LOCAL REGULARITY OF SOLUTIONS OF SOBOLEV-
GALPERN PARTIAL DIFFERENTIAL EQUATIONS

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Let $M$ and $L$ be elliptic differential operators of orders $2m$ and $2\ell$, respectively, with $m \leq \ell$. The existence and uniqueness of a solution to the abstract mixed initial and boundary value problem

$$Mu'(t) + Lu(t) = 0, \quad u(0) = u_0$$

was established for $u_0$ given in the domain of the infinitesimal generator of a strongly-continuous semi-group. The purpose of this paper is to show that this semi-group is holomorphic and then obtain differentiability results for the solution and convergence of this solution to the initial function $u_0$ as $t \downarrow 0$.

Let $G$ be a bounded open domain of $\mathbb{R}^n$ whose boundary $\partial G$ is an $(n-1)$-dimensional manifold with $G$ lying on one side of $\partial G$. $H^k = H^k(G)$ is the Hilbert space (of equivalence classes) of functions whose distributional derivatives through order $k$ belong to $L^2(G)$ with the usual inner-product and norm,

$$(f, g)_k = \sum \left\{ \int_G D^\alpha f \, D^\alpha g \, dx : |\alpha| \leq k \right\}$$

and

$$\|f\|_k = \sqrt{(f, f)_k}.$$

$H_0^k = H_0^k(G)$ is the closure in $H^k$ of $C^\infty(G)$, the space of infinitely differentiable functions with compact support in $G$.

We specify the problem by means of the bilinear forms

$$B_M(\phi, \psi) = \sum \{(m^\sigma D^\sigma \phi, D^\sigma \psi)_k : |\rho|, |\sigma| \leq m\}$$

and

$$B_L(\phi, \psi) = \sum \{(l^\sigma D^\sigma \phi, D^\sigma \psi)_k : |\rho|, |\sigma| \leq l\},$$

defined initially for $\phi$ and $\psi$ in $C_0^\infty(G)$. Furthermore, we require the following:

$P_1$: The coefficients $m^\sigma, l^\sigma$ are bounded and measurable.

$P_2$: $\text{Re} \; B_M(\phi, \psi) \geq k_m \|\phi\|_m^2, \; k_m > 0$

$\text{Re} \; B_L(\phi, \psi) \geq k_1 \|\phi\|_1^2, \; k_1 > 0$

for all $\phi$ in $C_0^\infty(G)$.

$P_3$: $M$ is symmetric; that is $m^\sigma = m^{\sigma}$ for all $\rho, \sigma$, (hence $B_M(\phi, \phi)$ is real for all $\phi$ in $C_0^\infty$).
From the assumptions \( P_1 \) and \( P_2 \) and the general theory of elliptic operators, \([1, 6, 7, 11, 12, 13]\), there are two operators, \( M_0 \) and \( L_0 \), which are topological isomorphisms of \( H^m_0 \) onto \( H^{-m} = (H^m_0)' \) and \( H_0' \) onto \( H^{-1} = (H_0^1)' \) (where "'" denotes the continuous linear dual), and these are determined by the respective identities

\[
B_M(\phi, \psi) = \langle M\phi, \psi \rangle
\]

and

\[
B_L(\phi, \psi) = \langle L\phi, \bar{\psi} \rangle
\]
on \( H^m_0 \) and \( H^i_0 \), respectively, where "\( \langle \cdot, \cdot \rangle " \) denotes \( \mathcal{D} - \mathcal{D}' \) duality, \( \mathcal{D}' \) being the space of distributions over \( G \).

Since \( l \geq m \) we have a topological inclusion \( H^i_0 \subset H^m_0 \), hence, by duality, \( H^{-m} \subset H^{-i} \). Thus the mapping \( L_0^{-1}M_0 \) is continuous from \( H^m_0 \) into \( H^i_0 \) and is a topological isomorphism only if \( l = m \). Letting \( D = L_0^{-1}M_0(H^m_0) = L_0^{-1}(H^{-m}) \), we have an unbounded operator \( A = M_0^{-1}L_0 \) on \( H^m_0 \) with domain \( D \) dense in \( H^i_0 \). In [16] we showed that \( A \) is the infinitesimal generator of an equicontinuous semi-group of bounded operators \([6, 9, 11]\) on \( H^m_0 \), denoted by \( \{ S(t) : t \geq 0 \} \). We shall prove that this semi-group is holomorphic.

We have already shown that the nonnegative real axis belongs to the resolvent set of \( A \) and, in fact,

\[
|R(\lambda, A)|_M = |(\lambda - A)^{-1}|_M \leq (\text{Re}(\lambda))^{-1}
\]
for all real \( \lambda \geq 0 \), where the norm \( |\cdot|_M \) defined by

\[
|\phi|_M = \sqrt{B_M(\phi, \phi)}
\]
on \( H^m_0 \) is equivalent to \( || \cdot ||_m \) by \( P_1 \) and \( P_2 \). Actually the whole right half of the complex plane belongs to the resolvent set of \( A \), and (1) is true there. This can be shown by noting that for \( \lambda = \sigma + i\tau \) we have

\[
B_M((A - \lambda)\phi, \phi) = B_M((A - \sigma)\phi, \phi) - i\tau B_M(\phi, \phi)
\]
and hence

\[
\text{Re} B_M((A - \lambda)\phi, \phi) = \text{Re} B_M((A - \sigma)\phi, \phi)
\]
in the argument leading to (1) for \( \lambda \) real. See [16] for details.

2. Our goal is to improve the estimate (1) to show that the family \( \{ \lambda R(\lambda, A) \} \) is uniformly bounded in \( \mathcal{L}(H^m_0) \) for \( \text{Re}(\lambda) > 0 \). First let \( \phi \) be in \( D \); then

\[
B_M((\lambda - A)\phi, \phi) = (\sigma + i\tau)B_M(\phi, \phi) + B_L(\phi, \phi)
\]
Since \( M \) is symmetric it follows that \( B_M(\phi, \phi) \) is real, so we obtain
(2) \( \Re B_M(\lambda - A)\phi, \phi) = \sigma B_M(\phi, \phi) + \Re B_L(\phi, \phi) \geq k_i \|\phi\|^2 \), since \( \sigma > 0 \). Similarly, from
\[
\Im B_M(\lambda - A)\phi, \phi) = \tau B_M(\phi, \phi) + \Im B_L(\phi, \phi)
\]
we obtain the estimate
(3) \( \|\Im B_M(\lambda - A)\phi, \phi\| \geq |\tau| \|\phi\|^2 - K_i \|\phi\|^2 \).

From (2) and (3) we conclude that either
(4) \( \|\Im B_M(\lambda - A)\phi, \phi\| \geq \frac{|\tau|}{2} \|\phi\|^2 \)

or
(5) \( \|\Re B_M(\lambda - A)\phi, \phi\| \geq \frac{k_i}{2K_i} |\tau| \|\phi\|^2 \),
for if (4) is not true then by (3)
\[
|\tau| \|\phi\|^2 - K_i \|\phi\|^2 \leq \frac{|\tau|}{2} \|\phi\|^2 ,
\]

hence
\[
\frac{|\tau|}{2} \|\phi\|^2 \leq K_i \|\phi\|^2 ,
\]
which with (2) implies (5). From (4) and (5) we obtain the estimate
(6) \( \| B_M(\lambda - A)\phi, \phi\) \geq \frac{k_i}{2K_i} |\tau| \|\phi\|^2 \)

for all \( \phi \) in \( D \), and this in turn yields
(7) \( |R(\lambda, A)|_M \leq \frac{2K_i}{k_i} \frac{1}{|\tau|} \),

whenever \( \Re (\lambda) > 0 \). The calculation is as follows:
\[
\frac{k_i}{2K_i} |\tau| \|\phi\|^2 \leq | B_M(\lambda - A)\phi, \phi| \leq |(\lambda - A)\phi|_M |\phi|_M
\]

implies
\[
|(\lambda - A)\phi|_M \geq |\tau| \frac{k_i}{2K_i} |\phi|_M
\]
for all \( \phi \) in \( D \), the domain of \( A \), so (7) follows. The estimates (1) and (7) imply that
\[ |\lambda R(\lambda, A)|_M \leq \frac{\tau}{\sigma} + 1 \]

when \( \sigma > 0 \) and, respectively, that
\[ |\lambda R(\lambda, A)|_M \leq \frac{2K_i}{k_i} \left( \frac{\sigma}{|\tau|} + 1 \right) \]

whenever \(|\tau| \neq 0\), where \( \lambda = \sigma + i\tau \). By considering the two cases, \(|\tau| \geq \sigma \) and \(|\tau| < \sigma \), we obtain, finally,
\[ (8) \]
\[ |\lambda R(\lambda, A)|_M \leq \frac{4K_i}{k_i} \]

for all \( \lambda \) in the right half of the complex plane. The estimate (8) yields the following result.

**Proposition [22].** The semi-group \( \{S(t): t \geq 0\} \) has a holomorphic extension into a sector of the complex plane. Furthermore, \( S(t) \) maps \( H^m_0 \) into \( D \) whenever \( t > 0 \), so \( S(t) \) is infinitely differentiable and \( S^{(p)}(t) = A^pS(t) \) for any integer \( p \geq 1 \).

The significance of this result for our problem is that, for each \( t > 0 \), \( S(t) \) maps \( H^m_0 \) into the domain of \( A^p \) for an arbitrary integer \( p \geq 1 \).

3. The differentiability of the semi-group yields differentiability of the solution to the problem being considered; the latter is obtained by means of the following.

Let \( H^k_{I_{oc}} \) denote those (equivalence classes of) functions on \( G \) which are locally in \( H^k \); that is,
\[ H^k_{I_{oc}} = \{f: f \in H^k(K) \text{ for each compact subset } K \text{ of } G\} . \]

The following result on the local regularity of solutions of elliptic equations is well known.

**Theorem [1, 4, 5, 7, 12, 13, 14].** Let \( p \) be an integer \( \geq -l \) for which \( l^{p\sigma} \) is \( \max \{1, |\rho| + p\} \) times continuously differentiable in \( G \) whenever \( |\rho| \) and \( |\sigma| \) are \( \leq l \). If \( u \) belongs to \( H^k_0 \), and if \( L_0u \) is in \( H^p_{I_{oc}} \), then \( u \) belongs to \( H^{2l+1+p}_{I_{oc}} \). That is, \( L_0 \) is a topological isomorphism of \( H^k_0 \cap H^{2l+1+p}_{I_{oc}} \) onto \( H^{-l} \cap H^p_{I_{oc}} \).

Let \( k \) be a nonnegative integer and assume that we have
\( P(k): m^{p\sigma} \) and \( l^{p\sigma} \) are \( \max \{1, |\rho| - m + k\} \) times continuously differentiable in \( G \).
From the above theorem it follows that \( M \) is a bijection of \( H'^{m} \cap H'_{\text{loc}}^{m+k} \) onto \( H'^{-m} \cap H'_{\text{loc}}^{-m} \). Also \( L^{-1} \) is a bijection of \( H'^{-i} \cap H'^{-m} \) onto \( H'^{i} \cap H'_{\text{loc}}^{i-m+k} \). Since \( H'^{-m} \subset H'^{-i} \), it follows that \( A^{-1} = -L^{-1}M \) maps \( H'^{m} \cap H'_{\text{loc}}^{m+k} \) into \( H'^{i} \cap H'_{\text{loc}}^{i-m+k} \).

**Corollary.** \( P(2(p - 1)(l - m)) \) implies that the domain of \( A^p \) is contained in \( H'^{i} \cap H'_{\text{loc}}^{i+2p(l-m)} \) for \( p \geq 1 \).

From § 2 we know that \( u(t) \) is in the domain of \( A^p \) for all \( t > 0 \) and \( p > 1 \). The corollary thus yields the following results.

**Theorem.** Assume \( P_1, P_2 \) and \( P_3 \) of § 2. Let the coefficients in \( M \) and \( L \) satisfy \( P(2(p - 1)(l - m)) \) for some integer \( p \geq 1 \). Then \( u(t) = S(t)u_0 \) belongs to \( H'^{i} \cap H'_{\text{loc}}^{i+2p(l-m)} \) for each \( t > 0 \), where \( u_0 \) is any element of \( H'^{m} \).

If \( p \) is sufficiently large we obtain pointwise-solutions by Sobolev’s Lemma [17]:

If \( m \) is an integer \( > (n/2) \), then \( H'^{m} \) is imbedded in \( C^{j}(G) \), \( j = m - [n/2] - 1 \), and the injection is continuous when the range space is given the topology of uniform convergence in all derivatives of order \( \leq j \) on compact of subsets of \( G \).

**Corollary.** Assume the hypotheses of the above theorem hold with \( m + 2p(l - m) - [n/2] - 1 = j \geq 0 \). Then, for \( t > 0 \), \( u(t) \) has \( j \) continuous derivatives in \( G \) and, for each point \( x \) in \( G \), the function \( t \to u(x, t) \) is infinitely differentiable.

**Proof.** Choose \( t' \) such that \( t > t' > 0 \). Since \( u(t') = S(t')u_0 \) belongs to \( D(A^p) \), the semi-group property yields

\[
\delta^{-1}[u(t + \delta) - u(t)] = A^{-p}\delta^{-1}[S(t + \delta - t') - S(t - t')]A^p u(t')
\]

for \( \delta \) sufficiently small. Since \( A^p u(t') \) belongs to \( D = D(A) \), the function to the right of \( A^{-p} \) has a limit in \( H'^{m} \) as \( \delta \to 0 \), so the function \( \delta^{-1}[u(t + \delta) - u(t)] \) has a limit in \( H'^{m+2p(l-m)}(K) \), where \( K \) is any compact subset of \( G \). By Sobolev’s Lemma, the function

\[
\delta \to \delta^{-1}[u(x, t + \delta) - u(x, t)]
\]

has a limit as \( \delta \to 0 \), so \( u(x, t) \) is differentiable. A repetition of this argument shows that \( u(x, t) \) is infinitely differentiable in \( t \) without any further assumptions on the coefficients.

All of the above results have been obtained for a solution with initial value \( u_0 \) in \( H'^{m} \). We note further that if \( u_0 \) is sufficiently
smooth then \( u(t) \rightarrow u_0 \) pointwise. (It is always true that \( u(t) \rightarrow u_0 \) in \( H^m \).)

**Corollary.** Assume the hypotheses of the above corollary and that \( u_0 \) belongs to the domain of \( A^p \). Then each \( u(t), t \geq 0 \) is a continuous function on \( G \), and for each point \( x \) in \( G \), \( u(x, t) \rightarrow u_0(x) = u(x, 0) \) as \( t \rightarrow 0 \).

**Proof.** This follows by an argument similar to the proof of the preceding corollary applied to the equation

\[
u(t) - u_0 = A^{-p}(S(t) - I)(A^p u_0) .\]

We note that a sufficient condition for \( u_0 \) to be in \( D = D(A) \) is that \( u_0 \) be in \( H^1_0 \cap H^{21-m} \). Also if the initial function and all coefficients in \( M \) and \( L \) are infinitely differentiable, then the solution is infinitely differentiable.

**Bibliography**


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