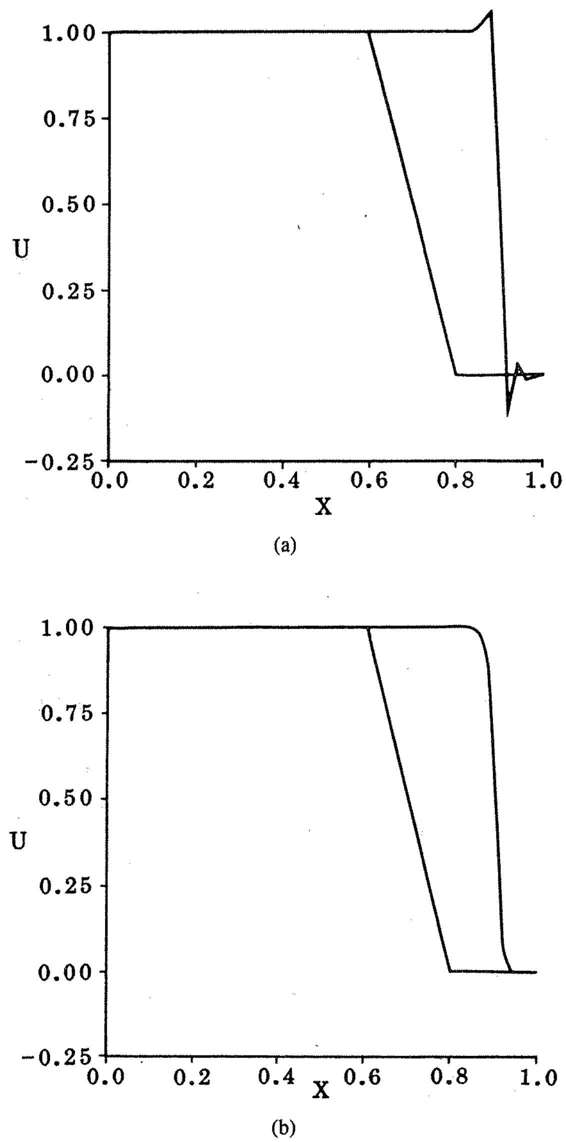


FIG. 1. Solution to Burgers' equation at $T = 0.4$ for initial slant step shown with $\Theta = 1$ (Backward Euler), $h = 0.02$, $\Delta t = 0.01$, with (a) $\varepsilon = 0.0$ and (b) $\varepsilon = 0.0075$.

"blows up" in the vicinity of the shock shortly thereafter. The regularized scheme with $\varepsilon = 0.0075$ behaves well—there is no overshoot, and the steepening front is approximated accurately. At $\varepsilon = 0.007$ and for lower values, the regularized scheme fails to stabilize the calculation.

The numerical experiment is repeated in Figure 2 for calculations with $\varepsilon = 0.0$ and 0.0125, $\Delta t = 0.016$, and $T = 0.16$. Similar behavior to that in Figure 1 is

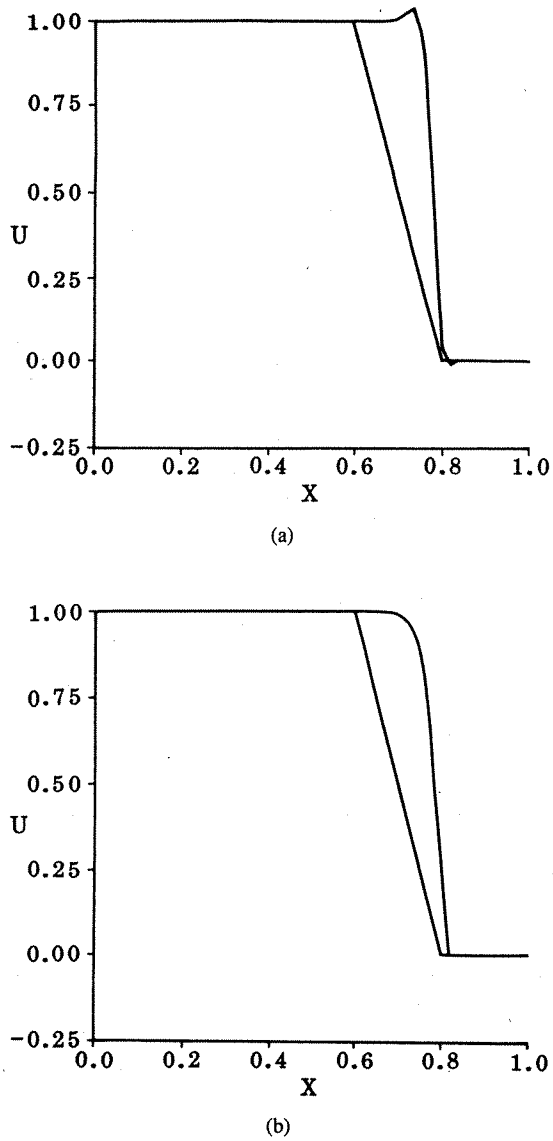
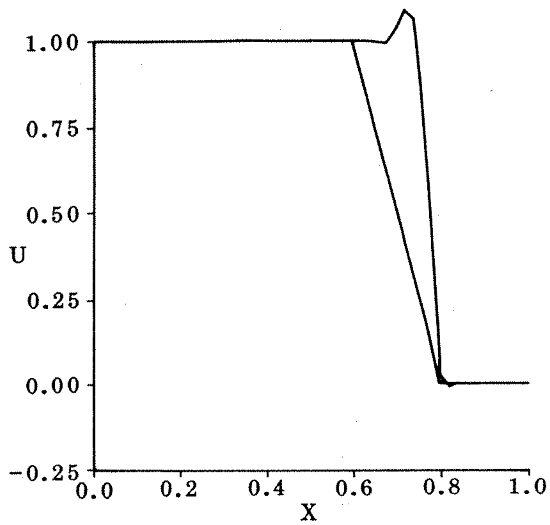
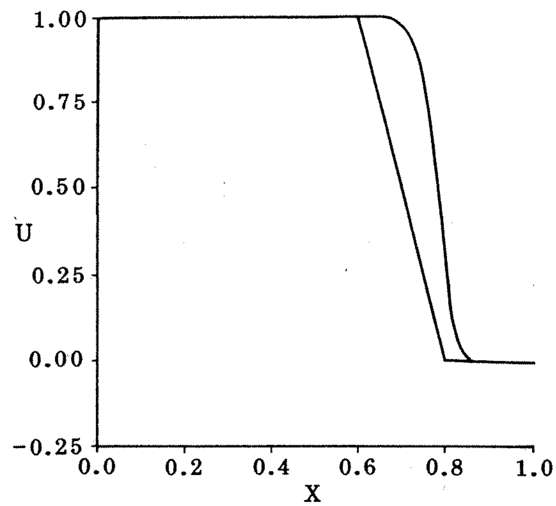


FIG. 2. Solution to Burgers' equation at $T = 0.16$ for initial slant step with $\Theta = 0.5$ (Crank–Nicolson), $h = 0.02$, $\Delta t = 0.016$, with (a) $\varepsilon = 0.0$ and (b) $\varepsilon = 0.0125$.

observed. Experiments indicate that the value $\varepsilon = 0.0125$ is approximately a lower bound for stable calculations with $\Delta t = 0.016$. If Δt is increased to 0.032, then we find it necessary to increase ε to 0.03 to stabilize the calculation and yield the results in Figure 3 for $\Theta = 0.5$ and $T = 0.16$ and Figure 4 for $T = 0.32$.



(a)



(b)

FIG. 3. Solution to Burgers' equation for initial slant step shown with $\Theta = 0.5$, $h = 0.02$, $\Delta t = 0.032$, with (a) $\varepsilon = 0.0$ and (b) $\varepsilon = 0.03$.

II. REMARKS

The regularization term εu_{xt} appears to be effective in stabilizing the nonlinear computations. The results for the standard model problem compare favorably with those obtained with other dissipative "nonlinear control" terms referenced previously. Empirically, we see that for the model problem there is a relationship

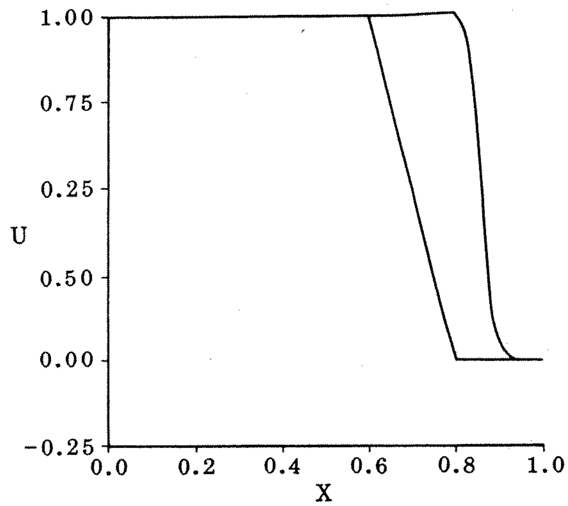


FIG. 4. Solution to problem in Figure 3b at $T = 0.32$.

between the minimum ε for stabilization and the integration step-size Δt. The results suggest that for this problem $\varepsilon_{\min} = 0(\Delta t)$. This might be inferred from the form of Eq. (8), where we have the coefficient $\varepsilon + \Delta t \Theta A$. Recall (Jiang and Carey [4]) that the least-squares approach naturally leads to a higher-order local dissipative term associated with $\Delta t \Theta A$ and, hence, that this dissipation increases with Δt. Including $\varepsilon > 0$ will increase the global dissipation, but the precise relation between ε and Δt is not yet clear.

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