PARTIAL DIFFERENTIAL EQUATIONS OF
SOBOLEV-GALPERN TYPE

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A mixed initial and boundary value problem is considered for
a partial differential equation of the form $M u_t(x, t) + L u(x, t) = 0$,
where $M$ and $L$ are elliptic differential operators of orders $2m$
and $2l$, respectively, with $m \leq l$. The existence and uniqueness
of a strong solution of this equation in $H^k_0(G)$ is proved by
semigroup methods.

We are concerned here with a mixed initial boundary value problem
for the equation

$$M u_t + L u = 0$$

in which $M$ and $L$ are elliptic differential operators. Equations of this
type have been studied using various methods in [2, 3, 4, 6, 7, 10, 11,
13, 14, 15, 17, 18]. We will make use of the $L^2$-estimates and related
results on elliptic operators to obtain a generalized solution to this
problem similar to that obtained for the parabolic equation

$$u_t + L u = 0$$
as in [7].

Let $G$ be a bounded open domain in $\mathbb{R}^n$ whose boundary $\partial G$
is an $(n - 1)$-dimensional manifold with $G$ lying on one side of $\partial G$. By
$H^k(G) = H^k$ we mean the Hilbert space (of equivalence classes) of
functions in $L^2(G)$ whose distributional derivatives through order $k$
belong to $L^2(G)$ with the inner product and norm given, respectively, by

$$(f, g)_k = \sum \left\{ \int_G D^\alpha f D^\gamma g dx : |\alpha| \leq k \right\}$$

and

$$\|f\|_k = \sqrt{(f, f)_k}.$$ 

$H^k_0 \equiv H^k_0(G)$ will denote the closure in $H^k$ of $C_0^\infty(G)$, the space of
infinitely differentiable functions with compact support in $G$. The
operators are of the form

$M = \sum \{(\pm)^\sigma D^\rho m^{\rho \sigma}(x) D^\sigma : |\rho|, |\sigma| \leq m\}$

and

$L = \sum \{(\pm)^\sigma D^\rho l^{\rho \sigma}(x) D^\sigma : |\rho|, |\sigma| \leq l\}$,
and they are uniformly strongly elliptic in $G$. We shall investigate the existence and uniqueness of solutions to (1) which coincide with the initial function $u_0$ in $H^l_0$ where $t = 0$ and vanish on $\partial G$ together with all derivatives of order less than or equal to $l - 1$.

If the order of $M$ is as high as that of $L (2m \geq 2l)$, then this problem can be handled as in [10] by forming the exponential of the bounded extension of $M^{-1}L$ on $H^m_0$ and thus obtaining a group of operators on $H^m_0$ and a corresponding solution for all $t$ in $R$. The case we shall consider is that of $m \leq l$, and this will include the parabolic equation as a special case. We obtain a semi-group of operators on $H^m_0$ and, hence, a solution for all $t \geq 0$.

2. In this section we shall formulate the problem. Assume temporarily the following.

$P'$: The coefficients $m^{\rho\sigma}$ in $M$ belong to $H^{\rho\sigma}$, and $D^\rho m^{\rho\sigma}$ is in $L^\sigma(G)$ whenever $|\rho| \leq m$. A similar statement is true for the coefficients in $L$. From $P'$ it follows that the sesqui-linear forms defined on $C^\omega_0(G)$ by

$$B_M(\varphi, \psi) = \sum \{(m^{\rho\sigma} D^\rho \varphi, D^\sigma \psi)_0 : |\rho|, |\sigma| \leq m\}$$

and

$$B_L(\varphi, \psi) = \sum \{(l^{\rho\sigma} D^\rho \varphi, D^\sigma \psi)_0 : |\rho|, |\sigma| \leq l\}$$

satisfy the identities

(2) \quad $B_M(\varphi, \psi) = (M \varphi, \psi)_0$

and

(2') \quad $B_L(\varphi, \psi) = (L \varphi, \psi)_0$

for all $\varphi, \psi$ in $C^{\omega}_0(G)$. Since $P'$ implies that

$$K_m = \sup \{\|m^{\rho\sigma}\|_\omega : |\rho|, |\sigma| \leq m\}$$

and

$$K_l = \sup \{\|l^{\rho\sigma}\|_\omega : |\rho|, |\sigma| \leq l\}$$

are finite, we see that

$$|B_M(\varphi, \psi)| \leq K_m \|\varphi\|_m \|\psi\|_m$$

and

$$|B_L(\varphi, \psi)| \leq K_l \|\varphi\|_l \|\psi\|_l$$

for all $\varphi, \psi$ in $C^{\omega}_0(G)$. Hence these sesqui-linear forms may be extended by continuity to all of $H^m_0$ and $H^l_0$, respectively.
The final properties which we shall assume are the following. For any \( \varphi, \psi \in C^\infty_c(G) \) we have

\[
P_2: \quad \text{Re } B_M(\varphi, \varphi) \geq k_m \|
\varphi\|_m, \quad k_m > 0,
\]

\[
\text{Re } B_L(\varphi, \varphi) \geq k_l \|
\varphi\|_l, \quad k_l > 0,
\]

and

\[
P_3: \quad |B_M(\varphi, \psi)|^2 \leq (\text{Re } B_M(\varphi, \varphi))(\text{Re } B_M(\psi, \psi)).
\]

These inequalities are valid for the respective extensions to \( H^m_0 \) and \( H^l_0 \). The assumptions of \( P_2 \) are inequalities of the Garding type which imply that \( M \) and \( L \) are uniformly strongly elliptic. Only the first of these is essential in applications, for the usual change of dependent variable \( u = ve^\lambda t \) changes our equation to one with \( L \) replaced by \( L + \lambda M \), and the Garding inequality is true for \( B_{L+\lambda M} \) if \( \lambda \) is sufficiently large and if the coefficients \( l^{p\sigma}(x), |\rho| = |\sigma| = l \) are uniformly continuous in \( G \). See [3, 8] for sufficient conditions that \( P_2 \) be true.

The assumption \( P_3 \) is a Cauchy-Schwarz inequality for the form \( B_M \). In view of the positivity of \( B_M \), a necessary and sufficient condition for \( P_3 \) is that \( M \) be symmetric, that is, \( m^{p\sigma} = \overline{m^{p\sigma}} \) for all \( \rho, \sigma \). Such is the case for the examples

(i) \( ku_t - \Delta u = 0 \) (\( m = 0 \)) and

(ii) \( -\gamma \Delta u_t + ku_t - \Delta u = 0 \),

where \( \Delta \) is the Laplacian and \( \gamma \) and \( k \) are positive. Example (i) is a parabolic equation, and examples like (ii) appear in various problems of fluid mechanics and soil mechanics, where a solution is sought which satisfies an initial condition \( u(x, 0) = u_0(x) \) on \( G \) and the Dirichlet condition \( u(x, t) = 0 \) on the boundary of \( G \). See [1, 11, 12, 13].

We shall not need the full strength of \( P_1 \) so we replace it with the following weaker assumption.

\( P_1: \) The coefficients \( m^{p\sigma} \) and \( l^{p\sigma} \) belong to \( L^\infty(G) \) for all \( \rho, \sigma \).

We shall proceed under the assumptions \( P_1, P_2 \) and \( P_3 \) and remark that \( P_1 \) is needed only when we wish to interpret our weak solutions by means of (2) and (2').

Under the hypotheses above there is by the theorem of Lax and Milgram [7] a closed linear operator \( M_0 \) with domain \( D_m \) dense in \( H^m_0 \) and range equal to \( H^\circ = L^2(G) \) such that

\[
(3) \quad B_M(\varphi, \psi) = (M_0 \varphi, \psi),
\]

whenever \( \varphi \) belongs to \( D_m \) and \( \psi \) to \( H^m_0 \). Furthermore, \( M_0^{-1} \) is a bounded operator from \( H^\circ \) into \( H^m_0 \). Similarly, there is a closed linear operator \( L_0 \) with domain \( D_l \) dense in \( H^l_0 \) and range equal to \( H^\circ \) with

\[
(3') \quad B_L(\varphi, \psi) = (L_0 \varphi, \psi),
\]
whenever \( \varphi \) belongs to \( D_t \) and \( \psi \) to \( H^1_t \). Also, \( L^{-1}_0 \) is bounded from \( H^0 \) into \( H^1_t \).

Consider the bijection \( A = -M^{-1}_0 L_0 \) from \( D_t \) onto \( D_m \). For any \( \varphi \) in \( D_m \) we have

\[
  k_l \| A^{-1} \varphi \| = k_l \| L_0^{-1} M_0 \varphi \| \\
  \leq \Re B_L(L_0^{-1} M_0 \varphi, L_0^{-1} M_0 \varphi) = \Re (M_0 \varphi, L_0^{-1} M_0 \varphi) \\
  = \Re B_L(\varphi, L_0^{-1} M_0 \varphi) \leq K_m \| \varphi \|_m \| A^{-1} \varphi \|_m \\
  \leq K_m \| \varphi \|_m \| A^{-1} \varphi \|_l
\]

which yields

\[
  \| A^{-1} \varphi \|_l \leq (K_m/k_l) \| \varphi \|_m
\]

for all \( \varphi \) in \( D_m \). But \( D_m \) is dense in \( H^m_0 \) so \( A^{-1} \) has a unique extension by continuity from \( H^m_0 \) onto the set \( D = A^{-1}(H^m_0) \) in \( H^1_t \), the domain of the closed extension of \( A \). The continuity of the injection of \( H^m_0 \) into \( H^m_0 \) implies that \( A^{-1} \) is a bounded operator on \( H^m_0 \), and this is the space in which we formulate the Generalized Problem:

For a given initial function \( u_0 \) in \( D \), find a differentiable map \( u(t) \) of \( R^+ \) into \( H^m_0 \) for which \( u(t) \) belongs to \( H^1_t \) for all \( t \geq 0 \), \( u(0) = u_0 \), and

\[
  B_M(u'(t), \varphi) + B_L(u(t), \varphi) = 0
\]

for all \( \varphi \) in \( C^\infty_0(G) \) and \( t \geq 0 \).

Sufficient conditions for a solution of this generalized problem to be a classical solution will be discussed in [9].

3. The objective of this section is to prove the following results.

THEOREM. There exists a unique solution of the generalized problem. If \( u(t) \) is in \( D_t \) then \( u(t) \) is in \( D_m \) and

\[
  M_0 u'(t) + L_0 u(t) = 0
\]

in \( H^0 \). The mapping of \( u_0 \) to \( u(t) \) is continuous from \( H^m_0 \) into itself for each \( t \geq 0 \) and furthermore satisfies

\[
  \| u(t) \|_m \leq \sqrt{(K_m/k_m)} \| u_0 \|_m \exp (-k_l t/K_m). \]

We first show that the operator \( A \) is the infinitesimal generator of a semi-group of bounded operators on \( H^m_0 \); this semi-group will provide a means of constructing a solution to the problem. From the assumptions on \( B_M \), it follows that the function defined by

\[
  | \varphi |_m = \sqrt{\Re B_M(\varphi, \varphi)}
\]

is a norm on \( H^m_0 \) that is equivalent to the norm \( \| \cdot \|_m \). In the following we shall use \( | \cdot |_m \) as the norm on \( H^m_0 \), noting further that
To obtain the necessary estimates we let \( \lambda \) be a nonnegative number and consider the operator \( \lambda M_0 + L_0 = N \) from the domain \( D_m \cap D_l \) into \( H^0 \). We can define a sesquilinear form on \( D_m \cap D_l \) by

\[
B_N(\phi, \psi) = ((\lambda M_0 + L_0)\phi, \psi) = \lambda B_M(\phi, \psi) + B_L(\phi, \psi)
\]

and then note that \( B_N \) is bounded as well as positive-definite with respect to the norm of \( H^0 \). We extend \( B_N \) by continuity to all of \( H^0 \), and then by the theorem of Lax and Milgram there is a closed linear operator \( N_0 \) from a domain \( D_n \) in \( H^0 \) onto \( H^0 \) for which

\[
B_N(\phi, \psi) = (N_0\phi, \psi)_0
\]

whenever \( \phi \) is in \( D_n \) and \( \psi \) in \( H^0 \). Clearly \( N_0 \) is an extension of \( N \) whose domain is \( D_m \cap D_l \).

For all \( \phi \) in \( D_m \) we have

\[
\Re (N_0\phi, \phi)_0 = \lambda \Re B_M(\phi, \phi) + \Re B_L(\phi, \phi) \\
\geq (\lambda + k_l/K_m) \Re B_M(\phi, \phi) \\
= (\lambda + k_l/K_m) |\phi|_H^2.
\]

Thus, for any \( \psi \) in \( D_n \) we see that \( N_0^{-1}M_0\psi \) belongs to \( D_n \) and from above

\[
(\lambda + k_l/K_m) |N_0^{-1}M_0\psi|_H^2 \leq \Re (M_0\psi, N_0^{-1}M_0\psi)_0 \\
= \Re B_M(\psi, N_0^{-1}M_0\psi) \leq |\psi|_H \|(N_0^{-1}M_0\psi)|_H
\]

by \( P_n \), so we have obtained the estimate

\[
|N_0^{-1}M_0\psi|_H \leq (\lambda + k_l/K_m)^{-1} |\psi|_H
\]

for all \( \psi \) in \( D_n \).

Letting \( \phi \) be an element of \( D_l \cap D_n \) we see

\[
(N_0^{-1}M_0)(\lambda + M_0^{-1}L_0)\phi = N_0^{-1}(\lambda M_0\phi + L_0\phi) \\
= N_0^{-1}N\phi = \phi,
\]

so \( \lambda + M_0^{-1}L_0 \) is injective and satisfies

\[
(\lambda + M_0^{-1}L_0)^{-1} = N_0^{-1}M_0
\]

on \( D_m \cap D_l \). Combining this with the estimate above we see that

\[
| (\lambda + M_0^{-1}L_0)^{-1}\psi |_H \leq (\lambda + k_l/K_m)^{-1} |\psi|_H
\]

for all \( \psi \) in \( D_l \cap D_m \). It follows by continuity that \( \lambda - A \) is invertible on \( H^0 \) and satisfies the estimate

\[
| (\lambda - A)^{-1}|_H \leq (\lambda + k_l/K_m)^{-1}.
\]
By the theorem of Hille and Yoshida [5, 16] on the characterization of the infinitesimal generators of semi-groups of class \( C_0 \) we have the following results: there exists a unique family of bounded operators \( \{ S(t): t \geq 0 \} \) on \( H_0^m \) for which

(i) \( S(t_1 + t_2) = S(t_1)S(t_2) \),
(ii) \( S(t)x \) is strongly continuous for each \( x \) in \( H_0^m \),
(iii) \( S(0) = I \) and \( |S(t)| \leq \exp \left(-kt_1/t_2\right) \) for all \( t \geq 0 \),
(iv) \( \lim_{n \to 0} h^{-i}(S(h) - I)x_0 = Ax_0 \) for each \( x_0 \) in \( D \), and
(v) \( S(t) \) commutes with \( (\lambda - A)^{-1} \) for all \( \lambda \geq 0 \).

The statement (v) implies in particular that \( D \) is invariant under each \( S(t) \).

Having been given the initial function \( u_0 \) in \( D \), we define

\[
    u(t) = S(t)u_0, \quad t \geq 0
\]

and show that \( u(t) \) is a solution of the generalized problem. Clearly we see \( u(t) \) belongs to \( H_0^m \) and \( u(0) = u_0 \). Furthermore, since \( S(t) \) leaves \( D \) invariant and \( u_0 \) is in \( D \), it follows that \( u(t) \) belongs to \( D \) and thus to \( H_0^m \). The function \( u(t) \) is differentiable with

\[
    u'(t) = Au(t)
\]

for all \( t \geq 0 \) by (i) and (iv), and hence \( u'(t) \) is in \( H_0^m \).

We shall verify that \( u(t) \) satisfies the equation (5). Since \( D_m \) is dense in \( H_0^m \) there is a sequence \( \{ \varphi_n \} \) in \( D_m \) for which \( \| \varphi_n - u'(t) \|_m \to 0 \) as \( n \to \infty \). Now \( \{ \varphi_n \} \) is a Cauchy sequence in \( H_0^m \) and it follows by (4) that \( \psi_n = A^{-1} \varphi_n \) is a Cauchy sequence in the complete space \( H_0^m \), so there is a \( \psi \) in \( H_0^m \) such that \( \| \psi_n - \psi \|_1 \to 0 \) as \( n \to \infty \). Since \( A^{-1} \) is continuous we have \( \psi = u(t) \). Each \( \psi_n \) belongs to \( D_1 \), since \( \varphi_n \) is in \( D_m \), and furthermore \( M_0 \varphi_n + L_0 \psi_n = 0 \). Now for each \( \varphi \) in \( C_0^m(G) \) we have by the continuity of \( B_M \) and \( B_L \)

\[
    B_M(u'(t), \varphi) + B_L(u(t), \varphi)
    = \lim_{n \to \infty} B_M(\varphi_n, \varphi) + \lim_{n \to \infty} B_L(\psi_n, \varphi)
    = \lim_{n \to \infty} \{ B_M(\varphi_n, \varphi) + B_L(\psi_n, \varphi) \} = \lim_{n \to \infty} \{ (M_0 \varphi_n, \varphi) + (L_0 \psi_n, \varphi) \} = 0,
\]

so the generalized problem does have a solution.

If \( u(t) \) is in \( D_1 \) then by (9) \( u'(t) \) is in \( D_m \). It follows from (5) that for every \( \varphi \) in \( C_0^m(G) \)

\[
    (M_0 u'(t) + L_0 u(t), \varphi)_0 = 0,
\]

and this implies (6). The estimate (7) is a consequence of (iii) and (8).

To show that the generalized problem has at most one solution, we let \( u(t) \) be a solution of the problem with \( u_0 = 0 \). By linearity it suffices to show that \( u(t) = 0 \). The differentiability of \( u(t) \) in \( H_0^m \)
implies that the real valued function
\[ \alpha(t) = \text{Re} B_M(u(t), u(t)) \]
is differentiable and

\[ \alpha'(t) = 2 \text{Re} B_M(u'(t), u(t)) . \]

Since (5) is true also for all \( \varphi \) in \( H_0^1 \) by continuity, we have from \( P_2 \)
\[ \alpha'(t) = -2 \text{Re} B_M(u(t), u(t)) \leq 0 . \]

But \( \alpha(0) = \text{Re} B_M(u(0), u(0)) = 0 \), so \( \alpha(t) = 0 \) for all \( t \geq 0 \). From \( P_2 \)
it follows that \( u(t) = 0 \) for \( t \geq 0 \).

REFERENCES


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