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ELLIPTIC-PARABOLIC EQUATIONS WITH HYSTERESIS BOUNDARY CONDITIONS*

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Abstract. A general porous-medium equation is uniquely solved subject to a pair of boundary conditions for the trace of the solution and a second function on the boundary. The use of maximal monotone graphs for the three nonlinearities permits not only the inclusion of the usual boundary conditions of Dirichlet, Neumann, or Robin type, including variational inequality constraints of Signorini type, but also dynamic boundary conditions and those that model hysteresis phenomena. It is shown that the dynamic is determined by a contraction semigroup in a product of L^1 spaces. Several examples and numerical results are described.

Key words. existence, uniqueness, porous-media equation, hysteresis, nonlinear boundary condition, semigroup

AMS subject classifications. 35K55, 35K65

1. Introduction. We shall consider a degenerate-parabolic initial boundary value problem in the form

$$\frac{\partial}{\partial t}a(u) - \Delta u \ni f\ , \qquad x \in \Omega\ , \tag{1.1.a}$$

$$\frac{\partial}{\partial t}b(v) + \frac{\partial u}{\partial \nu} \ni g, \quad \text{and} \tag{1.1.b}$$

$$\frac{\partial u}{\partial \nu} \in c(v-u)\ , \qquad s \in \Gamma \tag{1.1.c}$$

for each $t > 0$ with initial values specified at $t = 0$ for $a(u)$ and $b(v)$. At each $t > 0$, u is a function on the bounded domain Ω in $\mathbb{R}^n$ with smooth boundary Γ, and v is a function on Γ. Each $a(\cdot), b(\cdot), c(\cdot)$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ [7]. Our interest in (1.1) arises primarily from the fact that (1.1.b) together with (1.1.c) can represent *hysteresis* phenomena on the boundary. Specifically, consider the maximal monotone graph given by $\operatorname{sgn}(y) = \{-1\}$ for $y < 0$, $\operatorname{sgn}(0) = [-1, 1]$, and $\operatorname{sgn}(y) = \{1\}$ for $y > 0$. If we choose $c = \operatorname{sgn}^{-1}$, the *inverse* graph obtained by reflection of the coordinates, then (1.1.b) is an ordinary differential equation for $b(v)$ subject to the *constraint* (1.1.c),

$$u - 1 \le v \le u + 1\ .$$

If $g \equiv 0$, then the selection $w \in b(v)$, which realizes the equation (1.1.b), is constant except at the constraint; there the *control* $\frac{\partial u}{\partial \nu}$ forces the corresponding equality. Thus the relationship between u and $w \in b(v)$ is an example of a *generalized play* [14]. Furthermore, if we let $b = \operatorname{sgn}^+ \equiv \frac{1}{2}(1 + \operatorname{sgn})$, then (1.1.b) models a *perfect relay* [14]. Thus the system (1.1) consists of a generalized porous-media equation in the interior of Ω subject to a nonlinear dynamic Neumann constraint, which can contain

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hysteresis phenomena on the boundary. Here, w is the *internal state* of the hysteron, $v-u$ is the *order parameter*, and u is the external *input*. See [19] and [17] for further discussion of these terms and general perspectives on hysteresis.

Although the hysteresis effects obtained from the pair of graphs $b(\cdot)$, $c(\cdot)$ were our primary motivation, we were able to include the third graph $a(\cdot)$ with no essential additional difficulty. This is merely a reflection of the power of the method that was developed in [22]; this method permits the addition of gradient nonlinearities of p-Laplacian type in (1.1.a) as well as corresponding elliptic Laplace–Beltrami operators in (1.1.b) for the manifold Γ. See [18] for a treatment of the degenerate case $a(\cdot)=0$ corresponding to a Stefan problem on the boundary Γ. Adsorption in porous media may be governed by conditions on the surfaces of the solid material that are of hysteresis type. In that case, u is the concentration of a chemical species that is dissolved in the fluid occupying the pores, and w is its concentration on the surfaces once it has been adsorbed. If one assumes that the process is governed by certain thresholds, the adsorption rate shows a hysteresis phenomenon of the kind discussed in this paper. In [11] this idea is applied to homogenization of reactive transport through porous media. Additional papers that deal with problems closely related to those of the present paper are [2], [13], [24], [25], [26], [15], and [16], where parabolic problems with a hysteresis source term are studied.

A rather remarkable variety of boundary conditions is obtained in (1.1). For example, if $b\equiv 0$ we have an explicit *Neumann boundary condition*, and if $c\equiv 0$ it is homogeneous. (Clearly, any general solvability results cannot simultaneously allow $c=b=0$, because this forces $g=0$.) If $b(0)=\mathbb{R}$ (i.e., $b^{-1}=0$), then $v\equiv 0$ and we have a nonlinear Neumann constraint, and if $c(0)=\mathbb{R}$, we get $v=u$ on Γ and this satisfies a nonlinear *dynamic boundary condition* of Neumann type. If $b(0)=c(0)=\mathbb{R}$, we have the homogeneous *Dirichlet boundary condition*. For previous work on some of these various classes, we refer to [3], [4], [5], [6], [8], [20], and [23].

Our objective is to show that the dynamic of problem (1.1) is determined by a nonlinear semigroup of contractions on the Banach space $L^1(\Omega)\times L^1(\Gamma)$. The (negative of the) generator of this contraction semigroup is (the closure of) an operator $\mathbb{C}$ for which the *resolvent equation* $(I+\varepsilon\mathbb{C})([a,b])\ni[f,g]$ with $\varepsilon>0$ takes the form

$$a(u)-\varepsilon\Delta u\ni f\ ,\qquad x\in\Omega\ , \tag{1.2.a}$$

$$b(v)+\varepsilon\frac{\partial u}{\partial\nu}\ni g\ ,\quad\text{and} \tag{1.2.b}$$

$$\frac{\partial u}{\partial\nu}\in c(v-u)\ ,\qquad s\in\Gamma \tag{1.2.c}$$

in the state space $L^1(\Omega)\times L^1(\Gamma)$. In order to motivate the essential estimates that are needed, consider the (much simpler) case of *functions* $a(\cdot)$, $b(\cdot)$, $c(\cdot)$. Multiply the respective equations by appropriate functions φ on Ω and ψ on Γ, and integrate to obtain

$$\int_\Omega\big(a(u)\varphi+\varepsilon\vec{\nabla}u\cdot\vec{\nabla}\varphi\big)\,dx+\int_\Gamma\big(b(v)\psi+\varepsilon c(v-u)(\psi-\varphi)\big)\,ds=\int_\Omega f\varphi\,dx+\int_\Gamma g\psi\,ds\ . \tag{1.3}$$

This leads to the variational formulation of (1.2) and a priori estimates. For example, if we choose $\varphi=\operatorname{sgn}(u)$, $\psi=\operatorname{sgn}(v)$ and can simultaneously obtain $\varphi=\operatorname{sgn}(a(u))$, $\psi=\operatorname{sgn}(b(v))$, then we (formally) obtain the stability estimate

$$\|a(u)\|_{L^1(\Omega)}+\|b(v)\|_{L^1(\Gamma)}\le\|f\|_{L^1(\Omega)}+\|g\|_{L^1(\Gamma)}\ . \tag{1.4}$$

For the special case $a(u) = u$, $b(v) = v$, we could choose $\varphi = u$, $\psi = v$ and obtain corresponding L^2-estimates. For this special case we shall show that the corresponding evolution is *parabolic* in $L^2(\Omega) \times L^2(\Gamma)$; the same holds for its *additive perturbation* (see (5.1)). For the general case, estimate (1.4) suggests that the *resolvent* $[f, g] \mapsto [u, v] \to [a(u), b(v)]$ is a contraction. Of course we must obtain such estimates on *differences* of solutions.

Our plan is the following: In §2 we formulate the boundary value problem (1.2) as a variational problem in Sobolev space and give sufficient conditions for which it is well posed. In §3 we show that (1.1) is governed by a contraction semigroup on $L^1(\Omega) \times L^1(\Gamma)$ by constructing the operator $\mathbb{C}$, as suggested by our formal calculation above. Section 4 consists of some numerical examples which illustrate the hysteresis phenomena. Additional examples appear in [12]. Finally, we note in §5 that a corresponding *additive* perturbation of independent interest corresponds to a subgradient in Hilbert space from which one obtains parabolic regularizing effects.

2. The resolvent problem. Our objective is to make the boundary value problem (1.2) precise and give sufficient conditions for it to be well posed. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$. Denote by $L^p(\Omega)$ the usual space of Lebesgue pth-power integrable (equivalence classes of) functions on Ω when $1 \le p < \infty$, and denote by $L^\infty(\Omega)$ the essentially bounded measurable functions. Let $C_0^\infty(\Omega)$ be the infinitely differentiable functions with compact support in Ω, let $H^m(\Omega)$ be the Hilbert space of functions in $L^2(\Omega)$ for which each partial derivative up to order m belongs to $L^2(\Omega)$, and denote by $H_0^m(\Omega)$ the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. See [1] for information on these *Sobolev spaces*. Specifically, the *trace* map γ which assigns boundary values is well defined, continuous, and linear from $H^1(\Omega)$ into $L^2(\Gamma)$ with dense range $B = H^{1/2}(\Gamma)$.

We consider the Laplacian as an elliptic differential operator in divergence form from $H_0^1(\Omega)$ to its *dual* $H^{-1}(\Omega)$. Thus, assume we are given $a_{ij} \in L^\infty(\Omega)$, $1 \le i, j \le n$, which are uniformly positive definite; there is a $c_0 > 0$ for which

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge c_0|\xi|^2 , \qquad \xi \in \mathbb{R}^n, \tag{2.1}$$

where $|\xi|^2 = \sum_{j=1}^n |\xi_j|^2$. Then $\mathcal{A} : H^1(\Omega) \to H^1(\Omega)'$ is defined by

$$\mathcal{A}u(\varphi) = \int_\Omega \Big(\sum_{i,j=1}^{n} a_{ij}\partial_i u \partial_j \varphi \Big)\, dx , \qquad u, \varphi \in H^1(\Omega) .$$

The *formal part* of $\mathcal{A}$ is its restriction to $C_0^\infty(\Omega)$, the distribution

$$Au \equiv \mathcal{A}u|_{C_0^\infty(\Omega)} = -\sum_{i,j=1}^{n} \partial_j(a_{ij}\partial_i u) \in H^{-1}(\Omega) . \tag{2.2}$$

The monotone graphs in system (1.1) will be given as subgradients of convex functions [10]. Thus assume each $\zeta_a, \zeta_b, \zeta_c$ is a convex lower-semicontinuous function from $\mathbb{R}$ into the nonnegative extended reals $\mathbb{R}_\infty^+ = [0, +\infty]$, $\zeta_a(0) = \zeta_b(0) = \zeta_c(0) = 0$. Throughout most of the following we shall assume that ζ_c is quadratically bounded:

$$\zeta_c(s) \le C(|s|^2 + 1) , \qquad s \in \mathbb{R}; \tag{2.3.c}$$

hence ζ_c is continuous on *all* of $\mathbb{R}$. By defining

$$Z_a(u) \equiv \int_\Omega \zeta_a\big(u(x)\big)\,dx\ , \qquad u \in L^2(\Omega)\ , \tag{2.4.a}$$

$$Z_b(v) \equiv \int_\Gamma \zeta_b\big(v(s)\big)\,ds\ , \qquad v \in L^2(\Gamma)\ , \tag{2.4.b}$$

we obtain a pair of proper, convex, lower-semicontinuous functions, $Z_a : L^2(\Omega) \to \mathbb{R}^+_\infty$ and $Z_b : L^2(\Gamma) \to \mathbb{R}^+_\infty$. (By "proper" we mean that a function has a finite value somewhere.) Also, we define such a function Z_c on the product space $H^1(\Omega) \times L^2(\Gamma)$ by

$$Z_c\big([u,v]\big) \equiv \int_\Gamma \zeta_c\big(v(s) - \gamma u(s)\big)\,ds\ , \qquad u \in H^1(\Omega)\ ,\ v \in L^2(\Gamma)\ , \tag{2.4.c}$$

and Z_c is convex and continuous on $H^1(\Omega)\times L^2(\Gamma)$. The subgradients of these functions are easily computed by standard results [10]. Thus, we have $a \in \partial Z_a(u)$ in $L^2(\Omega)$ if and only if

$$a(x) \in \partial\zeta_a\big(u(x)\big) \qquad \text{a.e.}\ \ x \in \Omega\ , \tag{2.5.a}$$

and similarly we have $b \in \partial Z_b(v)$ in $L^2(\Gamma)$ exactly when

$$b(s) \in \partial\zeta_b\big(v(s)\big) \qquad \text{a.e.}\ \ s \in \Gamma. \tag{2.5.b}$$

Since imbedding $H^1(\Omega)$ into $L^2(\Omega)$ is continuous and dense and we identify $L^2(\Omega)$ with its dual, we have $L^2(\Omega) \subset H^1(\Omega)'$. Thus, $a \in \partial Z_a(u)$ in $L^2(\Omega)$ implies that the same holds in $H^1(\Omega)'$, but $a \in \partial Z_a(u)$ in $H^1(\Omega)'$ does *not necessarily* imply (2.5.a). We shall call a subgradient in $H^1(\Omega)'$ a *weak subgradient* and one in $L^2(\Omega)$ a *strong subgradient*. Finally, since $Z_c : H^1(\Omega) \times L^2(\Gamma) \to \mathbb{R}$ is a composition of continuous functions, we have from the *chain rule* [10] that its weak subgradient is characterized by $C \in \partial Z_c[u,v]$ in $H^1(\Omega)' \times L^2(\Gamma)$ if and only if $C = [-\gamma' c, c]$ with

$$c(s) \in \partial\zeta_c\big(v(s) - \gamma u(s)\big) \qquad \text{a.e.}\ \ s \in \Gamma\ . \tag{2.5.c}$$

The dual map γ' of $L^2(\Gamma)$ into $H^1(\Omega)'$ is given by

$$\gamma' g(\psi) = \int_\Gamma g \cdot \gamma\psi\,ds\ , \qquad g \in L^2(\Gamma)\ ,\ \psi \in H^1(\Omega)\ .$$

The boundary value problem (1.2) can now be realized as a subgradient equation. To this end, set

$$Z[u,v] \equiv Z_a(u) + Z_b(v) + \tfrac{1}{2}\mathcal{A}u(u) + Z_c[u,v]\ , \qquad u \in H^1(\Omega)\ ,\ v \in L^2(\Gamma). \tag{2.6}$$

Clearly, there is no loss of generality in taking $\varepsilon = 1$, so we shall do so for the remainder of this section. Then, Z is the sum of convex and lower-semicontinuous functions, Z is proper, the first two terms are independent, and the remaining two are continuous and defined everywhere. Thus, we can compute the weak subgradient term by term. From this it follows that

$$\partial Z\big([u,v]\big) \ni [f,g] \ \text{ in }\ H^1(\Omega)' \times L^2(\Gamma), \tag{2.7}$$

whenever we have $u \in H^1(\Omega)$, $v \in L^2(\Gamma)$, and there exists $a \in L^2(\Omega)$, b, and $c \in L^2(\Gamma)$ satisfying (2.5) and

$$a + \mathcal{A}u - \gamma' c = f \quad \text{in } H^1(\Omega)', \tag{2.8.a}$$

$$b + c = g \quad \text{in } L^2(\Gamma). \tag{2.8.b}$$

That is, the weak subgradient (2.7) follows from (2.5) and (2.8). Moreover, (2.7) is equivalent to (2.5) and (2.8) if the first two terms are both *strong* subgradients. This will always be the case (by the chain rule) when we assume bounds of the form

$$\zeta_a(s) \le C(|s|^2 + 1) \ , \tag{2.3.a}$$

$$\zeta_b(s) \le C(|s|^2 + 1) \ , \qquad s \in \mathbb{R} \ . \tag{2.3.b}$$

In order to show that (2.8.a) is equivalent to a partial differential equation in Ω and a boundary condition on Γ, we develop an appropriate *Green formula* for the operator $\mathcal{A}$ [21]. Use the formal part (2.2) to define the domain

$$D \equiv \{u \in H^1(\Omega) : Au \in L^2(\Omega)\} \ .$$

Note that if Γ and the coefficients in A are smooth, then $D = H^2(\Omega)$. Recall that we denote the range of the trace γ by B and that B is dense and continuously imbedded in $L^2(\Gamma)$. Thus we obtain the identification $L^2(\Gamma) \subset B'$.

LEMMA 1. *There is a unique linear operator $\partial_A : D \to B'$ such that $\mathcal{A}u = Au + \gamma'\partial_A u$ for $u \in D$. That is, we have for each $u \in D$,*

$$\mathcal{A}u(\varphi) = (Au, \varphi)_{L^2(\Omega)} + \partial_A u(\gamma\varphi) \ , \qquad \varphi \in H^1(\Omega) \ . \tag{2.9}$$

Proof. Since γ is a strict homomorphism of $H^1(\Omega)$ onto B, its dual γ' is an isomorphism of B' onto the annihilator $H_0^1(\Omega)^\perp$ in $H^1(\Omega)'$ of $H_0^1(\Omega)$, the kernel of γ. Thus, for each $u \in D$, the difference $\mathcal{A}u - Au$ belongs to $H_0^1(\Omega)^\perp$, so it equals $\gamma'(\partial_A u)$ for a unique $\partial_A u \in B'$. $\square$

The identity (2.9) is a generalization of the classical Green theorem. If Γ is sufficiently smooth and ν denotes the unit outward normal on Γ, and if $u \in H^2(\Omega)$ and $a_{ij} \in C^1(\overline{\Omega})$, $1 \le i, j \le n$, then

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j \varphi \, dx = \int_\Omega Au\varphi \, dx + \int_\Gamma \frac{\partial u}{\partial \nu}\gamma\varphi \, ds \ , \qquad \varphi \in H^1(\Omega) \ ,$$

where Au is given by (2.2) and the normal derivative is given by

$$\frac{\partial u}{\partial \nu} = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}\partial_i u \right) \nu_j \in L^2(\Gamma) \ .$$

We can thus regard ∂_A as an extension of $\frac{\partial u}{\partial v}$ to a (possibly) wider class of functions in D.

Consider (2.8.a) and assume $f \in L^2(\Omega)$. Applying it to $C_0^\infty(\Omega)$ shows that

$$a + Au = f \ \text{ in } \ L^2(\Omega) \ . \tag{2.10.a}$$

Since from (2.10.a) it follows that $u \in D$, we may use (2.8.a) and (2.9) to get

$$\partial_A u = c \ \text{ in } \ L^2(\Gamma) \ . \tag{2.10.c}$$

Then (2.8.b) is equivalent to

$$b + \partial_A u = g \quad \text{in} \quad L^2(\Gamma)\ . \tag{2.10.b}$$

This shows that (2.8) is equivalent to (2.10), and we have shown that the strong subgradient identity (2.7) is satisfied by a solution of the resolvent problem (1.2), namely, (2.5) and (2.10).

The following result gives sufficient conditions for the resolvent problem to be solvable and equivalent to (2.7) in $L^2(\Omega) \times L^2(\Gamma)$.

THEOREM 1. *Let the domain Ω with boundary $\Gamma = \partial\Omega$, the coefficients $\{a_{ij}\}$ satisfying* (2.1), *and the convex lower-semicontinuous functions $\zeta_a, \zeta_b, \zeta_c$ from $\mathbb{R}$ into $\mathbb{R}^+_\infty$ with $\zeta_a(0) = \zeta_b(0) = \zeta_c(0) = 0$ be given. Assume* (2.3.a)–(2.3.c) *and that for some $c_1 > 0$,* any two *of the following hold:*

$$\zeta_a(s) \geq c_1|s|^\alpha - C \quad \textit{with} \quad 1 < \alpha \leq 2\ , \tag{2.11.a}$$

$$\zeta_b(s) \geq c_1|s|^2 - C\ , \qquad s \in \mathbb{R}\ , \tag{2.11.b}$$

$$\zeta_c(s) \geq c_1|s|^2 - C\ , \qquad s \in \mathbb{R}\ . \tag{2.11.c}$$

Then, for the proper, convex, and lower-semicontinuous $Z : H^1(\Omega) \times L^2(\Gamma) \to \mathbb{R}^+_\infty$ given by (2.4) *and* (2.6), *it follows that the subgradient ∂Z is surjective onto $H^1(\Omega)' \times L^2(\Gamma)$. Thus, for each triple $f \in L^2(\Omega)$, $g, h \in L^2(\Gamma)$, there exists a solution pair $u \in H^1(\Omega)$, $v \in L^2(\Gamma)$ and corresponding selections $a \in L^2(\Omega)$, $b, c \in L^2(\Gamma)$ satisfying* (2.5) *and*

$$a + Au = f \quad \textit{in} \quad L^2(\Omega), \tag{2.12.a}$$

$$b + c = g \quad \textit{in} \quad L^2(\Gamma), \tag{2.12.b}$$

$$\partial_A u - c = h \quad \textit{in} \quad L^2(\Gamma). \tag{2.12.c}$$

Proof. From Green's identity it follows that (2.12) is equivalent to

$$\begin{aligned} &a + \mathcal{A}u - \gamma' c = f + \gamma' h && \text{in} \quad H^1(\Omega)', \\ &b + c = g && \text{in} \quad L^2(\Gamma), \end{aligned}$$

and this, in turn, is equivalent to

$$\partial Z\big([u, v]\big) \ni [f + \gamma' h, g] \quad \text{in} \quad H^1(\Omega)' \times L^2(\Gamma).$$

These equivalences follow by the same calculations relating (2.7), (2.8), and (2.10). Thus it suffices to show that Z is *coercive* on $H^1(\Omega) \times L^2(\Gamma)$, i.e.,

$$\frac{Z([u, v])}{\|u\|_{H^1(\Omega)} + \|v\|_{L^2(\Gamma)}} \to \infty \quad \text{as} \quad \|u\|_{H^1(\Omega)} + \|v\|_{L^2(\Gamma)} \to \infty. \tag{2.13}$$

We shall verify (2.13). If the fraction in (2.13) is bounded, then we obtain for some constant K,

$$\begin{aligned} \int_\Omega \Big(\zeta_a(u) + \frac{c_0}{2}|\nabla u|^2\Big)\, dx + \int_\Gamma \big(\zeta_b(v) + \zeta_c(v - \gamma u)\big)\, ds \\ \leq K\big\{\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Gamma)}\big\}\ . \end{aligned} \tag{2.14}$$

LEMMA 2. *There is a constant K_1 such that*

$$\|u\|_{L^2(\Omega)} \leq K_1\big(\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^1(\Omega)}\big)\ , \qquad u \in H^1(\Omega)\ . \tag{2.15}$$

Proof. Otherwise there is a sequence $\{u_n\}$ in $H^1(\Omega)$ for which $\|u_n\|_{L^2} = 1$, and the right side of (2.15) converges to zero. Then, $\{u_n\}$ is bounded in $H^1(\Omega)$, so (by passing to a subsequence) we have $u_n \rightharpoonup u$ in $H^1(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$ by compactness. But then $u_n \rightharpoonup u$ in the weaker norm on the right of (2.15), so by uniqueness of weak limits, we have $u = 0$. Thus $u_n \to 0$ in $L^2(\Omega)$, a contradiction. □

Suppose we have the case of (2.11.a) and (2.11.b). Using Lemma 2, we replace $\|u\|_{L^2(\Omega)}$ by $\|u\|_{L^1(\Omega)}$ in (2.14), and then we have

$$c_1\|u\|^{\alpha}_{L^1(\Omega)} + \frac{c_0}{2}\|\nabla u\|^2_{L^2(\Omega)} + c_1\|v\|^2_{L^2(\Gamma)} \le K_2\{\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^1(\Omega)} + \|v\|_{L^2(\Gamma)} + 1\} .$$

From here it follows that $\|u\|_{H^1(\Omega)} + \|v\|_{L^2(\Gamma)}$ is bounded, so we have (2.13).

Suppose we have the case of (2.11.a) and (2.11.c). As before, we obtain

$$\begin{aligned} c_1\|u\|^{\alpha}_{L^1(\Omega)} &+ \frac{c_0}{2}\|\nabla u\|^2_{L^2(\Omega)} + c_1\|v - \gamma u\|^2_{L^2(\Gamma)} \\ &\le K_2\{\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^1(\Omega)} + \|v - \gamma u\|_{L^2(\Gamma)} + \|\gamma u\|_{L^2(\Gamma)} + 1\} . \end{aligned}$$

Since γ is continuous from $H^1(\Omega)$ into $L^2(\Gamma)$, the term $\|\gamma u\|_{L^2(\Gamma)}$ can be absorbed in the first two terms by adjusting K_2, and then we are done.

Now consider the remaining case of (2.11.b) and (2.11.c); then, from (2.14) we have

$$\begin{aligned} \frac{c_0}{2}\|\nabla u\|^2_{L^2(\Omega)} &+ c_1\|v\|^2_{L^2(\Gamma)} + c_1\|v - \gamma u\|^2_{L^2(\Gamma)} \\ &\le K_2\{\|\nabla u\|_{L^2(\Omega)} + \|\gamma u\|_{L^2(\Gamma)} + \|v\|_{L^2(\Gamma)} + 1\} , \end{aligned} \tag{2.16}$$

in which we have used either a Poincaré inequality or the argument of Lemma 2 to replace $\|u\|_{L^2(\Omega)}$ by $\|\gamma u\|_{L^2(\Gamma)}$. Using the inequality $2\alpha\beta \le \varepsilon\alpha^2 + \frac{1}{\varepsilon}\beta^2$ with $\alpha = \|v\|_{L^2(\Gamma)}$, $\beta = \|\gamma u\|_{L^2(\Gamma)}$, and $1 < \varepsilon < 2$, we obtain

$$\begin{aligned} (2-\varepsilon)\|v\|^2_{L^2(\Gamma)} + \left(1 - \frac{1}{\varepsilon}\right)\|\gamma u\|^2_{L^2(\Gamma)} &\le 2\|v\|^2_{L^2(\Gamma)} - 2(v, \gamma u)_{L^2(\Gamma)} + \|\gamma u\|^2_{L^2(\Gamma)} \\ &= \|v\|^2_{L^2(\Gamma)} + \|v - \gamma u\|^2_{L^2(\Gamma)} . \end{aligned}$$

Thus, we can replace $\|v - \gamma u\|^2_{L^2(\Gamma)}$ with $\|\gamma u\|^2_{L^2(\Gamma)}$ in (2.16) by adjusting c_1, so we obtain the coercivity condition (2.13) as before. □

3. The evolution problem. The goal in this section is to construct the generator of the nonlinear semigroup which corresponds to system (1.1). We shall assume the domain Ω in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$, the coefficients $\{a_{ij}\}$ in $L^\infty(\Omega)$, and the function $\zeta_c : \mathbb{R} \to \mathbb{R}^+_\infty$ are given as in §2.

Define a single-valued operator $\mathbb{C}_2$ on the Hilbert space $L^2(\Omega) \times L^2(\Gamma)$ as follows: $\mathbb{C}_2([u, v]) = [f, g]$ if and only if $[f, g] \in L^2(\Omega) \times L^2(\Gamma)$, and

$$u \in H^1(\Omega),\ \mathcal{A}u = f + \gamma' g \ \text{ in } \ H^1(\Omega)' , \ \text{ and} \tag{3.1.a}$$

$$v \in L^2(\Gamma),\ \partial\zeta_c(v - \gamma u) \ni g \ \text{ in } \ L^2(\Gamma) . \tag{3.1.b}$$

This is just (2.5) and (2.12) with $\zeta_a = \zeta_b = 0$ and $h = 0$, and it can be written as

$$\mathcal{A}u(\varphi) + \int_\Gamma c(\psi - \gamma\varphi)\, ds = \int_\Omega f\varphi\, dx + \int_\Gamma g\psi , \qquad \varphi \in H^1(\Omega) ,\ \psi \in L^2(\Gamma) ,$$

$$c \in \partial\zeta_c(v - \gamma u) \ \text{ in } \ L^2(\Gamma) .$$

According to Lemma 1, its value is given explicitly by $\mathbb{C}_2([u,v]) = [Au, \partial_A u]$. We shall first show that $\mathbb{C}_2$ is m-accretive and also a *subgradient* in $L^2(\Omega) \times L^2(\Gamma)$; this implies that the special case of the system (1.1) with $a = b =$ identity is well posed and *parabolic* (see §5). Then we shall show that the closure of the operator $\mathbb{C}$ which corresponds to system (1.2) with nonlinear $a(\cdot)$, $b(\cdot)$ is m-accretive in $L^1(\Omega) \times L^1(\Gamma)$.

PROPOSITION 1. *The function $Z_2 : L^2(\Omega) \times L^2(\Gamma) \to \mathbb{R}^+_\infty$ given by* (2.4.c) *and*

$$Z_2([u,v]) \equiv \frac{1}{2}\mathcal{A}u(u) + Z_c([u,v])$$

is proper convex and lower-semicontinuous. The (*strong*) *subgradient is given by $\partial Z_2 = \mathbb{C}_2$.*

Proof. The function Z_2 is clearly proper and convex. To see that it is lower-semicontinuous, note that if $[u_n, v_n] \to [u,v]$ in $L^2(\Omega) \times L^2(\Gamma)$ and $\{Z_2([u_n,v_n])\}$ is bounded, then $\{[u_n,v_n]\}$ is bounded in $H^1(\Omega) \times L^2(\Gamma)$, so for some subsequence, $\gamma u_n \to \gamma u$ (strongly) in $L^2(\Gamma)$ and Fatou's lemma yields the desired result. To compute the subgradient, use the termwise weak subdifferentiability to see that if $[f,g] \in \partial Z_2([u,v])$, then there exists a $c \in L^2(\Gamma)$ with (2.5.c) and

$$f(\varphi-u)+g(\psi-v) \le \mathcal{A}u(\varphi-u)+\int_\Gamma c\big(\psi-v-\gamma(\varphi-u)\big)\,ds\,, \qquad \varphi \in H^1(\Omega)\,,\ \psi \in L^2(\Gamma)\,;$$

this is easily seen to be equivalent to (3.1).

We develop additional estimates on $\mathbb{C}_2$ and begin with the following lemma.

LEMMA 3. *If $\sigma : \mathbb{R} \to \mathbb{R}$ is monotone and Lipschitz, and $\sigma(0) = 0$, then for each pair*

$$\mathbb{C}_2([u_j,v_j]) = [f_j, g_j]\,, \qquad j = 1,2,$$

we have

$$\big(f_1 - f_2,\, \sigma(u_1-u_2)\big)_{L^2(\Omega)} + \big(g_1 - g_2,\, \sigma(v_1-v_2)\big)_{L^2(\Gamma)} \ge 0\,.$$

Proof. We use (3.1.a) to compute the above two terms. The composite $\sigma(u_1-u_2)$ belongs to $H^1(\Omega)$ and by the chain rule we obtain

$$\mathcal{A}(u_1-u_2)\big(\sigma(u_1-u_2)\big) = \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_i(u_1-u_2)\partial_j(u_1-u_2)\sigma'(u_1-u_2)\,dx\,,$$

and this is nonnegative in view of (2.1) and the monotonicity of σ. Also, we have to check the remaining term

$$\int_\Gamma (c_1-c_2)\big(\sigma(v_1-v_2) - \sigma(\gamma u_1 - \gamma u_2)\big)\,ds,$$

but this is nonnegative because of (3.1.b) since $\partial\zeta_c$ is a monotone graph and σ is a monotone function. □

The special case of $\sigma(s) = s$ is just the observation that $\mathbb{C}_2$ is monotone in the Hilbert space $L^2(\Omega) \times L^2(\Gamma)$. Since $\mathbb{C}_2$ is single valued, we can permit σ to be *multivalued*.

PROPOSITION 2. *Let the domain Ω with boundary Γ, the coefficients $\{a_{ij}\}$ in $L^\infty(\Omega)$ satisfying* (2.1), *and the convex continuous function $\zeta_c : \mathbb{R} \to \mathbb{R}^+_\infty$ with $\zeta_c(0) = 0$ and* (2.3.c) *be given. Let $j : \mathbb{R} \to \mathbb{R}^+_\infty$ be convex and lower-semicontinuous, and let $j(0) = 0$. Then we have*

$$\big(\mathbb{C}_2[u_1,v_1] - \mathbb{C}_2[u_2,v_2], [\sigma_1,\sigma_2]\big)_{L^2(\Omega)\times L^2(\Gamma)} \ge 0 \tag{3.2}$$

for any *selections* $\sigma_1 \in \partial j(u_1 - u_2)$ *in* $L^2(\Omega)$ *and* $\sigma_2 \in \partial j(v_1 - v_2)$ *in* $L^2(\Gamma)$.

Proof. Consider the lower-semicontinuous convex function J on $L^2(\Omega) \times L^2(\Gamma)$ given by

$$J([u,v]) = \int_\Omega j(u(x))\,dx + \int_\Gamma j(v(s))\,ds\ , \qquad u \in L^2(\Omega)\ ,\ v \in L^2(\Gamma)\ . \tag{3.3}$$

The subgradient of J is given by

$$\sigma \equiv [\sigma_1, \sigma_2] \in \partial J([u,v]) \ \text{ in } \ L^2(\Omega) \times L^2(\Gamma)$$

if and only if

$$\sigma([\varphi,\psi]) = \int_\Omega \sigma_1(x)\varphi(x)\,dx + \int_\Gamma \sigma_2(s)\psi(s)\,ds\ , \qquad \varphi \in L^2(\Omega)\ ,\ \psi \in L^2(\Gamma)\ ,$$

with $\sigma_1(x) \in \partial j(u(x))$ a.e. $x \in \Omega$, $\sigma_2(s) \in \partial j(v(s))$ a.e. $s \in \Gamma$. The Yoshida approximation J_ε of J is given by the same formula but with j replaced by j_ε. The derivative j'_ε is Lipschitz and monotone so Lemma 3 yields (3.2) in this special case. Thus, $\mathbb{C}_2$ is ∂J-monotone by Proposition 4.7 of [7] and the general case follows since the single-valued $\mathbb{C}_2$ is equal to its minimal section.

Remark. As a consequence of Proposition 2.17 of [7], we also obtain the following corollary.

COROLLARY 1. *Let j be given as above. Then $\partial(J + Z_2) = \partial J + \partial Z_2$.*

It follows that the special case of the boundary value problem (1.2) with $a = b = \partial j$ is well posed in $L^2(\Omega) \times L^2(\Gamma)$ when j satisfies an estimate of the form (2.11), because $J + Z_2$ is then coercive over $L^2(\Omega) \times L^2(\Gamma)$.

Next we construct the generator of the general system (1.1). This operator will be obtained by closing up the composition of $\mathbb{C}_2$ with the inverse of $[\partial\zeta_a, \partial\zeta_b]$ in $L^1(\Omega) \times L^1(\Gamma)$. As before, we shall always assume that (2.1) holds, $\zeta_a, \zeta_b, \zeta_c : \mathbb{R} \to \mathbb{R}^+_\infty$ are convex and lower-semicontinuous, and (2.3.c) holds.

DEFINITION. *The operator $\mathbb{C}$ in $L^2(\Omega) \times L^2(\Gamma)$ is defined as follows: $\mathbb{C}([a,b]) \ni [f,g]$ if there is a pair $[u,v]$ as in* (3.1) *and a pair $a \in L^2(\Omega)$, $b \in L^2(\Gamma)$ for which $\mathbb{C}_2([u,v]) = [f,g]$ and $a \in \partial\zeta_a(u)$ in $L^2(\Omega)$, $b \in \partial\zeta_b(v)$ in $L^2(\Gamma)$.*

Note that $Rg(I+\varepsilon\mathbb{C}) = L^2(\Omega)\times L^2(\Gamma)$ for $\varepsilon > 0$ in both the situation of Theorem 1 (i.e., (2.3.a) and (2.3.b)) and in the case of Corollary 1 with (2.11) and $\zeta_a = \zeta_b$.

LEMMA 4. *The operator $\mathbb{C}$ is accretive on $L^1(\Omega) \times L^1(\Gamma)$.*

Proof. Let $\varepsilon > 0$ and $(I + \varepsilon\mathbb{C})([a_j, b_j]) \ni [f_j, g_j]$ for $j = 1, 2$. Thus we have

$$\varepsilon\mathbb{C}_2([u_j, v_j]) = [f_j - a_j,\ g_j - b_j]\ , \qquad a_j \in \partial\zeta_a(u_j)\ ,\ b_j \in \partial\zeta_b(v_j)$$

as above. We choose $j(s) = |s|$ so that $\partial j = \operatorname{sgn}$; then we use (3.3) with

$$\sigma_1 = \operatorname{sgn}_0(u_1 - u_2 + a_1 - a_2) \in \operatorname{sgn}(u_1 - u_2) \cap \operatorname{sgn}(a_1 - a_2),$$
$$\sigma_2 = \operatorname{sgn}_0(v_1 - v_2 + b_1 - b_2) \in \operatorname{sgn}(v_1 - v_2) \cap \operatorname{sgn}(b_1 - b_2)$$

to obtain

$$\|a_1 - a_2\|_{L^1(\Omega)} + \|b_1 - b_2\|_{L^1(\Gamma)} \le \|f_1 - f_2\|_{L^1(\Omega)} + \|g_1 - g_2\|_{L^1(\Gamma)}\ . \tag{3.4}$$

Of course the same procedure with the function $j(s) = s^+$ and its subgradient $\partial j = \operatorname{sgn}^+$ yields the *comparison estimate*

$$\|(a_1 - a_2)^+\|_{L^1(\Omega)} + \|(b_1 - b_2)^+\|_{L^1(\Gamma)} \le \|(f_1 - f_2)^+\|_{L^1(\Omega)} + \|(g_1 - g_2)^+\|_{L^1(\Gamma)}\ . \tag{3.5}$$

This leads to the following L^∞ estimates.

COROLLARY 2. *If* $(I+\varepsilon\mathbb{C})([a,b]) \ni [f,g]$ *and* $\|f^+\|_{L^\infty(\Omega)}+\|g^+\|_{L^\infty(\Gamma)} \in Rg(\partial\zeta_a+\partial\zeta_b)$, *then*

$$\|a^+\|_{L^\infty(\Omega)} \le \|f^+\|_{L^\infty(\Omega)}, \quad \|b^+\|_{L^\infty(\Gamma)} \le \|g^+\|_{L^\infty(\Gamma)}. \tag{3.6}$$

Proof. Set $a_2 = \|f^+\|_{L^\infty(\Omega)}$, $b_2 = \|g^+\|_{L^\infty(\Gamma)}$, and choose k such that $\partial\zeta_a(k) \ni a_2$ and $\partial\zeta_b(k) \ni b_2$. With $u(x) = k$, $v(s) = k$ in the definition of $\mathbb{C}$, we have $(I + \varepsilon\mathbb{C})([a_2,b_2]) \ni [a_2,b_2]$, so we can apply (3.5) to get $\|(a-a_2)^+\|_{L^1(\Omega)}+\|(b-b_2)^+\|_{L^1(\Gamma)} = 0$. □

The same result holds for the "negative parts," and by adding the corresponding estimates, we obtain estimate (3.5) with the "positive part" deleted throughout.

LEMMA 5. *Assume that any two parts of* (2.11) *hold. Then for any* $\varepsilon > 0$ *and* $[f,g] \in L^\infty(\Omega) \times L^\infty(\Gamma)$ *with* $\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Gamma)} \in Rg(\partial\zeta_\alpha + \partial\zeta_b)$, *there exists a unique* $[a,b]$ *such that* $(I + \varepsilon\mathbb{C})([a,b]) \ni [f,g]$ *and*

$$\|a\|_{L^\infty(\Omega)} \le \|f\|_{L^\infty(\Omega)}, \quad \|b\|_{L^\infty(\Gamma)} \le \|g\|_{L^\infty(\Gamma)}. \tag{3.7}$$

Proof. Modify ζ_a to replace $\partial\zeta_a$ by its *truncation*

$$\partial\zeta_a^m(s) = \begin{cases} \{\min\{r,m\} : r \in \partial\zeta_a(s)\} & \text{if } s \ge 0, \\ \{\max\{r,-m\} : r \in \partial\zeta_a(s)\} & \text{if } s < 0, \end{cases}$$

where $m = \max\{\|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Gamma)}\}$. Thus $\partial\zeta_a^m$ has bounded range, so ζ_a^m satisfies (2.3.a). Likewise, modify ζ_b to obtain ζ_b^m satisfying (2.3.b). By Theorem 1, there is a unique solution $[a,b] \in L^2(\Omega) \times L^2(\Gamma)$ of $(I + \varepsilon\mathbb{C})([a,b]) \ni [f,g]$ with the modified functions ζ_a^m, ζ_b^m. This solution satisfies (3.7), so (2.5) holds since the modified functions agree with the original ones for these values of a and b. □

We summarize the above construction in the following.

THEOREM 2. *Assume we are given the domain* Ω *with boundary* Γ *as above, the coefficients* $\{a_{ij}\}$ *in* $L^\infty(\Omega)$ *satisfying* (2.1), *and the three convex, lower-semi-continuous functions* $\zeta_a, \zeta_b, \zeta_c$ *from* $\mathbb{R}$ *into* $\mathbb{R}^+_\infty$ *satisfying* $\zeta_a(0) = \zeta_b(0) = \zeta_c(0) = 0$ *and any two of* (2.11).

(a) *If either* (2.3.a)–(2.3.c) *holds or* $\zeta_a = \zeta_b$ *and* (2.3.c) *holds, then* $Rg(I + \varepsilon\mathbb{C}) = L^2(\Omega) \times L^2(\Gamma)$.

(b) *If* $Rg(\partial\zeta_a + \partial\zeta_b) = \mathbb{R}$, *then* $Rg(I + \varepsilon\mathbb{C}) \supset L^\infty(\Omega) \times L^\infty(\Gamma)$.

In both of these cases, the closure $\overline{\mathbb{C}}$ *of* $\mathbb{C}$ *in* $L^1(\Omega) \times L^1(\Gamma)$ *is m-accretive.*

Proof. Part (a) is implicit in Theorem 1 and Corollary 1. For part (b), we apply Lemma 5 and note that we have that $\|\partial_A u\|_{L^\infty(\Gamma)} \le \frac{2}{\varepsilon}\|g\|_{L^\infty(\Gamma)}$ from (3.7). Thus we may replace $\partial\zeta_c$ by its truncation $\partial\zeta_c^{2m/\varepsilon}$, and the corresponding convex $\zeta_c^{2m/\varepsilon}$ satisfies (2.3.c). □

Since $\overline{\mathbb{C}}$ is m-accretive, it follows from the Crandall–Liggett theorem [9] that the abstract Cauchy problem

$$\begin{aligned} &\tilde{a}'(t) + \overline{\mathbb{C}}(\tilde{a}(t)) \ni \tilde{f}(t)\,, \qquad 0 \le t \le T\,, \\ &\tilde{a}(0) = \tilde{a}_0 \end{aligned}$$

has an *integral solution* $\tilde{a}(t) = [a(t), b(t)]$ in $C([0,T], L^1(\Omega) \times L^1(\Gamma))$ which is unique; see also [3]. This solution can be obtained as the uniform limit of step functions obtained from the implicit difference scheme

$$[a^n, b^n] - [a^{n-1}, b^{n-1}] + h\,\overline{\mathbb{C}}([a^n, b^n]) \ni h[f^n, g^n]\,, \qquad 1 \le n \le N,$$

with step $h = T/N$ and $[a^0, b^0] = \tilde{a}_0 \in \mathrm{dom}(\overline{\mathbb{C}})$. This provides a generalized solution of the degenerate parabolic system

$$\frac{\partial a}{\partial t} + Au = f \ , \ a \in \partial\zeta_a(u) \ \text{ in } \ L^1(\Omega) \ ,$$
$$\frac{\partial b}{\partial t} + \frac{\partial u}{\partial \nu} = g \ , \ b \in \partial\zeta_b(v) \ , \ \frac{\partial u}{\partial \nu} \in \partial\zeta_c(v - \gamma u) \ \text{ in } \ L^1(\Gamma) \ ,$$

with initial data

$$a(x,0) = a^0(x) \quad \text{a.e.} \quad x \in \Omega,$$
$$b(s,0) = b^0(s) \quad \text{a.e.} \quad s \in \Gamma$$

as desired.

4. Examples. For the following numerical examples we have modified the initial boundary value problem (1) in that we assume the boundary Γ of the domain Ω is the union of two parts, namely, $\Gamma = \Gamma_D \cup \Gamma_H$. We prescribe Dirichlet data $u = u_D = h(t)$ on Γ_D and use the hysteresis boundary conditions (1.1.b), (1.1.c) on Γ_H. The modification of the theorems, such that this case is also covered, is obvious.

We consider a multiple of the signum function

$$b = \frac{1}{2}\,\mathrm{sgn}$$

($\varepsilon = 0$) or a smooth approximation thereof, namely,

$$b_\varepsilon(z) = \frac{1}{2}\frac{z}{\varepsilon + |z|},$$

and the inverse of the signum function

$$c(z) = \mathrm{sgn}^{-1}(z).$$

For the following examples we simplify by using $a(u) = u$ and $f, g = 0$. We are going to use the function

$$h(t) = \alpha 2^{-t/\beta} \sin(2\pi\omega t)$$

(with $\alpha, \beta, \omega > 0$). The initial values are all zero in the examples. As a numerical method, we have used the standard time-explicit difference scheme with constant step-sizes in x and t. Additional details and examples can be found in [12].

Example 1. As a one-dimensional example, let $\Omega = (0,1)$, $\Gamma_H = \{0\}$, $\Gamma_D = \{1\}$. We assume $u_D(t) = h(t)$ with $\alpha = 4$, $\beta = 10$, $\omega = 1/5$, and $\varepsilon = 0$. Figure 1 shows u and the selection $w \in b(u)$ at $x = 1$ as a function of time; the dotted line is the function h and w is the solid line bounded by $1/2$. Figure 2 shows w versus u; the oblique lines that cut the corners are a result of the discretization of time. This has the typical form of a *perfect relay*.

Example 2. The following is an example in two dimensions. We take $\Omega = \{(x_1, x_2) : \ 0 < x_1, x_2 < 1\}$ and assume $\Gamma_D = \{(x_1, x_2) : \ x_1 = 0\}$, $\Gamma_H = \partial\Omega \setminus \Gamma_D$. Again, we use $u_D(t) = h(t)$ for $x \in \Gamma_D$ with parameters $\alpha = 4$, $\beta = 2$, $\omega = 1$, and $\varepsilon = 0.1$. Figure 3 shows the profile of the solution u at time $t = 1.25$ with $\varepsilon = 0.1$.

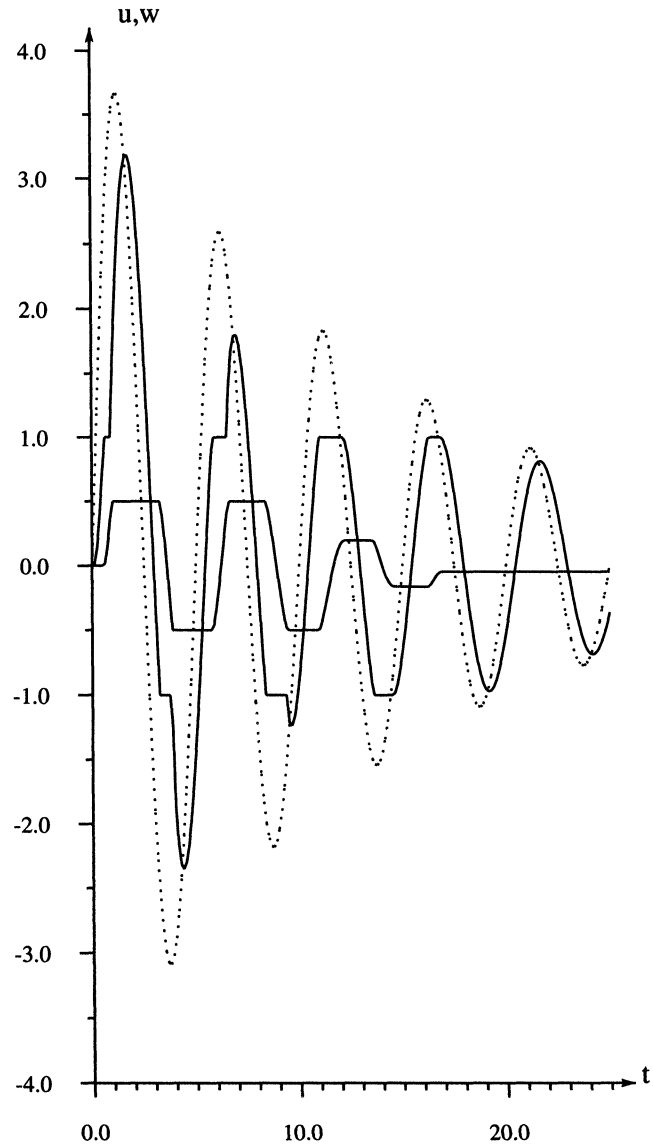

FIG. 1. *u and w as functions of t at $x = 1$ for Example 1 with $\varepsilon = 0$.*

5. A parabolic problem. We close with some remarks on a *parabolic* system obtained as an *additive* perturbation of $[\partial\zeta_a, \partial\zeta_b]$ instead of the composition $\mathbb{C}$ that was used in §3 to recover (1.1). The first is a corollary of Proposition 1.

COROLLARY 3. *Assume that ζ_a and ζ_b are given in Proposition* 1 *and* (2.3.a)–(2.3.c) *hold. For every $u_0 \in L^2(\Omega)$, $v_0 \in L^2(\Gamma)$ and $f \in L^2(0,T;L^2(\Omega))$, $g \in L^2(0,T;L^2(\Gamma))$, there is a unique solution $u \in C([0,T];L^2(\Omega))$, $v \in C([0,T];L^2(\Gamma))$*

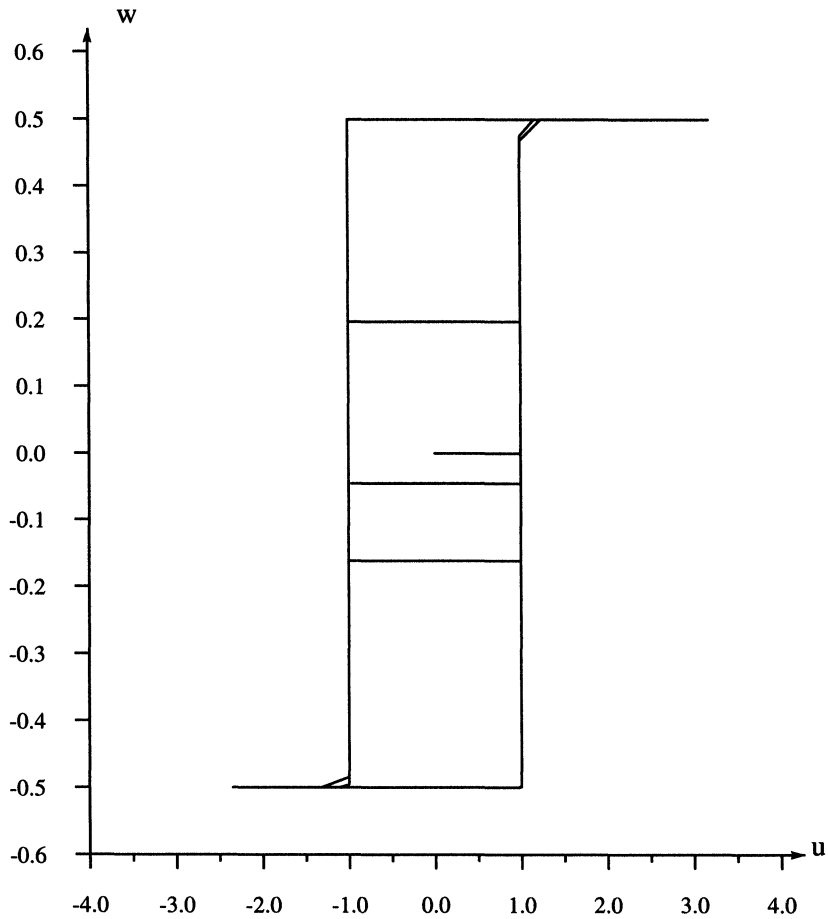

FIG. 2. *Relay: w versus u at $x = 1$ for Example 1 with $\varepsilon = 0$.*

of

$$\frac{\partial u}{\partial t} + a + Au = f \ , \quad a \in \partial\zeta_a(u) \ \ in \ \ L^2_{\mathrm{loc}}\big(0,T;L^2(\Omega)\big), \tag{5.1.a}$$

$$\frac{\partial v}{\partial t} + b + \partial_A u = g \ , \quad b \in \partial\zeta_b(v) \ , \ \ and \tag{5.1.b}$$

$$\partial_A u \in \partial\zeta_c(v - \gamma u) \ \ in \ \ L^2\big(0,T;L^2(\Gamma)\big) \ , \tag{5.1.c}$$

$$u(0) = u_0 \ \ in \ \ L^2(\Omega) \ , \quad v(0) = v_0 \ \ in \ \ L^2(\Gamma) \ . \tag{5.1.d}$$

Proof. Estimates (2.3.a)–(2.3.c) imply that ∂Z_a and ∂Z_b are defined everywhere, hence, by Corollary 2.7 of [7] we have $\partial Z = \partial Z_a + \partial Z_b + \partial Z_2$ in $L^2(\Omega) \times L^2(\Gamma)$. Then, (3.2) is the evolution generated by ∂Z. $\square$

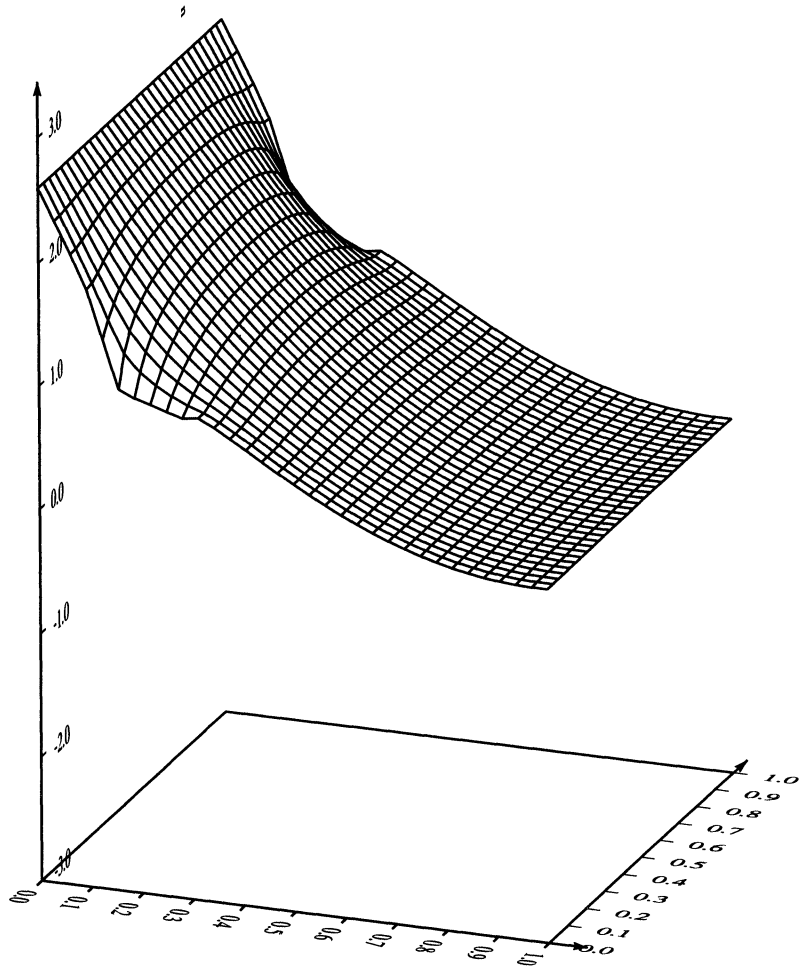

FIG. 3. *Profile at* $t = 1.25$ *for Example* 2 *with* $\varepsilon = 0.1$.

Such a subgradient induces a *parabolic regularizing effect* in the dynamics. Specifically, the solution of (3.2) is strongly differentiable and satisfies

$$u(t) \in D \quad \text{a.e.} \quad t \in (0, T) \ .$$

Also, we note from Theorem 1 that the stationary problem associated with (3.2) is well posed when two of the three parts of (2.11) hold.

The fact that $\mathbb{C}_2$ is ∂J-monotone for *any* J of the form (3.3) has many consequences for the special case of system (5.1) with $\zeta_a = \zeta_b = 0$. In particular, if

$(I+\varepsilon\mathbb{C}_2)([u_j, v_j]) = [f_j, g_j]$ for $j = 1, 2$ and $\varepsilon > 0$, then we have the *resolvent estimate*

$$J([u_1 - u_2,\, v_1 - v_2]) \le J([f_1 - f_2,\, g_1 - g_2]) \tag{5.2}$$

for any such J. Similar estimates hold for the evolution system, and any such J is a *Lyapunov function* for this special case of system (5.1). These lead to L^p-estimates and comparison theorems for solutions by taking appropriate choices of j. Finally, we note that $\partial(J + Z_2) = \partial J + \mathbb{C}_2$, and this leads to another parabolic case of (5.1).

COROLLARY 4. *Let j be given as in Proposition* 2 *of* §3 *and set $\zeta_a = \zeta_b = j$. Assume* (2.3.c) *holds. Then the result of Corollary* 3 *is valid.*

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