Hysteresis Models of Adsorption and Deformation

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Outline

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ADSORPTION HYSTERESIS
Examples of Play-Type Hysteresis

The Simple Linear-Sided Play: $w(t) \in \mathcal{H}(u(t))$
The Constraint Function

\[ \dot{w}(t) + c(w(t) - u(t)) \geq 0 \]
More generally, introduce a Side-Shape function $b(\cdot)$:

$$w(t) = b(v(t)) \in \mathcal{H}(u(t)) : \quad \dot{w}(t) + c(v(t) - u(t)) \geq 0$$

Regularized Relay

The Relay
Solution with Boundary Hysteresis

\[ \Upsilon \cap \Omega (t)n \mathcal{H} \ni ((t) \lambda) q = (t) m \]

\[ \Upsilon \cap \Omega (n - \lambda) e \ni \frac{u \zeta}{n \zeta} \]

\[ \text{pure } 0 \in \frac{u \zeta}{n \zeta} + (n \lambda) q \frac{u \zeta}{e} \]

\[ \Upsilon \text{ in } f = n \nabla - (n \lambda) \frac{u \zeta}{e} \]

Boundary Hysteresis.
The Relay Solution
The Relay Hysteresis Loops
Interior Hysteresis.

\[ \frac{\partial}{\partial t} a(u) - \Delta u - c(v - u) = f \] and
\[ \frac{\partial}{\partial t} b(v) + c(v - u) \ni 0 \text{ in } \Omega \]

Boundary Conditions on \( \partial \Omega \)

\[ w(t) = b(v(t)) \in \mathcal{H}(u(t)) \text{ in } \Omega \]
\[ \frac{\partial}{\partial t} (a(u) + \mathcal{H}(u)) - \Delta u = f \text{ in } \Omega \]

Super-Stefan Free-Boundary Problem

Sums of Relays → General Preisach models
Convective transport.

\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - c(v - u) = f \] and

\[ \frac{\partial v}{\partial t} + c(v - u) \geq 0 \]

\[ \frac{\partial}{\partial t} (u + \mathcal{H}(u)) + \frac{\partial u}{\partial x} = f \]
Sums of Linear-sided Plays.

Convex-Sided Hysteresis
Symmetric convex bounding cycles
Multi-linear Hysteresis.

\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \sum_{i=1}^{n} c_i (v_i - u) = f \text{ and} \]

\[ \frac{\partial v_j}{\partial t} + c_j (v_j - u) \ni 0, \quad 1 \leq j \leq n. \]

\[ \frac{\partial}{\partial t} (u + \mathcal{H}(u)) + \frac{\partial u}{\partial x} = f \]

\[ \mathcal{H}(u) = \sum_{i=1}^{n} v_i(t) \]
Multiple nonlinearities, play-type hysteresis, ...

\[
\frac{\partial a(u)}{\partial t} + \frac{\partial \alpha(u)}{\partial x} - c(v - u) = f \quad \text{and} \\
\frac{\partial b(v)}{\partial t} + c(v - u) \geq 0
\]

\[
\frac{\partial}{\partial t} (a(u) + \mathcal{H}(u)) + \frac{\partial \alpha(u)}{\partial x} = f
\]
Examples of Stop-Type Hysteresis

Elastic-Plastic Deformation.

\[ \sigma_t + c(\sigma) \ni \varepsilon_t , \quad \varepsilon = \frac{\partial u}{\partial x}. \]

Elastic-Plastic STOP Hysteresis

The momentum and constitutive equations are, respectively,

\[ v_t - \sigma_x = f , \quad \sigma_t + c(\sigma) \ni \varepsilon_t . \]
Kinematic hardening.

\[ \frac{\partial}{\partial t} \sigma - \sigma_x = f, \quad \sigma = \beta_1 \sigma_1 + \beta_2 \sigma_2, \]

\[ \frac{\partial}{\partial t} \sigma_1 + \partial \phi_1(\sigma_1) \geq \beta_1 \frac{\partial}{\partial x} \sigma, \quad \frac{\partial}{\partial t} \sigma_2 = \beta_2 \frac{\partial}{\partial x} \sigma. \]

Kinematic Hardening Hysteresis
Multi-linear Hysteresis
**Isotropic hardening.** Let \( K \) be a nonempty, convex, and closed subset of the product space \( \Sigma \times \mathbb{R} \)

\[
\begin{bmatrix}
M\dot{\sigma}(t) \\
\dot{s}(t)
\end{bmatrix}
+ \partial I_K \left( \begin{bmatrix}
\sigma(t) \\
 s(t)
\end{bmatrix} \right) \geq \begin{bmatrix}
\dot{\varepsilon}(u(t)) \\
0
\end{bmatrix}
\]

The parameter \( s \in \mathbb{R} \) represents an internal force associated with the size of the (expanding) set of admissible stresses. This *hidden variable* leads to additional *degeneracy*. 
A Multi-Scale Deforming-Adsorbing-Porous Medium

The Biot model of consolidation of a porous medium for the fluid pressure $p(\cdot)$ in the larger pores is extended to include the small-scale pore pressure $q(\cdot)$. It’s dependence on the pressure $p(\cdot)$ is given by a hysteresis relationship of play type. The solid velocity $v(\cdot)$ is in a hysteresis relationship of stop type with the stress tensor.

\[
\begin{align*}
ap(t) + A(p(t)) - c(q(t) - p(t)) + \alpha \nabla \cdot v(t) &= h(t), \\
bq(t) + c(q(t) - p(t)) &\geq 0, \\
\rho \ddot{v}(t) + F(v(t)) - \nabla \cdot (\sigma_1(t) + \sigma_2(t)) + \alpha \nabla p(t) &= f(t), \\
\mathcal{C} \dot{\sigma}_1(t) - \varepsilon(v(t)) &= 0, \\
\frac{1}{2\mu} \dot{\sigma}_2(t) - \varepsilon(v(t)) + \partial I_K(\sigma_2(t)) &= 0.
\end{align*}
\]
REFERENCES


(with T. Little) Semilinear parabolic equations with Preisach hysteresis, Differential and Integral Equations 7 (1994), 1021-1040.


