SEMILINEAR DEGENERATE PARABOLIC SYSTEMS
AND DISTRIBUTED CAPACITANCE MODELS

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Abstract. A two-scale microstructure model of current flow in a medium with continuously distributed capacitance is extended to include nonlinearities in the conductance across the interface between the local capacitors and the global conducting medium. The resulting degenerate system of partial differential equations is shown to be in the form of a semilinear parabolic evolution equation in Hilbert space. It is shown directly that such an equation is equivalent to a subgradient flow and, hence, displays the appropriate parabolic regularizing effects. Various limiting cases are identified and the corresponding convergence results obtained by letting selected parameters tend to infinity.

1. Introduction. As integrated circuits become smaller, distributed capacitors receive correspondingly more attention; they have become the "big" components in integrated circuits. An example of distributed capacitance is the *tantulum capacitor*. A porous slug pressed out of tantalum powder is sintered to make the metal particles cohere and then anodized to produce a film of tantalum oxide, which serves as the dielectric of the capacitor. Next the slug is immersed in a solution of manganese nitrate and heated; this process leaves deposits of semiconducting manganese dioxide in its pores. The manganese dioxide serves as one electrode, while the underlying tantalum serves as the other [24]. Due to the intricate fine scale of its geometry, the real model for a capacitor of this type is too singular to be useful. Thus we use the *distributed capacitance* model, in which a microcapacitor is identified with each point of a larger domain, as a continuous approximation to the actual situation. These microstructure models contain the fine scale geometry of the microcapacitors as well as the current flux across the intricate interface by which they are connected to the global field.

Another model for distributed capacitance is the *layered medium equation*. This model represents a continuous approximation to a medium consisting of alternating thin layers of conductive and dielectric materials and is given by the equation

$$-\frac{\partial}{\partial t}(\partial_z C(x,z)\partial_z u) - (\partial_z G_V(x,z)\partial_z u) - \vec{\nabla}_x \cdot (G_H(x,z)\vec{\nabla}_x u) = F(x,z,t) \quad (1.1)$$

Instead of the thin layers of dielectic and conducting materials modeled by the layered medium equation, we shall consider small horizontally aligned microcapacitors distributed continuously throughout the conducting medium. We will show that the layered medium equation can be obtained as a singular limit of the microstructure model by approximating the distributed capacitors by single points of charge storage.

The actual system of equations for the distributed capacitance model will be given in the next section. It will be shown that the system has the abstract form

$$\frac{d}{dt}(\mathcal{B}u) + A(u) \ni f \tag{1.2}$$

where the linear operator $\mathcal{B}$ is non-negative and symmetric but degenerate and the nonlinear operator A is monotone. The operator $\mathcal{B}$ is necessarily degenerate, since it arises as a consequence of the charging capacitor and thus is nonzero in the local cell equations but zero in the global field equations of the system. The layered medium equation (1.1) is also of this form.

In order to show that the Cauchy problem for the evolution equation (1.2) is well-posed, we will use some techniques of convex analysis. For details, see [4] and [9]. Let V be a Banach space, and let $\varphi : V \to (-\infty, +\infty]$ be convex, proper, and lower-semi-continuous. Then $w \in V'$, the dual space, is a *subgradient* of φ at $u \in V$ if

$$w(v - u) \le \varphi(v) - \varphi(u) \text{ for all } v \in V$$

The set of all subgradients of φ at u is denoted by $\partial\phi(u)$. The subgradient is a generalized notion of the derivative, comparable to a directional derivative. We regard $\partial\phi$ as a multivalued operator from V to V'; it is easily shown to be monotone.

Equations of the general form of (1.2) have been studied for a long time by a variety of methods. These equations are of interest not only for the sake of generalization but also because they arise naturally in a vast variety of applications. The case of linear $\mathcal{B}$ may have degenerate behaviour due to a (possibly spatially dependent) coefficient that vanishes somewhere. That situation is in essence the type encountered here.

The earliest general treatment of semilinear and degenerate evolution equations of the form of (1.2) in an abstract setting occurs in the work [22]. There the nonlinear $\mathcal{B}$ is monotone and continuous and A is structured after a family of linear elliptic operators; both are permitted to be time dependent. In a similar but simpler setting, results were obtained in [2] by a backward difference approximation technique, and such results were subsequently obtained directly from a characterization of maximal monotone operators in [5]. Thereafter this situation was shown in [21] to be attainable directly as an application of nonlinear semigroups in Hilbert space; the solution was stronger but required smoother data initially. (The 'parabolic' case with A being a subgradient could be handled this way if $\mathcal{B}$ were invertible, but this does not cover the degenerate case. See [7,11,23].) For a review of work prior to 1976 and many examples, we refer to Chapter 3 of the book [8]. Perturbation and continuous dependence results were demonstrated in [17,18] and [25], and additional extensions were subsequently developed in the series of papers [12,13,14,15,16].

Our purpose here is to develop a nonlinear microstructure model for distributed capacitance in a conducting medium, to formulate it in an abstract form of a system of degenerate semilinear parabolic equations, and to show that this system is well-posed. We will find that the operator A of (1.2) that arises from our distributed capacitance model is the subgradient of a convex function, i.e., there exists a convex function φ such that $A = \partial\phi$. Then we will show in the abstract setting that there exists another convex function Φ such that (1.2) is equivalent to an explicit equation of the standard form

$$\frac{d}{dt}w + \partial\Phi(w) \ni g, \tag{1.3}$$

so this problem is *parabolic* and has the corresponding *regularizing effects* on the data. See [4,6]. Our plan is as follows. In Section 2 we derive the equations in their variational form involving the subgradient of a convex function. In Section 3 we develop existence and uniqueness results for the Cauchy problem for (1.2); we then prove results in Section 4 concerning limiting cases. In Section 5 we apply these results to the distributed capacitance model.

2. The Distributed Capacitance Model. In this section we describe the distributed microstructure model. Let Ω be a bounded domain in $\mathbb{R}^3$; this domain represents the conducting material in which the capacitance is to be distributed. For each point $x \in \Omega$, there is given a bounded cylindrical domain of the form $\Omega_x = S_x \times [-\frac{h}{2}, \frac{h}{2}]$, where S_x is a cross section in $\mathbb{R}^2$ and $h > 0$ is the thickness. Each Ω_x represents the generic horizontally oriented microcapacitor embedded in the conductor Ω in the vicinity of the point x. Each $x \in \Omega$ represents a point in real space, and $y \in S_x$ is the local variable in the small scale. Let $u(x, t)$ be the voltage distribution in the global region Ω. We will represent the voltage *difference* across the capacitor Ω_x by $U(x, y, t)$; specifically, at any time t and point $y \in S_x$, $U(x, y, t)$ is equal to the voltage at the point $(y, \frac{h}{2})$ on the top minus the voltage at the bottom point $(y, -\frac{h}{2})$. Using the approximation $u(x_1, x_2, x_3 + \frac{h}{2}) - u(x_1, x_2, x_3 - \frac{h}{2}) \approx h\frac{\partial u}{\partial x_3}$, we have the voltage difference across the interface between the microcapacitor and the surrounding conducting medium given by $h\frac{\partial u}{\partial x_3} - U$.

In each cell Ω_x, the horizontal voltage gradient induces a current density given by Ohm's law as $-K\vec{\nabla}_y U$, where K is the horizontal conductance of the capacitor surfaces. We use the subscript y on the gradient symbol to indicate that the gradient is taken with respect to the local variable y. The gradient symbol with no subscript will be used to indicate a gradient taken with respect to the global variable x. The capacitor charges in time at a rate of $C(x, y)U_t$, where $C(x, y)$ is the distributed capacitance. A vertical current input to Ω_x from the surrounding conductor is induced by the voltage drop $h\frac{\partial u}{\partial x_3} - U$; we will assume that this current is given by $\partial\psi_G(h\frac{\partial u}{\partial x_3} - U)$, where $\partial\psi_G$ is the monotone nonlinear conductance function obtained as the derivative or subgradient of a convex function ψ_G. The principle of conservation of charge then yields

$$\frac{\partial}{\partial t}(C(x, y)U) - \vec{\nabla}_y \cdot (K\vec{\nabla}_y U) - \partial\psi_G\left(h\frac{\partial u}{\partial x_3} - U\right) \ni F, \qquad y \in S_x \qquad (2.1)$$

where the function F denotes any additional distributed current sources. Similarly, the surrounding conductor induces a current of magnitude $h\partial\psi_g(h\frac{\partial u}{\partial x_3} - U)$ across the boundary of the capacitor in the horizontal normal direction n, where $\partial\psi_g$ is the subgradient of a convex function ψ_g. Thus we have

$$K\frac{\partial U}{\partial n} - h\partial\psi_g\left(h\frac{\partial u}{\partial x_3} - \gamma U\right) \ni 0, \qquad s \in \partial S_x \qquad (2.2)$$

At the global level, the distributed current arises from two sources: the global voltage, $u(x, t)$, and the normal current exiting the microcapacitor at the point x. This current is given by

$$\vec{j} = -k\vec{\nabla}u - \frac{\vec{e_3}}{|S_x|}\left(\int_{\partial S_x} K\frac{\partial U}{\partial n}\, ds + \int_{S_x} \partial\psi_G\left(h\frac{\partial u}{\partial x_3} - U\right)\, dy\right),$$

where k is a positive definite and symmetric matrix representing the conductance of the surrounding material. This matrix will reflect the fact that the current will flow more easily in the horizontal directions than in the vertical one. Conservation of charge on the global scale gives $\vec{\nabla} \cdot \vec{j} = 0$. Thus we have

$$-\vec{\nabla} \cdot k\vec{\nabla}u - \frac{\partial}{\partial x_3}\left(\frac{1}{|S_x|}\int_{S_x}\partial\psi_G\left(h\frac{\partial u}{\partial x_3} - U\right)dy + \int_{\partial S_x}K\frac{\partial U}{\partial n}ds\right) \ni 0, \qquad x \in \Omega$$

(2.3)

We will assume a grounded boundary at the global level,

$$u(x) = 0, \, x \in \partial\Omega,$$

(2.4)

although any one of the usual boundary constraints can be handled similarly. Finally, we need to specify initial values for the charge distribution U,

$$C(x,y)U(x,y,0) = C(x,y)U_0(x,y), \qquad x \in \Omega, \quad y \in S_x.$$

(2.5)

The system given by (2.1)–(2.4) is our distributed RC network model for distributed capacitance. This system of partial differential equations is of mixed degenerate parabolic-elliptic type. It consists of a family of diffusion equations, given by (2.1), each of which describes the conduction and storage of charge on the local scale of an individual capacitor at a specific site on the global conduction medium, and the single elliptic equation (2.3) which governs the interconnection by conservation of charge on the global scale of the conductor. This model contains the geometry of the individual capacitors and the current flux across the intricate interface by which they are connected to the global current field.

Notice that the total charge rate of the capacitor is given by

$$\frac{d}{dt}\int_{S_*}CU\,dy = \int_{S_*}\left(F + \partial\psi_G\left(h\frac{\partial u}{\partial x_3} - U\right)\right)dy + \int_{\partial S_*}h\partial\psi_g\left(h\frac{\partial u}{\partial x_3} - \gamma U\right)dy$$

$$= \int_{S_*}F\,dy + \int_{\partial\Omega_x}J\,dS,$$

in which the function

$$J = \begin{cases} \partial\psi_G\left(h\dfrac{\partial u}{\partial x_3} - U\right)/2 & \text{at } \bar{y} = \left(y, \pm\dfrac{h}{2}\right), y \in S_x \\[4mm] \partial\psi_g\left(h\dfrac{\partial u}{\partial x_3} - \gamma U\right) & \text{at } \bar{y} = (y, y_3), y \in \partial S_x \end{cases}$$

(2.6)

is the current flux across the boundary $\partial\Omega_x$.

In this development we have permitted the current flux in (2.6) to be driven by a nonlinear conductance. Our hypotheses below require that the convex functions have at most quadratic growth, so the nonlinear terms in (2.6) will be linearly bounded. These restrictions are only technical and convenient; one can easily include more general nonlinearities and also quasilinear models arising from nonlinear conductances in (2.1) and (2.3).

In order to obtain the variational formulation of the distributed capacitance model, we specify the spaces to be used. Let $L^2(\Omega)$ be the Lebesgue space of

equivalence classes of functions that are square-integrable on Ω, and let $H^1(\Omega)$ be the Sobolev space consisting of those functions in $L^2(\Omega)$ having each of their partial derivatives also in $L^2(\Omega)$. Denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions with support contained in Ω; the space $H_0^1(\Omega)$ is the closure in $H^1(\Omega)$ of $C_0^\infty(\Omega)$. For more information on these spaces, see [1].

Let $Q \subset \Omega \times \mathbb{R}^2$ be a measurable set in $\mathbb{R}^5$, and set $S_x = \{y \in \mathbb{R}^2 : (x, y) \in Q\}$; this provides an explicit contruction of the measurable family of cells mentioned above. By zero-extension, we can identify $L^2(Q)$ as a subspace of $L^2(\Omega \times \mathbb{R}^2)$ and each $L^2(S_x)$ as a subspace of $L^2(\mathbb{R}^2)$. Thus we have the identification $L^2(Q) \cong \{U \in L^2(\Omega, L^2(\mathbb{R}^2)) : U(x) \in L^2(S_x) \text{ for a.e. } x \in \Omega\}$. We will denote this space by $L^2(\Omega, L^2(S_x))$, with the inner product

$$(U, \Theta)_{L^2(\Omega, L^2(S_x))} = \int_\Omega \left\{ \frac{1}{|\Omega_x|} \int_{S_x} U(x, y) \Theta(x, y) dy \right\} dx.$$

This is a continuous direct sum of Hilbert spaces, since a function that is in $L^2(\Omega, L^2(S_x))$ takes values in a different Hilbert space at each point $x \in \Omega$. We define Sobolev spaces in a similar manner. Define

$$L^2(\Omega, H^1(S_x)) \equiv \left\{ U \in L^2(\Omega, L^2(S_x)) : U(x) \in H^1(S_x) \text{ a.e. } x \in \Omega, \right.$$

$$\left. \text{and } \int_\Omega \|U(x)\|_{H^1(S_x)}^2 \, dx < \infty \right\};$$

this direct sum is a Hilbert space. The *state space* for our problem will be the product $H \equiv L^2(\Omega) \times L^2(\Omega, L^2(S_x))$, and the *energy space* will be $V \equiv H_0^1(\Omega) \times L^2(\Omega, H^1(S_x))$. We will denote an element of these product spaces by $\tilde{u} = [u, U]$. In order to define *trace maps* on these spaces, we require that each S_x is a bounded domain in $\mathbb{R}^2$ which lies locally on one side of its boundary, ∂S_x, and that ∂S_x be a smooth curve in $\mathbb{R}^2$. Let $\gamma_x : H^1(S_x) \to L^2(\partial S_x)$ be the trace map from each cell to its boundary. We assume these maps are uniformly bounded so that we may define the distributed trace $\gamma : L^2(\Omega, H^1(S_x)) \to L^2(\Omega, L^2(\partial S_x))$ by $\gamma(U)(x, s) = (\gamma_x U)(s)$; in this case, γ is bounded and linear.

These definitions enable us to state precisely the weak formulation of our system. Suppose $[u, U]$ is an appropriately smooth solution of (2.1)–(2.4), and let $[\theta, \Theta] \in V$ be corresponding test functions. Multiply (2.1) by Θ and integrate over S_x. Using Green's Theorem and (2.2), we obtain

$$\frac{1}{|\Omega_x|} \int_{S_x} \left\{ \frac{\partial}{\partial t}(C(x, y)U)\Theta + K \vec{\nabla}_y U \vec{\nabla}_y \Theta - \partial \psi_G \left(h \frac{\partial u}{\partial x_3} - U \right) \Theta \right\} dy$$
$$- \frac{1}{|S_x|} \int_{\partial S_x} \partial \psi_g \left(h \frac{\partial u}{\partial x_3} - U \right) \gamma \Theta \, ds = \frac{1}{|\Omega_x|} \int_{S_x} F\Theta \, dy \tag{2.7}$$

(Technically, an equation like this is an abuse of notation since $\partial \psi_G$ may be multivalued. When an equation like this is used, we mean that there exists a representative from the multivalued operator such that equality holds. We will use this notational convenience hereafter.)

Similarly, multiply (2.3) by θ and integrate over Ω to obtain

$$\int_\Omega \left\{ k \vec{\nabla} u \vec{\nabla} \theta + \frac{1}{|S_x|} \left(\int_{S_x} \partial \psi_G \left(h \frac{\partial u}{\partial x_3} - U \right) dy + \int_{\partial S_x} K \frac{\partial U}{\partial n} \, ds \right) \frac{\partial \theta}{\partial x_3} \right\} dx = 0 \tag{2.8}$$

Finally, use (2.2) and add the integral of (2.7) over Ω to (2.8) to obtain

$$
\begin{aligned}
\int_\Omega \Bigg\{ & k\vec{\nabla}u\vec{\nabla}\theta + \frac{1}{|\Omega_x|}\int_{S_x}\Bigg(\frac{\partial}{\partial t}(C(x,y)U)\Theta + K\vec{\nabla}_y U\vec{\nabla}_y\Theta \\
& +\partial\psi_G\Big(h\frac{\partial u}{\partial x_3}-U\Big)\Big(h\frac{\partial\theta}{\partial x_3}-\Theta\Big)\Bigg)\,dy \\
& +\frac{1}{|S_x|}\int_{\partial S_x}\partial\psi_g\Big(h\frac{\partial u}{\partial x_3}-\gamma U\Big)\Big(h\frac{\partial\theta}{\partial x_3}-\gamma\Theta\Big)\,ds\Bigg\}\,dx \\
& =\int_\Omega\frac{1}{|\Omega_x|}\int_{S_x}F\Theta\,dy\,dx
\end{aligned}
\tag{2.9}
$$

In Section 3 we shall define a *generalized solution* of (2.1)–(2.4) to be a pair of appropriate functions u and U such that (2.9) holds for all corresponding test functions θ and Θ. Conversely, a generalized solution $\tilde{u}=[u,U]\in V$ of (2.9) can be shown to satisfy (2.1)–(2.4).

We have shown that the variational form of the system (2.1)–(2.4) can be written succinctly as

$$
\begin{aligned}
&\frac{d}{dt}(\mathcal{B}\tilde{u}(t)) + A\tilde{u}(t)\ni\tilde{f}(t)\ \text{in}\ V' \\
&\mathcal{B}u(0)=\mathcal{B}u_0\ ,
\end{aligned}
\tag{2.10}
$$

where $\mathcal{B}:H\to H'$ and $A:V\to V'$ are the operators given by

$$
\mathcal{B}\tilde{u}(\tilde{\theta})=\int_\Omega\frac{1}{|\Omega_x|}\int_{S_x}C(x,y)U\Theta\,dy\,dx,\ \text{and,}
$$

$$
\begin{aligned}
A\tilde{u}(\tilde{\theta})=\int_\Omega\Bigg\{ & k\vec{\nabla}u\vec{\nabla}\theta + \frac{1}{|\Omega_x|}\int_{S_x}\Big[K\vec{\nabla}_y U\vec{\nabla}_y\Theta + \partial\psi_G\Big(h\frac{\partial u}{\partial x_3}-U\Big)\Big(h\frac{\partial\theta}{\partial x_3}-\Theta\Big)\Big]\,dy \\
& +\frac{1}{|S_x|}\int_{\partial S_x}\partial\psi_g\Big(h\frac{\partial u}{\partial x_3}-\gamma U\Big)\Big(h\frac{\partial\theta}{\partial x_3}-\gamma\Theta\Big)\,ds\Bigg\}\,dx\ .
\end{aligned}
$$

We assume that $C(x,y)$ is a bounded, nonnegative function, that k is a positive definite and symmetric matrix, and that K is a positive constant. We also assume that the convex functions ψ_G and ψ_g are lower semi-continuous, and that each satisfies the conditions

(i) $\psi(0)=0=\min(\psi)$, and

(ii) there exists $c>0$ such that $\psi(s)\le c(1+|s|^2)$, for $s\in\mathbb{R}$.

Under these assumptions, it follows that the operator A can be written as the subgradient of the convex function $\varphi:V\to\mathbb{R}$ given by

$$
\begin{aligned}
\varphi(\tilde{u})=\int_\Omega\Bigg\{ & \frac{1}{2}k(\vec{\nabla}u)\cdot\vec{\nabla}u + \frac{1}{|\Omega_x|}\int_{S_x}\Big(\frac{K}{2}|\vec{\nabla}_y U|^2+\psi_G\Big(h\frac{\partial u}{\partial x_3}-U\Big)\Big)\,dy \\
& +\frac{1}{|S_x|}\int_{\partial S_x}\psi_g\Big(h\frac{\partial u}{\partial x_3}-\gamma U\Big)\,ds\Bigg\}\,dx\ .
\end{aligned}
$$

We state this as the following.

Proposition 2.1. *The subgradient of $\varphi, \partial\phi : V \to V'$, is given by*

$$\langle \partial\phi(\tilde{u}), \tilde{\theta} \rangle = A\tilde{u}(\tilde{\theta}) \ ,$$

where A is given as above.

The proof follows by standard methods of convex analysis [4,9].

3. The Abstract Cauchy Problem. Let V be a separable reflexive Banach space, dense and continuously embedded in a Hilbert space H. Let $\mathcal{B}$ be a continuous linear operator from V to V'; we will assume that $\mathcal{B}$ is positive and self-adjoint. Define the semi-normed space W_b to be the completion of V with respect to the seminorm induced by the semiscalar product $(u, v)_{W_b} = \mathcal{B}u(v)$. Then the dual space W_b' is a Hilbert space, and $\mathcal{B}$ is a strict homomorphism from W_b into W_b'. Let $\varphi : V \to [0, \infty]$ be a proper, convex, and lower semi-continuous function. We consider the degenerate Cauchy problem

$$\frac{d}{dt}(\mathcal{B}u(t)) + \partial\phi(u(t)) \ni f(t) \text{ for a.e. } t \in [0, T] \tag{3.1}$$
$$\mathcal{B}u(0) = \mathcal{B}u_0 \ .$$

A *solution* of (3.1) is a function $u \in C([0, T], W_b)$ such that u is absolutely continuous on $[\delta, T]$ for all $\delta > 0$, and (3.1) holds in W_b' for almost every $t \in [0, T]$.

We will show that (3.1) is equivalent to an evolution equation in H in the explicit form

$$\frac{d}{dt}w(t) + C(w(t)) \ni g(t)$$

in which the operator C is the subgradient of a convex function, i.e., $C = \partial\Phi$. Standard results on maximal monotone operators in Hilbert space will then apply directly to yield existence and uniqueness results.

We will use the *square root* of the operator $\mathcal{B}$ [10]. Define the Hilbert space V_b to be the completion of V with respect to the scalar product

$$(u, v)_{V_b} = (u, v)_H + \mathcal{B}u(v) \ .$$

Since $\|u\|_{W_b} \le \|u\|_{V_b}$ for all $u \in V$, we have $V_b \subset W_b$, and V_b is dense in W_b. Also, the space V is dense and continuously embedded in V_b, which is dense and continuously embedded in H. Therefore, by extension, we can regard $\mathcal{B}$ as a continuous linear operator from V_b to V_b'. Since the bilinear form $(u, v)_{W_b} = \mathcal{B}u(v)$ is densely defined, closed, and symmetric on V_b, there exists a positive, self-adjoint, closed linear operator $B : \text{dom}(B) \subset V_b \to H$ such that

$$\mathcal{B}u(v) = (Bu, v)_H \text{ for } u \in \text{dom}(B) \ , \qquad v \in V_b \ .$$

Thus, B is obtained from $\mathcal{B}$ by restricting the range to $H \subset V_b'$; the domain of B is dense in V_b.

Next one constructs the positive, self-adjoint, continuous, linear operator $B^{1/2}$: $\text{dom}(B^{1/2}) = V_b \to H$ such that $B^{1/2}B^{1/2} = B$, and

$$(B^{1/2}u, B^{1/2}v)_H = \mathcal{B}u(v) = (u, v)_{W_b} \text{ for } u, v \in V_b \ .$$

It is the operator $B^{1/2}$ which will be of primary interest to us. Since $B^{1/2} \in \mathcal{L}(V_b, H)$, we can define the Banach space   adjoint, $B^{1/2*} \in \mathcal{L}(H, V_b')$, by

$$B^{1/2*} w(v) = (w, B^{1/2} v)_H \ .$$

It follows that $B^{1/2*} B^{1/2} = \mathcal{B}$ on V_b. For $v \in V_b$, $|B^{1/2} v|_H = \|v\|_{W_b}$, so $B^{1/2}$ is continuous on V_b with the W_b seminorm. Thus $B^{1/2}$ has a unique continuous extension (which we also denote by $B^{1/2}$) to a strict homomorphism from W_b to H, and so the adjoint $B^{1/2*} : H \to W_b'$ is continuous and onto. Since V_b is dense in W_b, the identity $B^{1/2*} B^{1/2} = \mathcal{B}$ also extends to W_b.

These properties of $B^{1/2}$ and $B^{1/2*}$ permit us to reformulate (3.1) in an explicit form.

Proposition 3.1. *Let $g \in B^{-1/2*} f$. Then the equation*

$$\frac{d}{dt}(w(t)) + B^{-1/2*} \partial \phi B^{-1/2}(w(t)) \ni g(t) \text{ for a.e. } t \in [0, T] \qquad (3.2)$$

is equivalent to (3.1) *in the following sense:*

- *If $w \in C([0, T]; H)$ is a solution to* (3.2)*, then there exists $u \in B^{-1/2} w$ such that $u \in C([0, T]; W_b)$ is a solution of* (3.1)*.*
- *If $u \in C([0, T]; W_b)$ is a solution to* (3.1)*, set $w = B^{1/2} u$. Then $w \in C([0, T]; H)$, is a solution to* (3.2)*.*

Next we show that the explicit operator $B^{-1/2*} \partial \phi B^{-1/2}$ is the subgradient of a convex function. Recall from [9, p.17]   that the *polar function* of φ, $\varphi^* : V' \to (-\infty, \infty]$, defined by

$$\varphi^*(f) = \sup_{v \in V} \{f(v) - \varphi(v)\} \ ,$$

is proper, convex, and lower semi-continuous; the relationship $\partial \varphi^* = (\partial \phi)^{-1}$ holds in the sense of multifunctions; and $(\varphi^*)^* = \varphi$. We will use the chain rule for subgradients, [9, p.27], on the composition $\varphi^* \circ B^{1/2*}$. The following coercivity condition on the function φ will guarantee the required continuity: there exist constants $c > 0$ and $k > 0$ such that

$$\text{if } v \in V \text{ with } \|v\|_V \geq k, \text{ then } \varphi(v) \geq c\|v\|_V \ .$$

Lemma 3.1. *If φ satisfies the coercivity condition, then φ^* is continuous at some point of the range of $B^{1/2*}$.*

Proof. We will show that φ^* is continuous at 0. It suffices to show that φ^* is bounded on the neighborhood $N = \{f \in V' : \|f\|_{V'} \leq c\}$, where c is the constant from the coercivity estimate. Define $V_1 = \{v \in V : \|v\|_V < k\}$, and $V_2 = \{v \in V : \|v\|_V \geq k\}$, and let f be an element of N. Since $\varphi(v) \geq 0$,

$$\sup_{v \in V_1} \{f(v) - \varphi(v)\} \leq \sup_{v \in V_1} (\|f\|_{V'} \|v\|_V) \leq k\|f\|_{V'} \leq c \cdot k \ .$$

Using the coercivity condition,

$$\sup_{v \in V_2} \{f(v) - \varphi(v)\} \leq \sup_{v \in V_2} \{\|f\|_{V'} \|v\|_V - c\|v\|_V\} \leq 0 \ .$$

Hence $\varphi^*(f) \leq c \cdot k$ on the neighborhood N. $\square$

Proposition 3.2. *The function* $\Phi \equiv (\varphi^* \circ B^{1/2*})^* : H \to [0, \infty]$ *is proper, convex, and lower semi-continuous with* $\partial\Phi = B^{-1/2*}\partial\phi B^{-1/2}$.

Proof. Consider the composite function $\varphi^* \circ B^{1/2*} : H \to (-\infty, \infty]$. Using Lemma 3.1, an application of the chain rule yields

$$\partial(\varphi^* \circ B^{1/2*}) = B^{1/2}\partial\varphi^* B^{1/2*} .$$

But the subgradient of the polar is given by

$$\begin{aligned}
\partial((\varphi^* \circ B^{1/2*})^*) &= (\partial(\varphi^* \circ B^{1/2*}))^{-1} \\
&= B^{-1/2*}(\partial\varphi^*)^{-1}B^{-1/2} \\
&= B^{-1/2*}\partial\phi B^{-1/2} .
\end{aligned}$$

Finally, note that Φ is proper, convex and lower semi-continuous , and

$$\mathrm{rg}(\Phi) \subset \mathrm{rg}((\varphi^*)^*) = \mathrm{rg}(\varphi) \subset [0, \infty] . \qquad \square$$

Thus we can write (3.2) in the form

$$\frac{d}{dt}w(t) + \partial\Phi(w(t)) \ni B^{-1/2*}f(t) \text{ in } H . \qquad (3.3)$$

We now relate the initial conditions for (3.1) to those appropriate for (3.3).

Lemma 3.2. $B^{1/2}(\overline{\mathrm{dom}(\varphi)}^{W_b}) \subset \overline{\mathrm{dom}(\Phi)}^H$.

Proof. We first show that $B^{1/2}(\mathrm{dom}(\varphi)) \subset \mathrm{dom}(\Phi)$. Suppose that $h = B^{1/2}u$ for some $u \in \mathrm{dom}(\varphi)$. Then for every $g \in H$,

$$\begin{aligned}
\varphi(u) = \varphi^{**}(u) &= \sup_{f \in V'}\{f(u) - \varphi^*(f)\} \\
&\geq B^{1/2*}g(u) - \varphi^*(B^{1/2*}g) ,
\end{aligned}$$

and so

$$\begin{aligned}
(\varphi^* \circ B^{1/2*})^*(h) &= (\varphi^* \circ B^{1/2*})^*(B^{1/2}u) \\
&= \sup_{g \in H}\{(B^{1/2}u, g)_H - (\varphi^* \circ B^{1/2*})(g)\} \\
&\leq \sup_{g \in H}\{(B^{1/2}u, g)_H + \varphi(u) - B^{1/2*}g(u)\} \\
&= \varphi(u) .
\end{aligned}$$

Since $u \in \mathrm{dom}(\varphi)$, we have $h \in \mathrm{dom}((\varphi^* \circ B^{1/2*})^*)$. Thus $\overline{B^{1/2}(\mathrm{dom}(\varphi))}^H \subset \overline{\mathrm{dom}(\Phi)}^H$.

Next, we show that $B^{1/2}(\overline{\mathrm{dom}\,\varphi}^{W_b}) = \overline{B^{1/2}(\mathrm{dom}(\varphi))}^H$; this will complete the proof. Let $\{u_n\}$ be a sequence in $\mathrm{dom}(\varphi)$ such that $u_n \to u$ in W_b, and suppose that $h = B^{1/2}u$, i.e., $h \in B^{1/2}(\overline{\mathrm{dom}\,\varphi}^{W_b})$. Set $h_n = B^{1/2}u_n$. Then, since

$$\|h_n - h\|_H = \|B^{1/2}(u_n - u)\|_H = \|u_n - u\|_{W_b} \to 0 ,$$

$h_n \to h$ in H, and so $h \in \overline{B^{1/2}(\mathrm{dom}(\varphi))}^H$. The converse follows similarly. $\square$

Finally, we note that if $f \in L^2(0, T; W_b')$, then there exists $g \in L^2(0, T; H)$ such that $g \in B^{-1/2*}f$.

We have thus completed all the steps necessary to reduce the Cauchy problem (3.1) to the form

$$w'(t) + \partial\Phi w(t) \ni g(t) \text{ in } H \text{ for a.e. } t \in [0, T] \qquad (3.4)$$
$$w(0) = w_0 ,$$

for which there is a complete theory [4, p.131].

Theorem 3.1 (Existence). *Let $u_0 \in \overline{\mathrm{dom}(\varphi)}^{W_b}$, $f \in L^2(0, T; W_b')$, and $\mathcal{B}$ and φ be given as above. Then there exists a solution $u \in C([0, T]; W_b)$ to*

$$\frac{d}{dt}(\mathcal{B}u(t)) + \partial\phi(u(t)) \ni f(t) \text{ in } W_b', \text{ for a.e. } t \in [0, T] \qquad (3.5)$$
$$\mathcal{B}u(0) = \mathcal{B}u_0 ,$$

and $u(t) \in \mathrm{dom}(\partial\phi)$ for a.e. $t \in [0, T]$.

Proof. By Lemma 3.2, if $u_0 \in \overline{\mathrm{dom}(\varphi)}^{W_b}$ then

$$w_0 \equiv B^{1/2}u_0 \in B^{1/2}(\overline{\mathrm{dom}(\varphi)}^{W_b}) \subset \overline{\mathrm{dom}(\Phi)}^H .$$

Also, there exists $g \in L^2(0, T; H)$ such that $g \in B^{-1/2*}(f)$.

Choosing $\Phi = (\varphi^* \circ B^{1/2*})^*$ and g as above, we obtain the existence of a unique solution $w \in C([0, T]; H)$ of (3.4). Proposition 3.1 and Proposition 3.2 show that there exists a function $u \in C([0, T]; W_b)$ satisfying (3.5) and we have

$$\mathcal{B}u(0) = B^{1/2*}B^{1/2}u(0) = B^{1/2*}w(0) = B^{1/2*}w_0 = B^{1/2*}B^{1/2}u_0 = \mathcal{B}u_0 . \quad \square$$

Even though the solution to the explicit equation is unique, the choice of $u(t)$ as an element of the set $B^{-1/2}w(t)$ could introduce nonuniqueness. To insure uniqueness, we impose an additional condition.

Theorem 3.2 (Uniqueness). *In the situation of Theorem 3.1, if $\mathcal{B}+\partial\phi$ is strictly monotone, then the solution $u(t)$ is unique.*

Thus we have sufficient conditions for existence and uniqueness for the Cauchy problem (3.5). Continuous dependence on the data u_0 and f can be shown using standard methods. Also see [25].

In the next section, we will need the solution u to be slightly more regular. In anticipation of this, we have the following result.

Theorem 3.3. *If, in addition to the previous hypotheses, $u_0 \in \mathrm{dom}(\varphi)$, then the solution u satisfies*

$$2\int_0^t \frac{d}{ds}(\mathcal{B}u(s))u(s)ds = \|u(t)\|_{W_b}^2 - \|u(0)\|_{W_b}^2$$

for $t \in [0, T]$.

Proof. From the proof of Lemma 3.2, we have $B^{1/2}(\text{dom}(\varphi)) \subset \text{dom}(\Phi)$. Thus $w_0 = B^{1/2}u_0 \in \text{dom}(\Phi)$, and so we have $\frac{dw}{dt} \in L^2(0, T; H)$. It follows that $\|w(t)\|_H^2$ is absolutely continuous on $[0, T]$ and

$$\frac{d}{dt}(\|w(t)\|_H^2) = \frac{d}{dt}(\|B^{1/2}u(t)\|_H^2) = 2\left\langle \frac{d}{dt}(\mathcal{B}u(t)), u(t) \right\rangle_{W_b', W_b} . \qquad \square$$

Note that the solution of (3.5) satisfies $u(t) \in \text{dom}(\partial\varphi) \subset V$ and the equation holds in $V_b' \subset V'$ at a.e. $t \in (0, T]$, even though $u(0)$ is given in $\overline{\text{dom}(\varphi)}^{W_b}$. This is the *parabolic regularizing effect*.

4. Limiting Cases. Suppose that $\varphi = \phi_0 + \phi_1$, with ϕ_0 and ϕ_1 proper, convex, and continuous from V into $[0, \infty]$, and set $\phi_\epsilon = \phi_0 + \frac{1}{\epsilon}\phi_1$ with $0 < \epsilon \leq 1$. In this section we will consider the limiting problem obtained by replacing $\partial\phi$ with $\partial\phi_\epsilon$ in (3.1) and letting $\epsilon \to 0$.

We assume that $V_0 = \{v \in V : \phi_1(v) = 0\}$ is a linear subspace of V. Let $\mathcal{V} = L^2(0, T; V)$, $\mathcal{H} = L^2(0, T; H)$, and $\mathcal{V}_0 = L^2(0, T; V_0)$; let H_0 be the closure in H of V_0 and $\mathcal{H}_0 = L^2(0, T; H_0)$. Similarly, let W_0 be the closure in W_b of V_0 and $\mathcal{W}_0 = L^2(0, T; W_0)$. In addition, we assume that φ (and hence ϕ_ϵ) is V-coercive; this implies that ϕ_ϵ satisfies the coercivity condition of Section 3. We also assume that $\mathcal{B} + \partial\phi$ (and hence $\mathcal{B} + \partial\phi_\epsilon$) is strictly monotone.

Let $f \in L^2(0, T; W_b')$ and $u_\epsilon^0 \in \text{dom}(\phi_\epsilon)$ for each $\epsilon \in (0, 1)$. From Theorem 3.1 and Theorem 3.2, there exists a unique solution $u_\epsilon \in C([0, T]; W_b)$ of

$$\frac{d}{dt}(\mathcal{B}u_\epsilon) + \partial\phi_\epsilon(u_\epsilon) \ni f \text{ in } W_b' \text{ for a.e. } t \in [0, T] \tag{4.1}$$
$$\mathcal{B}u_\epsilon(0) = \mathcal{B}u_\epsilon^0 .$$

Similarly, let $f_0 = f|_{W_0}$ and let $u_0^0 \in \text{dom}(\phi_0)$. Then another application of Theorem 3.1 and Theorem 3.2 gives a unique solution $u_0 \in C([0, T]; W_0)$ of

$$\frac{d}{dt}(\mathcal{B}u_0) + \partial\phi_0(u_0) \ni f_0 \text{ in } W_0' \text{ for a.e. } t \in [0, T] \tag{4.2}$$
$$\mathcal{B}u_0(0) = \mathcal{B}u_0^0 .$$

Our goal is to show that, with appropriate hypotheses on the initial conditions, $u_\epsilon \rightharpoonup u_0$ in $\mathcal{V}$. Attaining this will require several lemmas.

Lemma 4.1. *If $\{u_\epsilon^0\}$ is bounded in W_b, then $\{u_\epsilon\}$ is bounded in $\mathcal{V}$.*

Proof. Apply (4.1) to u_ϵ and integrate to obtain:

$$\int_0^t \phi_\epsilon(u_\epsilon) \, ds + \int_0^t \left\langle \frac{d}{ds}(\mathcal{B}u_\epsilon)u_\epsilon \right\rangle ds \leq \int_0^t \langle f, u_\epsilon \rangle \, ds .$$

Theorem 3.3 yields

$$2\int_0^t \phi_0(u_\epsilon) \, ds + \frac{2}{\epsilon}\int_0^t \phi_1(u_\epsilon) \, ds + \|u_\epsilon(t)\|_{W_b}^2$$
$$\leq \|f\|_{\mathcal{W}_b'}\left(\int_0^t \|u_\epsilon\|_{W_b}^2 \, ds\right)^{1/2} + \|u_\epsilon^0\|_{W_b}^2 .$$

A Gronwall-type inequality then shows that $\|u_\epsilon\|_{C([0,T];W_b)}$ is bounded. Since ϕ_ϵ is V-coercive, this implies that $\{u_\epsilon\}$ is also bounded in $\mathcal{V}$. $\square$

Thus there exists a subsequence, which we will also denote by $\{u_\epsilon\}$, such that $u_\epsilon \rightharpoonup u_1$ for some $u_1 \in \mathcal{V}$. From the proof of Lemma 4.1 we see that $\int_0^T \frac{1}{\epsilon}\phi_1(u_\epsilon)\,dt$ is bounded, so Fatou's Lemma and the weak lower semi-continuity of ϕ_1 imply that $u_1 \in \mathcal{V}_0$.

Lemma 4.2. *If $\{\frac{1}{\epsilon}\phi_1(u_\epsilon^0)\}$ is bounded, then $\{\frac{d}{dt}\mathcal{B}u_\epsilon\}$ is bounded in $\mathcal{W}_b'$.*

Proof. Let $w_\epsilon(t) \equiv B^{1/2}u_\epsilon(t)$ and $\Phi_\epsilon \equiv (\varphi_\epsilon^* \circ B^{1/2*})^*$, so that

$$w_\epsilon' + \partial\Phi_\epsilon(w_\epsilon) \ni B^{-1/2*}f \text{ in } H \text{ , for a.e. } t \in [0,T] \text{ .}$$

Taking the scalar product in H with w_ϵ' and integrating yields

$$\|w_\epsilon'\|_{\mathcal{H}}^2 + \Phi_\epsilon(w_\epsilon(T)) \le \|g\|_{\mathcal{H}}\|w_\epsilon'\|_{\mathcal{H}} + \Phi_\epsilon(w_\epsilon(0)) \text{ .}$$

Since $\inf_{v \in V}\{\phi_\epsilon(v)\} \ge 0$, we also have $\inf_{w \in H}\{\Phi_\epsilon(w)\} \ge 0$, so $\Phi_\epsilon(w_\epsilon(T))$ is positive. Also, from the proof of Lemma 3.2, we see that $\Phi_\epsilon(w_\epsilon(0)) \le \phi_\epsilon(u_\epsilon^0)$, which is bounded since $\{\frac{1}{\epsilon}\phi_1(u_\epsilon^0)\}$ is bounded, so we have $\|w_\epsilon'\|_{\mathcal{H}}$ bounded. Since $B^{1/2}$ is an isomorphism from $\mathcal{H}$ to $\mathcal{W}_b'$, the result follows. $\square$

Thus we can choose a further subsequence of $\{u_\epsilon\}$ such that $\frac{d}{dt}\mathcal{B}u_\epsilon \rightharpoonup \frac{d}{dt}\mathcal{B}u_1$ in $\mathcal{W}_b'$ and $\mathcal{B}u_\epsilon(T) \rightharpoonup \mathcal{B}u_1(T)$ in W_b'.

Lemma 4.3. *Assume that $\mathcal{B}u_\epsilon^0 \to \mathcal{B}u_0^0$ in W_b'. Then the equation*

$$\frac{d}{dt}(\mathcal{B}u_1) + \partial\phi_0(u_1) \ni f_0$$

holds in W_0' for a.e. $t \in [0,T]$.

Proof. For every $v \in \mathcal{V}$ and almost every $t \in [0,T]$, (4.1) yields

$$\left\langle f - \frac{d}{dt}(\mathcal{B}u_\epsilon), v - u_\epsilon \right\rangle \le \phi_\epsilon(v) - \phi_\epsilon(u_\epsilon) \text{ .}$$

Restricting this to $v \in \mathcal{V}_0$, applying Theorem 3.3, and noting that $\phi_\epsilon \ge \phi_0$ gives

$$\int_0^T \langle f, v - u_\epsilon \rangle - \left\langle \frac{d}{dt}(\mathcal{B}u_\epsilon), v \right\rangle dt + \frac{1}{2}\|u_\epsilon(T)\|_{W_b}^2 - \frac{1}{2}\|u_\epsilon^0\|_{W_b}^2 \tag{4.3}$$
$$\le \int_0^T \phi_0(v)\,dt - \int_0^T \phi_0(u_\epsilon)\,dt \text{ .}$$

Using weak lower semi-continuity, we have $\int_0^T \phi_0(u_1)\,dt \le \liminf_{\epsilon \to 0}\int_0^T \phi_0(u_\epsilon)\,dt$ and $\|\mathcal{B}u_1(T)\|_{W_b}^2 \le \liminf_{\epsilon \to 0}\|\mathcal{B}u_\epsilon(T)\|_{W_b}^2$. Since $\mathcal{B}u_\epsilon^0 \to \mathcal{B}u_0^0$ in W_b' and $\mathcal{B}$ is an isomorphism from W_b to W_b', $u_\epsilon^0 \to u_0^0$ in W_b. Taking the lim inf of (4.3) thus gives

$$\int_0^T \left\langle f_0 - \frac{d}{dt}(\mathcal{B}u_1), v - u_1 \right\rangle dt \le \int_0^T \phi_0(v) - \phi_0(u_1)\,dt$$

for every $v \in \mathcal{V}_0$. Thus we obtain

$$f_0 - \frac{d}{dt}(\mathcal{B}u_1) \in \partial\phi_0(u_1) \text{ in } V_0', \text{ hence, in } W_0' \text{ for a.e. } t \in [0,T] \text{ .} \qquad \square$$

Theorem 4.1. *Let u_ϵ and u_0 be the generalized solutions to (4.1) and (4.2), respectively. If $\{\frac{1}{\epsilon}\phi_1(u_\epsilon^0)\}$ is bounded and $\mathcal{B}u_\epsilon^0 \to \mathcal{B}u_0^0$ in W_b', then $u_\epsilon \rightharpoonup u_0$ in $\mathcal{V}$.*

Proof. We have shown in the previous lemmas that a subsequence of $\{u_\epsilon\}$ converges weakly in $\mathcal{V}$ to $u_1 \in \mathcal{V}_0$ satisfying

$$\frac{d}{dt}(\mathcal{B}u_1) + \partial\phi_0(u_1) \ni f_0 \text{ in } W_0' \text{ for a.e. } t \in [0, T] .$$

To show that u_1 is a solution to (4.2), we assert that $\mathcal{B}u_1(0) = \mathcal{B}u_0^0$. Using the continuous embedding of $H^1(0, T; H)$ into $C([0, T], H)$, we have the estimate

$$\|B^{1/2}u_\epsilon(t)\|_H \leq c\left(\|B^{1/2}u_\epsilon\|_{\mathcal{H}} + \|\frac{d}{dt}(B^{1/2}u_\epsilon)\|_{\mathcal{H}}\right)$$

for a.e. $t \in [0, T]$. This implies that the map from $H^1(0, T; H)$ to H which takes $(B^{1/2}u_\epsilon, \frac{d}{dt}(B^{1/2}u_\epsilon))$ to $B^{1/2}u_\epsilon(0)$ is strongly continuous; it is also linear and hence weakly continuous. The same reasoning shows that the operator $B^{1/2} : \mathcal{V} \to \mathcal{H}$ is weakly continuous as well, so $B^{1/2}u_\epsilon \rightharpoonup B^{1/2}u_1$ in $\mathcal{H}$. Lemma 4.2 shows that the derivatives also converge weakly in $\mathcal{H}$, and so we have $B^{1/2}u_\epsilon(0) \rightharpoonup B^{1/2}u_1(0)$. It follows that $\mathcal{B}u_\epsilon(0) \rightharpoonup \mathcal{B}u_1(0)$ since $B^{1/2}$ is weakly continuous. But

$$\mathcal{B}u_\epsilon(0) = \mathcal{B}u_\epsilon^0 \to \mathcal{B}u_0^0 ,$$

and so, since weak limits are unique, $\mathcal{B}u_1(0) = \mathcal{B}u_0^0$. Thus u_1 is a generalized solution of (4.2), and, since the solution is unique, it follows that $u_1 = u_0$ and the original sequence satisfies $u_\epsilon \rightharpoonup u_0$. $\square$

5. Examples. We will apply the preceeding results to the distributed capacitance model to obtain existence and uniqueness of a generalized solution and characterize three limiting problems. As in Section 2, define the spaces $V = H_0^1(\Omega) \times L^2(\Omega, H^1(S_x))$ and $H = L^2(\Omega) \times L^2(\Omega, L^2(S_x))$. We have shown that, with $\mathcal{B} : V \to V'$ given by

$$\mathcal{B}\tilde{u}(\tilde{\theta}) = \int_\Omega \frac{1}{|\Omega_x|} \int_{S_*} C(x, y) U\Theta \, dy \, dx , \qquad \tilde{u} = [u, U] , \ \tilde{\theta} = [\theta, \Theta] ,$$

and $\varphi : V \to [0, \infty)$ defined by

$$\varphi(\tilde{u}) = \int_\Omega \left\{ \frac{1}{2}k(\vec{\nabla}u) \cdot \vec{\nabla}u + \frac{1}{|\Omega_x|}\int_{S_*}\left(\frac{K}{2}|\vec{\nabla}_y U|^2 + \psi_G\left(h\frac{\partial u}{\partial x_3} - U\right)\right) dy \right.$$
$$\left. + \frac{1}{|S_x|}\int_{\partial S_*} \psi_g\left(h\frac{\partial u}{\partial x_3} - \gamma U\right) ds \right\} dx , \qquad \tilde{u} = [u, U] ,$$

the distributed capacitance model can be written in the abstract form

$$\frac{d}{dt}(\mathcal{B}\tilde{u}) + \partial\phi(\tilde{u}) \ni \tilde{f} \text{ in } V' \tag{5.1}$$
$$\mathcal{B}\tilde{u}(0) = \mathcal{B}\tilde{u}_0 .$$

The generalized solution that we obtain for this equation will be in the sense of that defined in Section 3. Notice that the space W_b is a set of pairs of functions $[u, U]$ whose second component U satisfies $C^{\frac{1}{2}}U \in L^2(\Omega, L^2(S_x))$, and W_b' is the set of pairs of functionals of the form $[0, C^{\frac{1}{2}}f]$ with $f \in L^2(\Omega, L^2(S_x))$. A precise statement of our notion of solution is found in the existence theorem below.

We will use two versions of Poincaré's inequality.

Lemma 5.1. *Assume that G is an open set in $\mathbb{R}^n$ with $\sup\{|x_1| : (x_1, x_2, \ldots, x_n) \in G\} = D < \infty$. If $v \in H_0^1(G)$, then $\int_G |v|^2 \, dx \leq 4D^2 \int_G |\partial_1 v|^2 \, dx$.*

Lemma 5.2. *Assume in addition that G lies locally on one side of its boundary, ∂G, and that ∂G is a smooth surface in $\mathbb{R}^n$. If $v \in H^1(G)$, then*

$$\int_G |v|^2 \, dx \leq 2D \int_{\partial G} |\gamma v(s)|^2 \, ds + 4D^2 \int_G |\partial_1 v|^2 \, dx .$$

Recall that each S_x satisfies the hypotheses for Lemma (5.2), and furthermore these sets are uniformly bounded, say $|S_x| \leq d$ for all $x \in \Omega$. Also, we have the estimate

$$\frac{|\partial S_x|}{|S_x|} \geq \frac{|\partial S_x|}{d} \geq \frac{2\sqrt{\pi}}{\sqrt{d}} \equiv d_1 ,$$

since the boundary of minimum length enclosing a given area is a circle.

Theorem 5.1 (Existence for the Distributed Capacitance Model). *Let the measurable set Q in $\Omega \times \mathbb{R}^2$ and the corresponding sets S_x, $x \in \Omega$, be given as in Section 2. Let the positive definite matrix k, the constant $K > 0$, the non-negative function $C \in L^\infty(Q)$, and the convex functions ψ_G and ψ_g be given as in Section 2. Suppose that there exists a number $a > 0$ for which either $\psi_G(s) \geq as^2$ for all $s \in \mathbb{R}$ or $\psi_g(s) \geq as^2$ for all $s \in \mathbb{R}$. Let $T > 0$ and assume the measurable functions $F : Q \times (0, T) \to \mathbb{R}$ and $U_0 : Q \to \mathbb{R}$ are given with $C^{\frac{1}{2}} F \in L^2((0, T), L^2(Q))$ and $C^{1/2} U_0 \in L^2(Q)$. Then there exist measurable functions $u : \Omega \times (0, T) \to \mathbb{R}$ and $U : Q \times (0, T) \to \mathbb{R}$ for which*

- *$u(t) \in H_0^1(\Omega)$ and $U(t) \in L^2(\Omega, H^1(S_x))$ for a.e. $t \in (0, T)$,*
- *$C^{1/2} U \in C([0, T], L^2(Q))$ and is locally absolutely continuous,*
- *(2.9) holds at a.e. $t \in (0, T)$ for every $\theta \in H_0^1(\Omega)$, $\Theta \in L^2(\Omega, H^1(S_x))$,*
- *and $\lim_{t \to 0} C^{1/2} U(t) = C^{1/2} U_0$ in $L^2(Q)$.*

Proof. This is a direct application of Theorem 3.1; we only need to show that φ satisfies the coercivity condition of Section 3.3. We write φ in four positive parts as $\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$. Let k_0 be the coercivity constant for the matrix k. That is,

$$k(\vec{\nabla} u) \vec{\nabla} u \geq k_0 |\vec{\nabla} u|^2 \text{ for all } u \in H_0^1(\Omega) .$$

From Lemma 5.1, we have

$$\frac{1}{2} \varphi_1(u) \geq c_1 \|u\|_{H^1(\Omega)}^2 .$$

Assume now that $\psi_G(s) \geq as^2$ for all $s \in \mathbb{R}$. Then

$$\frac{1}{2} \varphi_1(\tilde{u}) + \varphi_3(\tilde{u})$$

$$\geq \int_\Omega \left\{ \frac{k_0}{4} \left|\frac{\partial u}{\partial x_3}\right|^2 + \frac{a}{|\Omega_x|} \int_{S_*} \left(\left(h\frac{\partial u}{\partial x_3} \right)^2 - 2h\frac{\partial u}{\partial x_3} U + U^2 \right) dy \right\} dx$$

$$\geq \int_\Omega \left\{ \frac{k_0}{4} \left|\frac{\partial u}{\partial x_3}\right|^2 + \frac{a}{|\Omega_x|} \int_{S_*} \left(\left(h\frac{\partial u}{\partial x_3} \right)^2 - h\left(\epsilon\frac{\partial u}{\partial x_3} \right)^2 - \frac{h}{\epsilon^2} U^2 + U^2 \right) dy \right\} dx$$

$$= \int_\Omega \left\{ \left(\frac{k_0}{4} + ha - a\epsilon^2 \right) \left|\frac{\partial u}{\partial x_3}\right|^2 + \frac{a}{|\Omega_x|} \int_{S_*} \left(1 - \frac{h}{\epsilon^2} \right) U^2 \, dy \right\} dx$$

If ϵ is chosen so that $\sqrt{h} < \epsilon < \sqrt{h + \frac{k_0}{4a}}$ then this is bounded below by $c_2 \int_\Omega \frac{1}{|\Omega_x|} \int_{S_*} U^2 \, dy \, dx$. Thus for each $\tilde{u} = [u, U]$ in V we have

$$\varphi(\tilde{u}) \geq c_1 \|u\|^2_{H^1(\Omega)} + \int_\Omega \left\{ \frac{1}{|\Omega_x|} \int_{S_*} \left(\frac{K}{2} \|\vec{\nabla}_y U\|^2 + c_2 U^2 \right) dy \right\} dx$$
$$\geq c \|\tilde{u}\|^2_V \ .$$

If instead we assume that $\psi_g(s) \geq as^2$ for all $s \in \mathbb{R}$, then, as above, we have by another appropriate choice of ϵ

$$\frac{1}{2} \varphi_1(\tilde{u}) + \varphi_4(\tilde{u}) \geq c_2 \int_\Omega \frac{1}{|\Omega_x|} (\gamma(U))^2 \, ds \, dx \ .$$

Using Lemma 5.2 and the fact that $|S_x| \leq d$ for all $x \in \Omega$, we obtain

$$\varphi(\tilde{u}) \geq c_1 \|u\|^2_{H^1(\Omega)} + \int_\Omega \left\{ \frac{1}{|\Omega_x|} \int_{S_*} \frac{K}{2} |\vec{\nabla}_y U|^2 \, dy + \frac{c_2}{|S_x|} \int_{\partial S_*} (\gamma(U))^2 \, ds \right\} dx$$
$$\geq c \|\tilde{u}\|^2_V \ . \hspace{3cm} \square$$

Note that the solution is smooth enough for the equation to hold in W_b' whereas only the minimal requirements are asked of the initial function U_0.

In order to insure that the solution is unique, it is sufficient for $\mathcal{B} + \partial\phi : V \to V'$ to be strictly monotone.

Theorem 5.2 (Uniqueness for the Distributed Capacitance Model). *If, in addition to the hypotheses of Theorem 5.1, either $\partial\psi_G$ or $\partial\psi_g$ is strictly monotone, or if $\int_{S_*} C(x, y) \, dy > 0$ for a.e. $x \in \Omega$, then the solution to (5.1) is unique.*

Proof. Assume that $\tilde{u}$ and $\tilde{w}$ are elements of V, and that $\mathcal{B}(\tilde{u} - \tilde{w})(\tilde{u} - \tilde{w}) + \langle \partial\phi\tilde{u} - \partial\phi\tilde{w}, \tilde{u} - \tilde{w} \rangle = 0$. Since this expression is a sum of positive terms, we have

$$\langle \partial\phi_j\tilde{u} - \partial\phi_j\tilde{w}, \tilde{u} - \tilde{w} \rangle = 0 \hspace{2cm} (5.2)$$

for each $j \in \{1, 2, 3, 4\}$. When $j = 1$, this gives $\int_\Omega |\vec{\nabla}(u - w)|^2 \, dx = 0$, which implies that $u = w$ in $H_0^1(\Omega)$. Also the case $j = 2$ implies that $U(x, y) - W(x, y) = (U - W)(x)$, i.e., this difference does not depend on y. If $\partial\psi_G$ is strictly monotone, then (5.2) with $j = 3$ implies that $U = W$ and thus $\tilde{u} = \tilde{w}$ in V. The same follows from $\mathcal{B}(\tilde{u} - \tilde{w})(\tilde{u} - \tilde{w}) = 0$ if instead we assume the above condition on $C(x, y)$. Alternatively, assume that $\partial\psi_g$ is strictly monotone. Then (5.2) with $j = 4$ shows that $\gamma(U) = \gamma(W)$, and hence $U = W$, so in this case also $\tilde{u} = \tilde{w}$ in V. $\square$

Notice that we could have equivalently assumed that either ψ_G or ψ_g is strictly convex, since a convex function ψ is strictly convex if and only if its subgradient $\partial\psi$ is strictly monotone.

Finally, apply the results of Section 4 to the distributed capacitance model. Specifically, we will multiply ψ_g, ψ_G, or K by $\frac{1}{\epsilon}$ and use Theorem 4.1 to characterize each of the three corresponding limiting problems.

For the first case set

$$\varphi^1(\tilde{u}) = \int_\Omega \frac{1}{|S_x|} \int_{\partial S_*} \psi_g \left(h\frac{\partial u}{\partial x_3} - \gamma U \right) ds \, dx \ ,$$

and let $\varphi^0 = \varphi - \varphi^1$. We originally assumed that $\psi_g(0) = 0$; here we will also need to assume that $\psi_g(s) = 0$ only if $s = 0$. (This is guaranteed if $\psi_g(s) \geq as^2$ as in Theorem 5.1.) In this case, (4.1) is the weak form of the distributed capacitance model with ψ_g replaced by $\frac{1}{\epsilon}\psi_g$, and Theorem 4.1 shows that, with appropriate initial conditions, its solution converges to that of (4.2). Using the above assumption on ψ_g, we see that

$$V_0 \equiv \ker(\varphi^1) = \left\{ \tilde{u} \in V : h\frac{\partial u}{\partial x_3} = \gamma U \text{ in } L^2(\Omega, L^2(\partial S_x)) \right\}.$$

From calculations similar to those in Section 2.4, we see that (4.2) is a weak form of the system

$$\frac{\partial}{\partial t}(C(x,y)U) - K\triangle_y U - \partial\psi_G\left(h\frac{\partial u}{\partial x_3} - U\right) \ni F, \qquad y \in S_x$$

$$h\frac{\partial u}{\partial x_3} = \gamma U , \qquad s \in \partial S_x$$

$$-\vec{\nabla} \cdot k(\vec{\nabla}u) - \frac{\partial}{\partial x_3}\left(\frac{1}{|S_x|}\int_{S_*} \partial\psi_G(h\frac{\partial u}{\partial x_3} - U)\,dy\right) \ni 0, \qquad x \in \Omega$$

$$u(x,t) = 0, \qquad x \in \partial\Omega , \ t \in [0,T].$$

That is, (2.2) is replaced by the Dirichlet condition above. This limiting problem is the *matched model* in which the distributed voltage differences on the capacitor boundaries, γU, are in perfect contact with the global voltage gradient, $h\frac{\partial u}{\partial x_3}$.

Next we consider the case where ψ_G is replaced by $\frac{1}{\epsilon}\psi_G$; we set

$$\varphi^1(\tilde{u}) = \int_\Omega \frac{1}{|\Omega_x|} \int_{S_*} \psi_G\left(h\frac{\partial u}{\partial x_3} - U\right) dy\,ds .$$

As above, we assume additionally that $\psi_G(s) = 0$ only if $s = 0$. Then

$$V_0 \equiv \ker(\varphi^1) = \left\{ \tilde{u} \in V : h\frac{\partial u}{\partial x_3} - U = 0 \text{ in } L^2(\Omega, L^2(S_x)) \right\},$$

and we find that (4.2) is a generalized form of

$$-\nabla \cdot \left(k\nabla u + \frac{\vec{e_3}}{|S_x|}\int_{S_*} \frac{\partial}{\partial t}\left(C(x,y)\left(h\frac{\partial u}{\partial x_3}\right)\right) dy\right) = -\frac{\partial}{\partial x_3}\left(\frac{1}{|S_x|}\int_{S_*} F(x,y)\,dy\right).$$

This is a degenerate form of the layered medium equation (1.1), and Theorem 4.1 shows that the solution of (4.1) converges to its solution as $\epsilon \to 0$.

Finally, we consider the case where $K \to \infty$. Define

$$\varphi^1(\tilde{u}) = \int_\Omega \frac{1}{|\Omega_x|} \int_{S_*} \frac{K}{2}|\nabla_y U|^2\,dy\,dx.$$

Then we have

$$V_0 = \{\tilde{u} \in V : U(x,y) = v(x) \text{ for some } v(x) \in L^2(\Omega)\} V_0 \cong H_0^1(\Omega) \times L^2(\Omega).$$

As above, we find that (4.2) is the weak formulation of the system

$$0 = -\vec{\nabla} \cdot k(\vec{\nabla} u) - \frac{\partial}{\partial x_3}\left(\partial\psi_G\left(h\frac{\partial u}{\partial x_3} - v\right) + \left(\int_{\partial S_*} ds\right)\left(h\partial\psi_g\left(h\frac{\partial u}{\partial x_3} - v\right)\right)\right),$$

$$\frac{\partial}{\partial t}\left(\frac{1}{|S_x|}\int_{S_x} C(x,y)\,dy\right)v - \partial\psi_G\left(h\frac{\partial u}{\partial x_e} - v\right)$$

$$-\left(\frac{h}{|S_x|}\int_{\partial S_*} ds\right)\left(\partial\psi_g\left(h\frac{\partial u}{\partial x_3} - v\right)\right) = \frac{1}{|S_x|}\int_{S_*} F\,dy.$$

Again, Theorem 4.1 guarantees that the solution of (4.1) converges to the solution of this system as $\epsilon \to 0$ (i.e., as $K \to \infty$).

In summary, we have described a PDE model of current flow in a medium with continuously distributed capacitance and nonlinear connections between the local capacitors and the global conducting medium. Clearly one could obtain corresponding results for more general situations. For example, one could permit monotone nonlinearities in the conductance in both the capacitors and in the medium and resolve as above the corresponding quasilinear system. Additionally, one could supplement (2.1) and (2.3) with terms representing losses due to leakage of current between capacitor plates or connections, and one could permit the convex functions to be more general, specifically, to include unilateral constraints such as arise in diode nonlinearities. Our techniques apply to such models after some technical work to bring them to the form of the semilinear degenerate evolution (1.2).

Finally, our main result, Theorem 3.1, shows that (1.2) is really *parabolic* when the operator A is a subgradient. The improvement over earlier work is to show that a very strong solution is obtained when one begins the evolution with very general data.

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