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A Darcy–Brinkman model of fractures in porous media ☆

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ABSTRACT

For a fully-coupled Darcy–Stokes system describing the exchange of fluid and stress balance across the interface between a saturated porous medium and an open very narrow channel, the limiting problem is characterized as the width of the channel converges to zero. It is proven that the limit problem is a fully-coupled system of Darcy flow in the porous medium with Brinkman flow in tangential coordinates of the lower dimensional interface.

1. Introduction

We consider the limiting form of a system of equations describing incompressible fluid flow in a fully-saturated region Ω^ϵ which consists of two parts, a porous medium Ω_1 and a very narrow channel Ω_2^ϵ of width $\epsilon > 0$ along part of its boundary, $\Gamma = \partial\Omega_1 \cap \partial\Omega_2^\epsilon$. That is, we have $\Omega^\epsilon \equiv \Omega_1 \cup \Gamma \cup \Omega_2^\epsilon$. The filtration flow in the porous medium is governed by Darcy's law on Ω_1 and the faster flow of the fluid in the narrow open channel by Stokes' system on Ω_2^ϵ. For simplicity, we assume that the channel is *flat*, that is, $\Omega_2^\epsilon \equiv \Gamma \times (0, \epsilon)$, where $\Gamma \subset \mathbb{R}^{n-1}$, $\mathbb{R}^{n-1}$ is identified with $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$, and $\Gamma = \partial\Omega_1 \cap \partial\Omega_2^\epsilon$ is the interface. See Fig. 1. We assume that $\partial\Omega_1 - \Gamma$ is smooth. The Darcy and Stokes systems have very different regularity properties, and both the tangential velocity and pressure of the fluid are discontinuous across the interface, so the analysis is delicate. Our goal is to establish the existence of a limit problem as the width $\epsilon \to 0$ and to characterize it. This limit is a fully-coupled system consisting of Darcy flow in the porous medium Ω_1 and Brinkman flow on the part Γ of its boundary.

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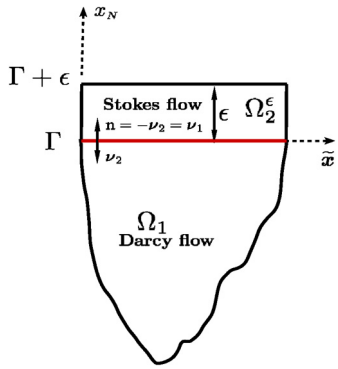


Fig. 1. The porous medium Ω_1 below the thin channel Ω_2^ϵ.

The Darcy–Stokes system above has two types of singularities: the geometric one coming from the narrowness of the channel $\mathcal{O}(\epsilon)$ with respect to the dimensions, and the physical one of high fluid flow velocity $\mathcal{O}(1/\epsilon)$ in the channel with respect to the porous medium. These singularities introduce multiple scales in the system which have an impact on the numerical simulation. Some of these consequences are ill-conditioned matrices, problems of numerical stability, poor quality of the numerical solutions and high computational costs. Earlier modeling of fractures was based on Darcy–Darcy models for slower flow in 'debris-filled' channels [10,22,33,24,25,34]. Since the original analytic and numerical treatment of the Darcy–Stokes system [18], much progress has been made to handle such issues [5,13,11,7,14,23], and for the use of Brinkman flow to couple numerically the Darcy and Stokes flow models [35,17,29,12]. See especially Quarteroni et al. [19] for additional issues, references and perspectives.

The Brinkman system has nothing to do with the usual models of porous media flow, but rather describes Stokes flow through a sparse array of particles for which the porosity is more than 0.8 [16,8,9,26]. This requirement is highly restrictive since most naturally occurring porous media have a porosity less than 0.6. Lévy [20,21] showed that the Brinkman system holds only for arrays of particles whose size is precisely of order η^3, where $\eta << 1$ is the distance between neighboring particles. Larger particles impede the fluid flow sufficiently to be described by a Darcy system, and smaller particles do not change the flow from the Stokes system. Allaire [2–4] proved and developed this homogenization result by means of two-scale convergence, and Arbogast & Lehr [6] confirmed that a Darcy–Stokes vugular medium homogenizes to a Darcy medium. But in the situations considered here the singular geometry of the problem with small $\epsilon > 0$ keeps all of the fluid in the channel very close to the interface where it is slowed by viscous resistance forces from the porous medium. This suggests that there is a very narrow region along the interface between Stokes flow and a porous medium where the fluid velocity is well approximated by a Brinkman law in the tangential coordinates. (Of course, the normal component of velocity is determined independently by the conservation of fluid mass across the interface.) The convergence of the ϵ-model established below provides an explanation for the success of numerical approximations that use an intermediate Brinkman system to connect Darcy and Stokes flows across an interface by adjusting the coefficients.

Our model describes two fundamental situations. The first is the rapid tangential flow near the boundary of a porous medium where the porosity becomes large due to the inefficiency of the packing of the particles of the medium. If the particles in this *boundary channel* are sufficiently sparse, the less impeded flow begins to follow this Stokes-like model in the substantial space between particles. See Nield & Bejan [27] for additional discussion and perspectives. The second and more common situation is obtained by reflecting Ω^ϵ about the outer wall of the channel, $\Gamma \times \{\epsilon\}$. This provides a model for a narrow *interior fracture* of width 2ϵ in a porous medium. (See Remark 1.) Such a fracture is assumed to be open, so fluid flow follows the Stokes system; debris-filled fractures have been modeled as regions of Darcy flow with very high permeability [10,22,24,25]. In the limiting problem below, the fracture is described by

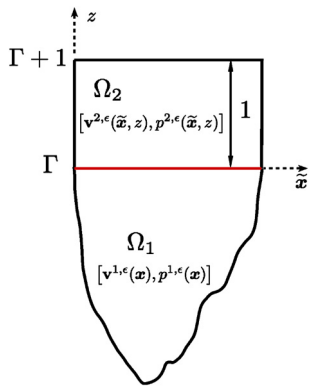


Fig. 2. The domain of reference for asymptotic analysis.

Brinkman flow in tangential coordinates coupled on *both* sides to the surrounding Darcy flow of the porous medium.

In this work we present the full asymptotic analysis for this coupled Darcy–Stokes system in order to derive a new model, free of singularities. The *limit problem* consists of a Darcy–Brinkman fully coupled system with Darcy flow on the original porous medium and Brinkman flow on the surface approximating the adjacent channel or internal fracture; see Fig. 2. The spaces of convergence will be found and the convergence of solutions will be established. It is worthwhile to stress that the method is remarkably simple with respect to other techniques as it uses only scaling, standard weak convergence methods and general Hilbert space theory. It is precisely this *simplicity* that gives the method its power and success in handling simultaneously the asymptotic analysis, the multiple scales and the substantially different structures of Darcy and Stokes systems. In particular, we obtain explicitly the correspondence between the coefficients in the Beavers–Joseph–Saffman interface condition and those in the limiting Brinkman system.

1.1. Notation

We shall use standard function spaces (see [32,1]). For any smooth bounded region G in $\mathbb{R}^N$ with boundary ∂G, the space of square integrable functions is denoted by $L^2(G)$, and the Sobolev space $H^1(G)$ consists of those functions in $L^2(G)$ for which each of the first-order weak partial derivatives belongs to $L^2(G)$. The *trace* is the continuous linear function $\gamma : H^1(G) \to L^2(\partial G)$ which agrees with restriction to the boundary, i.e., $\gamma(w) = w\big|_{\partial G}$ on smooth functions. Its kernel is $H^1_0(G) \stackrel{\mathbf{def}}{=} \{w \in H^1(G) : \gamma(w) = 0\}$. The trace space is $H^{1/2}(\partial G) \stackrel{\mathbf{def}}{=} \gamma(H^1(G))$, the range of γ endowed with the usual norm from the quotient space $H^1(G)/H^1_0(G)$, and we denote by $H^{-1/2}(\partial G)$ its topological dual. Column vectors and corresponding vector-valued functions will be denoted by boldface symbols, e.g., we denote the product space $\big[L^2(G)\big]^N$ by $\mathbf{L}^2(G)$ and the respective N-tuple of Sobolev spaces by $\mathbf{H}^1(G) \stackrel{\mathbf{def}}{=} \big[H^1(G)\big]^N$. Each $w \in H^1(G)$ has *gradient* $\boldsymbol{\nabla} w = \big(\frac{\partial w}{\partial x_1}, \dots, \frac{\partial w}{\partial x_N}\big) \in \mathbf{L}^2(G)$. We shall also use the space $\mathbf{H}_{\mathrm{div}}(G)$ of vector functions $\mathbf{w} \in \mathbf{L}^2(G)$ whose weak divergence $\boldsymbol{\nabla} \cdot \mathbf{w}$ belongs to $L^2(G)$. Let $\mathbf{n}$ be the unit outward normal vector on ∂G. If $\mathbf{w}$ is a vector function on ∂G, we denote its normal component by $w_n = \gamma(\mathbf{w}) \cdot \mathbf{n}$ and the normal projection by $w_n \mathbf{n}$. The tangential component is $\mathbf{w}^2_\tau = \mathbf{w} - w_n\, \mathbf{n}$. For the functions $\mathbf{w} \in \mathbf{H}_{\mathrm{div}}(G)$, there is a *normal trace* defined on the boundary values, which will be denoted by $\mathbf{w} \cdot \mathbf{n} \in H^{-1/2}(\partial G)$. For those $\mathbf{w} \in \mathbf{H}^1(G)$ this agrees with $\gamma(\mathbf{w}) \cdot \mathbf{n}$. Greek letters are used to denote general second-order tensors. The contraction of two tensors is given by $\sigma : \tau = \sum_{i,\,j} \sigma_{ij}\tau_{ij}$. For a tensor-valued function τ on ∂G, we denote the normal component (vector) by $\tau(\mathbf{n}) \stackrel{\mathbf{def}}{=} \sum_j \tau_{ij}\, \mathbf{n}_j \in \mathbb{R}^N$, and its normal and tangential parts by $\big(\tau(\mathbf{n})\big) \cdot \mathbf{n} = \tau_n \stackrel{\mathbf{def}}{=} \sum_{i,\,j} \tau_{ij} \mathbf{n}_i \mathbf{n}_j$ and $\tau(\mathbf{n})_\tau \stackrel{\mathbf{def}}{=} \tau(\mathbf{n}) - \tau_n \mathbf{n}$, respectively. For a vector function $\mathbf{w} \in \mathbf{H}^1(G)$, the tensor $(\boldsymbol{\nabla}\mathbf{w})_{ij} = \frac{\partial w_i}{\partial x_j}$ is the *gradient* of $\mathbf{w}$ and $\big(\boldsymbol{\mathcal{E}}(\mathbf{w})\big)_{ij} = \frac{1}{2}\big(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\big)$ is the *symmetric gradient*.

Next we describe the geometry of the domains to be used in the present work; see Fig. 1 for the case $N = 2$. The disjoint bounded domains Ω_1 and Ω_2^ϵ in $\mathbb{R}^N$ share the common *interface*, $\Gamma \stackrel{\mathbf{def}}{=} \partial\Omega_1 \cap \partial\Omega_2^\epsilon \subset \mathbb{R}^{N-1} \times \{0\}$, and we define $\Omega^\epsilon \stackrel{\mathbf{def}}{=} \Omega_1 \cup \Gamma \cup \Omega_2^\epsilon$. For simplicity of notation we have assumed that the interface is flat and, moreover, that the domain Ω_2^ϵ is a cylinder: $\Omega_2^\epsilon \stackrel{\mathbf{def}}{=} \Gamma \times (0, \epsilon)$. We denote by $\mathbf{n}(\cdot)$ the unit outward normal vector on $\partial\Omega_1$ and on $\partial\Omega_2^\epsilon - \Gamma$. The domain Ω_1 is the porous medium, and Ω_2^ϵ is the free fluid region. We focus on the case where Ω_2^ϵ is the lower half of a symmetric narrow horizontal fracture of width ϵ, $0 < \epsilon \ll 1$, and Ω_1 is the porous medium below the fracture. By modifying boundary conditions on $\Gamma + \epsilon$, we recover the case of a free-fluid region adjacent to (a flat part of) $\partial\Omega_1$.

For a column vector $\mathbf{x} = \big(x_1, \ldots, x_{N-1}, x_N\big) \in \mathbb{R}^N$ we denote the corresponding vector in $\mathbb{R}^{N-1}$ consisting of the first $N-1$ components by $\widetilde{\mathbf{x}} = \big(x_1, \ldots, x_{N-1}\big)$, and we identify $\mathbb{R}^{N-1} \times \{0\}$ with $\mathbb{R}^{N-1}$ by $\mathbf{x} = (\widetilde{\mathbf{x}}, x_N)$. For a vector function $\mathbf{w}$ on Γ we see $\mathbf{w}_T = \widetilde{\mathbf{w}}$ is the first $N-1$ components and $\mathbf{w_n} = w_N$ is the last component of the function. The operators $\boldsymbol{\nabla}_T$, $\boldsymbol{\nabla}_T\cdot$ denote respectively the $\mathbb{R}^{N-1}$-gradient and the $\mathbb{R}^{N-1}$-divergence in directions tangent to Γ, i.e. $\boldsymbol{\nabla}_T = \big(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{N-1}}\big)$, $\boldsymbol{\nabla}_T\cdot = \big(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{N-1}}\big)\cdot$.

1.2. The equations

We determine the fluid flow through the porous medium Ω_1 by the *Darcy system*, i.e.

$$\boldsymbol{\nabla} \cdot \mathbf{v}^1 = h_1\,, \tag{1a}$$

$$\mathcal{Q}\,\mathbf{v}^1 + \boldsymbol{\nabla} p^1 = \mathbf{0}\,, \quad \text{in}\,\Omega_1. \tag{1b}$$

The functions p^1, $\mathbf{v}^1$ are respectively, the *pressure* and *filtration velocity* of the incompressible viscous fluid in the pores. The resistance tensor $\mathcal{Q}$ is the shear viscosity μ of the fluid times the reciprocal of the *permeability* of the structure. The flow of the fluid in the adjacent open channel Ω_2^ϵ is described by the *Stokes system* [32,28]

$$\boldsymbol{\nabla} \cdot \mathbf{v}^2 = 0, \tag{2a}$$

$$-\boldsymbol{\nabla} \cdot \sigma^2 + \boldsymbol{\nabla} p^2 = \mathbf{f}_2, \tag{2b}$$

$$\sigma^2 = 2\,\epsilon\,\mu\,\boldsymbol{\mathcal{E}}(\mathbf{v}^2)\,, \quad \text{in}\,\Omega_2^\epsilon. \tag{2c}$$

Here, $\mathbf{v}^2$, σ^2, p^2 are respectively, the *velocity*, *stress tensor* and the *pressure* of the fluid in Ω_2^ϵ, while $\boldsymbol{\mathcal{E}}(\mathbf{v}^2)$ denotes the symmetric gradient of the velocity field. Among the equations above, only (1b) and (2c) are constitutive and subject to scaling. Darcy's law (1b) describes the fluid on the part of the domain with fixed geometry, hence, it is not scaled. The law (2c) establishes the relationship between the strain rate and the stress for the fluid in the thin channel, therefore it is scaled according to the geometry. Finally, recalling that $\boldsymbol{\nabla} \cdot \mathbf{v}^2 = 0$, we have

$$\boldsymbol{\nabla} \cdot \sigma^2 = \boldsymbol{\nabla} \cdot \big[2\,\epsilon\,\mu\,\boldsymbol{\mathcal{E}}\big(\mathbf{v}^2\big)\big] = \epsilon\,\mu \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \mathbf{v}^2. \tag{3}$$

This observation transforms the system (2a), (2b) and (2c) to the classical form of *Stokes flow* system.

1.3. Interface conditions

The interface coupling conditions account for the stress and mass conservation. For the stress balance, the tangential and normal components are given by the *Beavers–Joseph–Saffman* (4a) and the classical *Robin* boundary condition (4b) respectively, i.e.

$$\sigma_T^2 = \epsilon^2 \beta \sqrt{\mathcal{Q}}\, \mathbf{v}_T^2 \,, \tag{4a}$$

$$\sigma_{\mathbf{n}}^2 - p^2 + p^1 = \alpha\, \mathbf{v}^1 \cdot \mathbf{n} \text{ on } \Gamma. \tag{4b}$$

In the expression (4a) above, ϵ^2 is a scaling factor destined to balance out the geometric singularity introduced by the thinness of the channel. In addition, the coefficient $\alpha \geq 0$ in (4b) is the *fluid entry resistance*. In the present work it is assumed that the velocity is curl-free on the interface, so the conditions (4a) and (4b) are equivalent to

$$\epsilon\, \mu \frac{\partial}{\partial \mathbf{n}} \mathbf{v}_T^2 = \epsilon\, \mu \frac{\partial}{\partial x_N} \mathbf{v}_T^2 = \epsilon^2 \beta \sqrt{\mathcal{Q}}\, \mathbf{v}_T^2 \,, \tag{5a}$$

$$\epsilon\, \mu \Big(\frac{\partial \mathbf{v}^2}{\partial \mathbf{n}} \cdot \mathbf{n} \Big) - p^2 + p^1 = \epsilon\, \mu \frac{\partial \mathbf{v}_N^2}{\partial x_N} - p^2 + p^1 = \alpha\, \mathbf{v}^1 \cdot \mathbf{n} \text{ on } \Gamma. \tag{5b}$$

The conservation of fluid across the interface gives the normal fluid flow balance

$$\mathbf{v}^1 \cdot \mathbf{n} = \mathbf{v}^2 \cdot \mathbf{n} \text{ on } \Gamma. \tag{5c}$$

The *interface conditions* (5) will suffice precisely to couple the Darcy system (1) in Ω_1 to the Stokes system (2) in Ω_2^ϵ.

In the homogenization to Darcy flow of a Stokes-solid region with length scale ϵ, both viscosity μ and permeability K are scaled by ϵ^2. This maintains the flow rate while the volume of pore space decreases by ϵ^3 (see [21]). Here, only the width of the fracture is scaled by ϵ. In order to maintain the flow rate and two-way coupling, it is sufficient to scale the viscosity and permeability by ϵ, so the ratio $\mathcal{Q}$ remains constant, and the tangential friction by an extra power, ϵ^2. Therefore, the scaling of Equations (2c)/(3) and (4a) follows.

1.4. Boundary conditions

We choose the *boundary conditions* on $\partial\Omega^\epsilon = \partial\Omega_1 \cup \partial\Omega_2^\epsilon - \Gamma$ in a classical simple form, since they play no essential role here. On the exterior boundary of the porous medium, $\partial\Omega_1 - \Gamma$, we impose the *drained* conditions,

$$p^1 = 0 \quad \text{on } \partial\Omega_1 - \Gamma. \tag{6a}$$

As for the exterior boundary of the free fluid, $\partial\Omega_2^\epsilon - \Gamma$, we choose no-slip conditions on the wall of the cylinder,

$$\mathbf{v}^2 = 0 \quad \text{on } \partial\Gamma \times (0, \epsilon). \tag{6b}$$

On the top of the cylinder $\Gamma + \epsilon \stackrel{\mathbf{def}}{=} \big\{ (\widetilde{x}, \epsilon) : \widetilde{x} \in \Gamma \big\}$, the hyper-plane of symmetry, we have mixed boundary conditions, a Neumann-type condition on the tangential component of the normal stress

$$\frac{\partial \mathbf{v}^2}{\partial \mathbf{n}} - \Big(\frac{\partial \mathbf{v}^2}{\partial \mathbf{n}} \cdot \mathbf{n} \Big) \mathbf{n} = \frac{\partial \mathbf{v}_T^2}{\partial x_N} = 0 \quad \text{on } \Gamma + \epsilon, \tag{6c}$$

and a null normal flux condition, i.e.

$$\mathbf{v}^2 \cdot \mathbf{n} = \mathbf{v}_N^2 = 0 \quad \text{on } \Gamma + \epsilon. \tag{6d}$$

Remark 1. The boundary conditions (6b) and (6c) are appropriate for the mid-line of an *internal fracture* with symmetric geometry. In that case, the interface conditions (5) hold on both sides of the fracture. If Ω_2^ϵ

is an adjacent open channel along the boundary of Ω_1, then we extend the no-slip condition (6b) to hold on all of $\partial\Omega_2^\epsilon - \Gamma$.

Remark 2. For a detailed exposition on the system's adopted scaling namely, the fluid stress tensor (2c) and the Beavers–Joseph–Saffman condition (4a), together with the **formal** asymptotic analysis see [23].

1.5. Preliminary results

We close this section by recalling some classic results.

Lemma 1. *Let $G \subset \mathbb{R}^N$ be an open set with Lipschitz boundary, let $\mathbf{n}$ be the unit outward normal vector on ∂G. The normal trace operator $\mathbf{u} \in \mathbf{H}_{\mathrm{div}}(G) \mapsto \mathbf{u}\cdot\mathbf{n} \in H^{-1/2}(\partial G)$ is defined by*

$$\left\langle \mathbf{u}\cdot\mathbf{n}, \phi \right\rangle_{H^{-1/2}(\partial G),\, H^{1/2}(\partial G)} \overset{\mathbf{def}}{=} \int_G \left(\mathbf{u}\cdot\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\cdot\mathbf{u}\,\phi \right) dx, \quad \phi \in H^1(G). \tag{7}$$

For any $g \in H^{-1/2}(\partial G)$ there exists $\mathbf{u} \in \mathbf{H}_{\mathrm{div}}(G)$ such that $\mathbf{u}\cdot\mathbf{n} = g$ on ∂G and $\|\mathbf{u}\|_{\mathbf{H}_{\mathrm{div}}(G)} \le K\|g\|_{H^{-1/2}(\partial G)}$, with K depending only on the domain G. In particular, if g belongs to $L^2(\partial G)$, the function $\mathbf{u}$ satisfies the estimate $\|\mathbf{u}\|_{\mathbf{H}_{\mathrm{div}}(G)} \le K\|g\|_{0,\partial G}$.

Proof. See Lemma 20.2 in [31]. □

We shall recall in Section 2 that the boundary-value problem consisting of the Darcy system (1), the Stokes system (2), the interface coupling conditions (5) and the boundary conditions (6) can be formulated as a constrained minimization problem. Let $\mathbf{X}$ and $\mathbf{Y}$ be Hilbert spaces and let $\mathcal{A} : \mathbf{X} \to \mathbf{X}'$, $\mathcal{B} : \mathbf{X} \to \mathbf{Y}'$ and $\mathcal{C} : \mathbf{Y} \to \mathbf{Y}'$ be continuous linear operators. The problem is to find a pair satisfying

$$\begin{aligned} (\mathbf{x},\mathbf{y}) \in \mathbf{X}\times\mathbf{Y} : \quad & \mathcal{A}\mathbf{x} + \mathcal{B}'\mathbf{y} = F_1 \quad \text{in } \mathbf{X}', \\ & -\mathcal{B}\mathbf{x} + \mathcal{C}\mathbf{y} = F_2 \quad \text{in } \mathbf{Y}' \end{aligned} \tag{8}$$

with $F_1 \in \mathbf{X}'$ and $F_2 \in \mathbf{Y}'$. We present a well-known result [15] to be used in this work.

Theorem 2. *Assume that the linear operators $\mathcal{A} : \mathbf{X} \to \mathbf{X}'$, $\mathcal{B} : \mathbf{X} \to \mathbf{Y}'$, $\mathcal{C} : \mathbf{Y} \to \mathbf{Y}'$ are continuous and*

(i) *$\mathcal{A}$ is non-negative and $\mathbf{X}$-coercive on* $\ker(\mathcal{B})$,
(ii) *$\mathcal{B}$ satisfies the inf-sup condition*

$$\inf_{\mathbf{y}\in\mathbf{Y}} \sup_{\mathbf{x}\in\mathbf{X}} \frac{|\mathcal{B}\mathbf{x}(\mathbf{y})|}{\|\mathbf{x}\|_{\mathbf{X}}\,\|\mathbf{y}\|_{\mathbf{Y}}} > 0\,, \tag{9}$$

(iii) *C is non-negative and symmetric.*

Then, for every $F_1 \in \mathbf{X}'$ and $F_2 \in \mathbf{Y}'$ the problem (8) has a unique solution $(\mathbf{x},\mathbf{y}) \in \mathbf{X}\times\mathbf{Y}$, and it satisfies the estimate

$$\|\mathbf{x}\|_{\mathbf{X}} + \|\mathbf{y}\|_{\mathbf{Y}} \le c\left(\|F_1\|_{\mathbf{X}'} + \|F_2\|_{\mathbf{Y}'}\right) \tag{10}$$

for a positive constant c depending only on the preceding assumptions on $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$.

Several variations of such systems have been extensively developed, e.g., see [30] for nonlinear degenerate and time-dependent cases.

2. A well-posed formulation

In this section we present a mixed formulation for the problem on the domain Ω^ϵ described in Section 1 and show it is well-posed. In order to remove the dependence of the domain Ω^ϵ on the parameter $\epsilon > 0$, we rescale Ω_2^ϵ and get an equivalent problem on the domain Ω^1.

The abstract problem is built on the function spaces

$$\mathbf{X}_2^\epsilon \stackrel{\mathbf{def}}{=} \left\{\mathbf{v} \in \mathbf{H^1}(\Omega_2^\epsilon) : \mathbf{v} = 0 \text{ on } \partial\Gamma \times (0,\epsilon),\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma + \epsilon\right\}, \tag{11a}$$

$$\mathbf{X}^\epsilon \stackrel{\mathbf{def}}{=} \left\{[\mathbf{v}^1, \mathbf{v}^2] \in \mathbf{H}_{\mathrm{div}}(\Omega_1) \times \mathbf{X}_2^\epsilon : \mathbf{v}^1\cdot\mathbf{n} = \mathbf{v}^2\cdot\mathbf{n} \text{ on } \Gamma\right\} = \left\{\mathbf{v} \in \mathbf{H}_{\mathrm{div}}(\Omega^\epsilon) : \mathbf{v}^2 \in \mathbf{X}_2^\epsilon\right\}, \tag{11b}$$

$$\mathbf{Y}^\epsilon \stackrel{\mathbf{def}}{=} L^2(\Omega^\epsilon), \tag{11c}$$

endowed with their respective natural norms. We shall use the following hypothesis.

Hypothesis 1. It will be assumed that $\mu > 0$ and the coefficients β and α are nonnegative and bounded almost everywhere. Moreover, the tensor $\mathcal{Q}$ is elliptic, i.e., there exists a $C_{\mathcal{Q}} > 0$ such that $(\mathcal{Q}\,\mathbf{x})\cdot\mathbf{x} \geq C_{\mathcal{Q}}\|\mathbf{x}\|^2$ for all $\mathbf{x} \in \mathbb{R}^N$.

Proposition 3. *The boundary-value problem consisting of the equations (1), (2), the interface coupling conditions (5) and the boundary conditions (6) has the constrained variational formulation*

$\left[\mathbf{v}^\epsilon, p^\epsilon\right] \in \mathbf{X}^\epsilon \times \mathbf{Y}^\epsilon$:

$$\int_{\Omega_1} \left(\mathcal{Q}\,\mathbf{v}^{1,\epsilon}\cdot\mathbf{w}^1 - p^{1,\epsilon}\,\boldsymbol{\nabla}\cdot\mathbf{w}^1\right) dx + \int_{\Omega_2^\epsilon} \left(\epsilon\,\mu\boldsymbol{\nabla}\,\mathbf{v}^{2,\epsilon} - p^{2,\epsilon}\delta\ \right) : \boldsymbol{\nabla}\mathbf{w}^2\, d\widetilde{x}dx_N \tag{12a}$$

$$+\alpha\int_\Gamma \left(\mathbf{v}^{2,\epsilon}\cdot\mathbf{n}\right)\left(\mathbf{w}^2\cdot\mathbf{n}\right) dS + \int_\Gamma \epsilon^2\,\beta\,\sqrt{\mathcal{Q}}\,\mathbf{v}_T^{2,\epsilon}\cdot\mathbf{w}_T^2\, dS = \int_{\Omega_2^\epsilon} \mathbf{f}^{2,\epsilon}\cdot\mathbf{w}^2\, d\widetilde{x}\, dx_N,$$

$$\int_{\Omega_1} \boldsymbol{\nabla}\cdot\mathbf{v}^{1,\epsilon}\,\varphi^1\, dx + \int_{\Omega_2^\epsilon} \boldsymbol{\nabla}\cdot\mathbf{v}^{2,\epsilon}\,\varphi^2\, d\widetilde{x}\, dx_N = \int_{\Omega_1} h^{1,\epsilon}\,\varphi^1\, dx, \tag{12b}$$

for all $\left[\mathbf{w}, \varphi\right] \in \mathbf{X}^\epsilon \times \mathbf{Y}^\epsilon$.

Proof. Let $\mathbf{v}^\epsilon = \left[\mathbf{v}^{1,\epsilon}, \mathbf{v}^{2,\epsilon}\right]$, $p^\epsilon = \left[p^{1,\epsilon}, p^{2,\epsilon}\right]$ be a solution and choose a test function $\mathbf{w} = [\mathbf{w}^1, \mathbf{w}^2] \in \mathbf{X}^\epsilon$. Substitute the relationship (3) in the momentum equation (2b) and multiply the outcome by $\mathbf{w}^2$. Multiply the Darcy law (1b) by $\mathbf{w}^1$. Integrating both expressions and adding them together, we obtain

$$\int_{\Omega_1} \left(\mathcal{Q}\,\mathbf{v}^{1,\epsilon}\cdot\mathbf{w}^1 - p^{1,\epsilon}\,\delta : \boldsymbol{\mathcal{E}}\left(\mathbf{w}^1\right)\right) dx + \int_{\Omega_2^\epsilon} \left(\epsilon\,\mu\boldsymbol{\nabla}\mathbf{v}^2 - p^{2,\epsilon}\,\delta\right) : \boldsymbol{\nabla}\mathbf{w}^2\, dx$$

$$+\int_\Gamma \left(p^{1,\epsilon}\,\mathbf{n}\cdot\mathbf{w}^1 + \epsilon\left(\boldsymbol{\nabla}\,\mathbf{v}^{2,\epsilon}\,(\mathbf{n})\right)\cdot\mathbf{w}^2 - p^{2,\epsilon}\left(\mathbf{w}^2\cdot\mathbf{n}\right)\right) dS = \int_{\Omega_2^\epsilon} \mathbf{f}^2\cdot\mathbf{w}^2\, dx. \tag{13}$$

Since $\mathbf{w}$ satisfies the admissibility constraint (5c), $\mathbf{w}^1 \cdot \mathbf{n} = \mathbf{w}^2 \cdot \mathbf{n}$ on Γ, the interface integral reduces to

$$\int_\Gamma \Big(\epsilon \frac{\partial \mathbf{v}^{2,\epsilon}}{\partial \mathbf{n}} \cdot \mathbf{w}^2 + \big(p^{1,\epsilon} - p^{2,\epsilon}\big) \big(\mathbf{w}^1 \cdot \mathbf{n}\big) \Big)\, dS.$$

Decomposing the velocity terms into their normal and tangential components, we obtain

$$\int_\Gamma \Big\{ \epsilon \Big(\frac{\partial \mathbf{v}^{2,\epsilon}}{\partial \mathbf{n}} \Big)_T \cdot \mathbf{w}^2_T + \Big(\epsilon \Big(\frac{\partial \mathbf{v}^{2,\epsilon}}{\partial \mathbf{n}} \cdot \mathbf{n} \Big) + p^{1,\epsilon} - p^{2,\epsilon} \Big) \big(\mathbf{w}^2 \cdot \mathbf{n}\big) \Big\}\, dS.$$

Therefore, the interface conditions (5a) and (5b) yield

$$\int_\Gamma \epsilon^2 \beta \sqrt{\mathcal{Q}}\, \mathbf{v}^{2,\epsilon}_T \cdot \mathbf{w}^2_T\, dS + \alpha \int_\Gamma \big(\mathbf{v}^{1,\epsilon} \cdot \mathbf{n}\big) \big(\mathbf{w}^1 \cdot \mathbf{n}\big)\, dS,$$

and inserting this in (13) yields the variational statement (12a). Next, multiply the fluid conservation equations with a test function $\varphi = [\varphi^1, \varphi^2] \in L^2(\Omega^\epsilon)$, integrate over the corresponding regions and add them together to obtain the variational statement (12b). Conversely, by making appropriate choices of test functions in (12) and reversing the preceding calculations, it follows that these formulations are equivalent. □

2.1. The mixed formulation

Define the operators $A^\epsilon : \mathbf{X}^\epsilon \to (\mathbf{X}^\epsilon)'$, $B^\epsilon : \mathbf{X}^\epsilon \to (\mathbf{Y}^\epsilon)'$ by

$$A^\epsilon \mathbf{v}(\mathbf{w}) \stackrel{\mathbf{def}}{=} \int_{\Omega_1} \big(\mathcal{Q}\, \mathbf{v}^1 \cdot \mathbf{w}^1\big)\, dx + \alpha \int_\Gamma \big(\mathbf{v}^1 \cdot \mathbf{n}\big) \big(\mathbf{w}^1 \cdot \mathbf{n}\big)\, dS + \int_\Gamma \epsilon^2 \beta \sqrt{\mathcal{Q}} \mathbf{v}^2_T \cdot \mathbf{w}^2_T\, dS + \int_{\Omega_2^\epsilon} \big(\epsilon\, \mu \boldsymbol{\nabla}\, \mathbf{v}^2 : \boldsymbol{\nabla} \mathbf{w}^2\big)\, d\widetilde{x} dx_N\,, \tag{14a}$$

$$B^\epsilon \mathbf{v}(\varphi) \stackrel{\mathbf{def}}{=} \int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{v}^1\, \varphi^1\, dx + \int_{\Omega_2^\epsilon} \boldsymbol{\nabla} \cdot \mathbf{v}^2\, \varphi^2\, d\widetilde{x}\, dx_N, \tag{14b}$$

for all $\mathbf{v}, \mathbf{w} \in \mathbf{X}^\epsilon$, $\varphi \in \mathbf{Y}^\epsilon$.

These are denoted also by matrix operators

$$A^\epsilon = \begin{pmatrix} \mathcal{Q} + \gamma'_{\mathbf{n}}\, \alpha\, \gamma_{\mathbf{n}} & \mathbf{0} \\ \mathbf{0} & \epsilon^2\, \gamma'_T\, \beta \sqrt{\mathcal{Q}}\, \gamma_T + \epsilon\, (\boldsymbol{\nabla})'\, \mu\, \boldsymbol{\nabla} \end{pmatrix} \tag{15a}$$

and

$$B^\epsilon = \begin{pmatrix} \boldsymbol{\nabla} \cdot & 0 \\ 0 & \boldsymbol{\nabla} \cdot \end{pmatrix} = \begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix}. \tag{15b}$$

With these operators, the variational formulation (12) for the boundary-value problem takes the form

$$\begin{aligned} [\mathbf{v}^\epsilon, p^\epsilon] \in \mathbf{X}^\epsilon \times \mathbf{Y}^\epsilon : A^\epsilon\, \mathbf{v}^\epsilon - (B^\epsilon)'\, p^\epsilon &= \mathbf{f}^{2,\epsilon}, \\ B^\epsilon\, \mathbf{v}^\epsilon &= h^{1,\epsilon}. \end{aligned} \tag{16}$$

Here, the unknowns are $\mathbf{v}^\epsilon \stackrel{\mathbf{def}}{=} [\mathbf{v}^{1,\epsilon}, \mathbf{v}^{2,\epsilon}] \in \mathbf{X}^\epsilon$, $p^\epsilon \stackrel{\mathbf{def}}{=} [p^{1,\epsilon}, p^{2,\epsilon}] \in \mathbf{Y}^\epsilon$. Next, we show that the Problem (16) is well-posed by verifying that the hypotheses of Theorem 2 are satisfied.

Lemma 4. *The operator A^ϵ is $\mathbf{X}^\epsilon$-coercive over $\mathbf{X}^\epsilon \cap \ker(B^\epsilon)$.*

Proof. The form $A^\epsilon\,\mathbf{v}\,(\mathbf{v}) + \int\limits_{\Omega_1} \big(\boldsymbol{\nabla}\cdot\mathbf{v}\big)^2$ is $\mathbf{X}^\epsilon$-coercive, and $\boldsymbol{\nabla}\cdot\mathbf{v}\,|_{\Omega_1} = 0$ whenever $\mathbf{v} \in \ker(B^\epsilon)$. □

In order to verify the inf-sup condition for the operator B^ϵ we introduce the space

$$\mathbf{F}(\Omega^\epsilon) \stackrel{\mathbf{def}}{=} \big\{\mathbf{v} \in \mathbf{H^1}(\Omega^\epsilon) : \mathbf{v} = 0 \text{ on } \partial\Omega_2^\epsilon - \Gamma\big\}, \tag{17}$$

endowed with the $\mathbf{H^1}(\Omega^\epsilon)$-norm.

Lemma 5. *The operator B^ϵ has closed range.*

Proof. Since $\mathbf{F}(\Omega^\epsilon) \subseteq \mathbf{X}^\epsilon$ and the Poincaré inequality gives a constant $C > 0$ such that $\|\mathbf{v}\|_{\mathbf{X}} \le C\|\mathbf{v}\|_{\mathbf{H}^1(\Omega^\epsilon)}$ for all $\mathbf{v} \in \mathbf{F}(\Omega^\epsilon)$, we have

$$\begin{aligned}\inf_{\varphi\in L^2(\Omega^\epsilon)} \sup_{\mathbf{v}\in\mathbf{X}} \frac{B^\epsilon\,\mathbf{v}\,(\varphi)}{\|\mathbf{v}\|_{\mathbf{X}}\|\varphi\|_{L^2(\Omega^\epsilon)}} &\ge \inf_{\varphi\in L^2(\Omega^\epsilon)} \sup_{\mathbf{v}\in\mathbf{F}(\Omega^\epsilon)} \frac{B^\epsilon\,\mathbf{v}\,(\varphi)}{\|\mathbf{v}\|_{\mathbf{X}}\|\varphi\|_{L^2(\Omega^\epsilon)}} \\ &\ge \frac{1}{C} \inf_{\varphi\in L^2(\Omega^\epsilon)} \sup_{\mathbf{v}\in\mathbf{F}(\Omega^\epsilon)} \frac{B^\epsilon\,\mathbf{v}\,(\varphi)}{\|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega^\epsilon)}\|\varphi\|_{L^2(\Omega^\epsilon)}}.\end{aligned} \tag{18}$$

The last term above is known to be positive (see Theorem 3.7 in [15]), since it corresponds to the inf-sup condition for the Stokes problem with mixed boundary conditions:

$$-\boldsymbol{\nabla}\cdot\big(\mu\,\boldsymbol{\nabla}\mathbf{w}\big) + \boldsymbol{\nabla}q = \mathbf{g}_1\,, \qquad \boldsymbol{\nabla}\cdot\mathbf{w} = 0 \ \text{ in } \ \Omega^\epsilon, \tag{19}$$

$$\mathbf{w} = 0 \ \text{ on } \ \partial\Omega_2^\epsilon - \Gamma\,, \qquad \mu\,\frac{\partial\mathbf{w}}{\partial\mathbf{n}} - q\,\mathbf{n} = \mathbf{g}_2 \ \text{ on } \ \partial\Omega^\epsilon - \partial\Omega_2^\epsilon, \tag{20}$$

and forcing terms $\mathbf{g}_1$ and $\mathbf{g}_2$ satisfying the necessary hypotheses of duality. □

Theorem 6. *The Problem (16) is well-posed.*

Proof. Due to Lemmas 4 and 5 above, the operators A^ϵ and B^ϵ satisfy the hypotheses of Theorem 2 and the result follows. □

2.2. The reference domain

The solutions $\big\{\big[\mathbf{v}^\epsilon, p^\epsilon\big] : \epsilon > 0\big\}$ to the Problem (12) (equivalently Problem (16)), have different geometric domains of definition and therefore no convergence statements can be stated. In addition, the a priori estimates given from the well-posedness of the Problem (16) depend on the geometry of the domain where the problem is defined. Therefore, a domain of reference will be established; since the only part that is changing is the thickness of the channel, this suffices for the appropriate change of variable. Given $\mathbf{x} = (\widetilde{\boldsymbol{x}}, x_N) \in \Omega_2^\epsilon$, define $x_N = \epsilon\,z$, hence $\frac{\partial}{\partial x_N} = \frac{1}{\epsilon}\frac{\partial}{\partial z}$, see Fig. 2. For any $\mathbf{w} \in \mathbf{X}_2^\epsilon$ we have the following changes on the structure of the gradient and the divergence, respectively,

$$\boldsymbol{\nabla}\,\mathbf{w}\left(\widetilde{\boldsymbol{x}}, x_N\right) = \begin{pmatrix} [\boldsymbol{\nabla}_T\,\mathbf{w}_T] & \dfrac{1}{\epsilon}\,\partial_z \mathbf{w}_T \\ \left(\boldsymbol{\nabla}_T\,\mathbf{w}_N\right)' & \dfrac{1}{\epsilon}\,\partial_z \mathbf{w}_N \end{pmatrix}(\widetilde{x}, z), \tag{21a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{w}\left(\widetilde{\boldsymbol{x}}, x_N\right) = \left(\boldsymbol{\nabla}_T\cdot\,\mathbf{w}_T + \frac{1}{\epsilon}\,\partial_z \mathbf{w}_N\right)(\widetilde{x}, z). \tag{21b}$$

Taking in consideration (21a), (21b) and combining it with (12) we obtain a family of ϵ-problems in a common domain of definition (see Fig. 1) given by $\Omega \overset{\mathbf{def}}{=} \Omega_1 \cup \Omega_2$, where $\Omega_1, \Omega_2 \subseteq \mathbb{R}^3$ are bounded open sets, with $\Omega_2 \overset{\mathbf{def}}{=} \Gamma \times (0,1)$ and $\Gamma = \partial\Omega_1 \cap \partial\Omega_2 \subseteq \mathbb{R}^2$. Letting $\Gamma + 1 \overset{\mathbf{def}}{=} \left\{(\widetilde{\boldsymbol{x}}, 1) : \widetilde{\boldsymbol{x}} \in \Gamma\right\}$, the functional setting is now independent of ϵ and defined by

$$\begin{gathered} \mathbf{X}_2 \overset{\mathbf{def}}{=} \left\{\mathbf{v} \in \mathbf{H^1}(\Omega_2) : \mathbf{v} = \mathbf{0} \text{ on } \partial\Gamma \times (0,1),\ \ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma + 1\right\}, \\ \mathbf{X} \overset{\mathbf{def}}{=} \left\{[\mathbf{v}^1, \mathbf{v}^2] \in \mathbf{H}_{\mathrm{div}}(\Omega_1) \times \mathbf{X}_2 : \mathbf{v}^1\cdot\mathbf{n} = \mathbf{v}^2\cdot\mathbf{n} \text{ on } \Gamma\right\} = \left\{\mathbf{v} \in \mathbf{H}_{\mathrm{div}}(\Omega) : \mathbf{v}^2 \in \mathbf{X}_2\right\}, \\ \mathbf{Y} \overset{\mathbf{def}}{=} L^2(\Omega). \end{gathered} \tag{22}$$

Moreover, we have the following result.

Proposition 7. *Under the change of variable* $(\widetilde{x}, x_N) \mapsto (\widetilde{x}, x_N)\mathbb{1}_{\Omega_1} + (\widetilde{x}, \epsilon\, z)\mathbb{1}_{\Omega_2}$, *the Problem* (12) *is equivalent to the corresponding problem*

$[\mathbf{v}^\epsilon, p^\epsilon] \in \mathbf{X} \times \mathbf{Y}$:

$$\begin{aligned} &\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^{1,\epsilon}\cdot\mathbf{w}^1\,dx - \int_{\Omega_1} p^{1,\epsilon}\,\boldsymbol{\nabla}\cdot\mathbf{w}^1\,dx - \epsilon\int_{\Omega_2} p^{2,\epsilon}\,\boldsymbol{\nabla}_T\cdot\mathbf{w}^2_T\,d\widetilde{x}\,dz - \int_{\Omega_2} p^{2,\epsilon}\,\partial_z\mathbf{w}^2_N\,d\widetilde{x}\,dz \\ &\qquad + \epsilon^2\int_{\Omega_2} \mu\boldsymbol{\nabla}_T\,\mathbf{v}^{2,\epsilon}_T : \boldsymbol{\nabla}_T\,\mathbf{w}^2_T\,d\widetilde{x}\,dz + \int_{\Omega_2} \mu\,\partial_z\mathbf{v}^{2,\epsilon}_T\cdot\,\partial_z\mathbf{w}^2_T\,d\widetilde{x}\,dz \\ &\qquad + \epsilon^2\int_{\Omega_2} \mu\,\boldsymbol{\nabla}_T\mathbf{v}^{2,\epsilon}_N\cdot\boldsymbol{\nabla}_T\mathbf{w}^2_N\,d\widetilde{x}\,dz + \int_{\Omega_2} \mu\,\partial_z\mathbf{v}^{2,\epsilon}_N\,\partial_z\mathbf{w}^2_N\,d\widetilde{x}\,dz \\ &\qquad + \alpha\int_{\Gamma} \left(\mathbf{v}^{1,\epsilon}\cdot\mathbf{n}\right)\left(\mathbf{w}^1\cdot\mathbf{n}\right)dS + \epsilon^2\int_{\Gamma} \beta\sqrt{\mathcal{Q}}\,\mathbf{v}^{2,\epsilon}_T\cdot\mathbf{w}^2_T\,dS = \epsilon\int_{\Omega_2} \mathbf{f}^{2,\epsilon}\cdot\mathbf{w}^2\,d\widetilde{x}\,dz, \end{aligned} \tag{23a}$$

$$\int_{\Omega_1} \boldsymbol{\nabla}\cdot\mathbf{v}^{1,\epsilon}\varphi^1\,dx + \epsilon\int_{\Omega_2} \boldsymbol{\nabla}_T\cdot\mathbf{v}^{2,\epsilon}_T\,\varphi^2\,d\widetilde{x}\,dz + \int_{\Omega_2} \partial_z\mathbf{v}^{2,\epsilon}_N\,\varphi^2\,d\widetilde{x}\,dz = \int_{\Omega_1} h^{1,\epsilon}\,\varphi^1\,dx,$$

$$\textit{for all } [\mathbf{w}, \Phi] \in \mathbf{X} \times \mathbf{Y}, \tag{23b}$$

defined on the common domain of reference Ω.

Proof. The proof follows from direct substitution together with the identities (21a) and (21b). □

Remark 3. In order to avoid overloaded notation, from now on, we denote the volume integrals by $\int_{\Omega_1} F = \int_{\Omega_1} F\,dx$ and $\int_{\Omega_2} F = \int_{\Omega_2} F\,d\widetilde{x}\,dz$. The explicit notation $\int_{\Omega_2} F\,d\widetilde{x}\,dz$ will be used only for those cases where specific calculations are needed. Both notations will be clear from the context.

Proposition 8. *The Problem (23) is a weak formulation of the strong form*

$$\mathcal{Q}\,\mathbf{v}^{1,\epsilon} + \boldsymbol{\nabla} p^{1,\epsilon} = \mathbf{0}\,, \tag{24a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{v}^{1,\epsilon} = h^{1,\epsilon} \quad in \quad \Omega_1, \tag{24b}$$

$$\epsilon\,\boldsymbol{\nabla}_T p^{2,\epsilon} - \epsilon^2\,\boldsymbol{\nabla}_T\cdot\mu\boldsymbol{\nabla}_T\,\mathbf{v}_T^{2,\epsilon} - \partial_z\,\mu\,\partial_z\,\mathbf{v}_T^{2,\epsilon} = \epsilon\,\mathbf{f}_T^{2,\,\epsilon}\,, \tag{24c}$$

$$\partial_z\,p^{2,\epsilon} - \epsilon^2\,\boldsymbol{\nabla}_T\,\cdot\,\mu\boldsymbol{\nabla}_T\mathbf{v}_N^{2,\epsilon} - \partial_z\,\mu\,\partial_z\mathbf{v}_N^{2,\epsilon} = \epsilon\,\mathbf{f}_N^{2,\,\epsilon}\,, \tag{24d}$$

$$\epsilon\,\boldsymbol{\nabla}_T\cdot\mathbf{v}_T^{2,\epsilon} + \partial_z\mathbf{v}_N^{2,\epsilon} = 0 \quad in \quad \Omega_2, \tag{24e}$$

$$\epsilon\,\mu\,\partial_z\,\mathbf{v}_N^{2,\epsilon} - p^{2,\epsilon} + p^{1,\epsilon} = \alpha\,\mathbf{v}^{1,\epsilon}\cdot\mathbf{n}\,, \tag{24f}$$

$$\epsilon\,\mu\,\frac{\partial\,\mathbf{v}_\tau^{2,\epsilon}}{\partial\,\mathbf{n}} = \epsilon\,\mu\,\partial_z\,\mathbf{v}_T^{2,\epsilon} = \epsilon^2\beta\,\sqrt{\mathcal{Q}}\,\mathbf{v}_T^{2,\epsilon}\,, \tag{24g}$$

$$\mathbf{v}^{1,\epsilon}\cdot\mathbf{n} = \mathbf{v}^{2,\epsilon}\cdot\mathbf{n} \quad on \quad \Gamma, \tag{24h}$$

$$p^{1,\epsilon} = 0 \quad on \quad \partial\Omega_1 - \Gamma, \tag{24i}$$

$$\mathbf{v}^{2,\epsilon} = \mathbf{0} \quad on \quad \partial\Gamma\times(0,1), \tag{24j}$$

$$\mathbf{v}^{2,\epsilon}\cdot\mathbf{n} = \mathbf{v}_N^{2,\epsilon} = 0\,, \tag{24k}$$

$$\mu\,\frac{\partial\,\mathbf{v}_\tau^{2,\epsilon}}{\partial\,\mathbf{n}} = \mu\,\partial_z\,\mathbf{v}_T^{2,\epsilon} = \mathbf{0} \quad on \quad \Gamma + 1. \tag{24l}$$

Sketch of the Proof. The strong Problem (24) is obtained using the standard procedure for recovering strong forms. First the strong equations (24a), (24b), (24c), (24d) and (24e) are recovered by testing the weak variational Problem (23) with compactly supported functions. Next, the standard integration by parts with suitable test functions recovers the boundary conditions (24i), (24j), (24k), (24l) and the interface conditions (24f), (24g), respectively. Finally, the admissibility constraint (24h) comes from the modeling space $\mathbf{X}$ defined in (22). □

3. Convergence statements

We begin this section recalling a classical space.

Definition 1. Let $\Omega_2 \stackrel{\text{def}}{=} \Gamma\times(0,1)$, define the Hilbert space

$$H(\partial_z,\Omega_2) \stackrel{\text{def}}{=} \left\{u\in L^2(\Omega_2) : \partial_z\,u \in\, L^2(\Omega_2)\right\}, \tag{25a}$$

endowed with the inner product

$$\left\langle\, u, v\,\right\rangle_{H(\partial_z,\Omega_2)} \stackrel{\text{def}}{=} \int_{\Omega_2} (\,u\,v + \partial_z u\,\partial_z v\,)\,dx. \tag{25b}$$

Lemma 9. *Let $H(\partial_z,\Omega_2)$ be the space introduced in Definition 1, then the trace map $u\mapsto u\big|_\Gamma$ from $H(\partial_z,\Omega_2)$ to $L^2(\Gamma)$ is well-defined. Moreover, the following Poincaré-type inequalities hold in this space*

$$\left\|u\right\|_{0,\Gamma} \le \sqrt{2}\left(\|u\|_{0,\Omega_2} + \left\|\partial_z\,u\right\|_{0,\Omega_2}\right), \tag{26a}$$

$$\left\|u\right\|_{0,\Omega_2} \le \sqrt{2}\left(\left\|\partial_z\,u\right\|_{0,\Omega_2} + \|u\|_{0,\Gamma}\right), \tag{26b}$$

for all $u\in H(\partial_z,\Omega_2)$.

Proof. The proof is a direct application of the fundamental theorem of calculus on the smooth functions $C^\infty(\Omega_2)$ which is a dense subspace in $H(\partial_z, \Omega_2)$. $\square$

In order to derive convergence statements, it will be shown, accepting the next hypothesis, that the sequence of solutions is globally bounded.

Hypothesis 2. In the following, it will be assumed that the sequences $\{\mathbf{f}^{2,\epsilon} : \epsilon > 0\} \subseteq \mathbf{L}^2(\Omega_2)$ and $\{h^{1,\epsilon} : \epsilon > 0\} \subseteq L^2(\Omega_1)$ are bounded, i.e., there exists $C > 0$ such that

$$\left\|\mathbf{f}^{2,\epsilon}\right\|_{0,\Omega_2} \leq C, \qquad \left\|h^{1,\epsilon}\right\|_{0,\Omega_1} \leq C, \qquad \text{for all } \epsilon > 0. \tag{27}$$

Theorem 10. *Let $[\mathbf{v}^\epsilon, p^\epsilon] \in \mathbf{X} \times \mathbf{Y}$ be the solution to the Problem (23). There exists a $K > 0$ such that*

$$\begin{aligned} &\left\|\mathbf{v}^{1,\epsilon}\right\|^2_{0,\Omega_1} + \left\|\boldsymbol{\nabla}_T\left(\epsilon\, \mathbf{v}^{2,\epsilon}_T\right)\right\|^2_{0,\Omega_2} + \left\|\partial_z \mathbf{v}^{2,\epsilon}_T\right\|^2_{0,\Omega_2} \\ &\qquad + \left\|\epsilon\, \boldsymbol{\nabla}_T \mathbf{v}^{2,\epsilon}_N\right\|^2_{0,\Omega_2} + \left\|\partial_z \mathbf{v}^{2,\epsilon}_N\right\|^2_{0,\Omega_2} + \left\|\mathbf{v}^{2,\epsilon}_N\right\|^2_{0,\Gamma} + \left\|\epsilon\, \mathbf{v}^{2,\epsilon}_T\right\|^2_{0,\Gamma} \leq K, \qquad \textit{for all } \epsilon > 0. \end{aligned} \tag{28}$$

Proof. Set $\mathbf{w} = \mathbf{v}^\epsilon$ in (23a) and $\varphi = p^\epsilon$ in (23b); add them together to get

$$\begin{aligned} &\int_{\Omega_1} \mathcal{Q}\, \mathbf{v}^{1,\epsilon} \cdot \mathbf{v}^{1,\epsilon} + \int_{\Omega_2} \mu \boldsymbol{\nabla}_T\left(\epsilon\, \mathbf{v}^{2,\epsilon}_T\right) : \boldsymbol{\nabla}_T\left(\epsilon\, \mathbf{v}^{2,\epsilon}_T\right) + \int_{\Omega_2} \mu\, \partial_z \mathbf{v}^{2,\epsilon}_T \cdot \partial_z \mathbf{v}^{2,\epsilon}_T \\ &\qquad + \epsilon^2 \int_{\Omega_2} \mu\, \boldsymbol{\nabla}_T \mathbf{v}^{2,\epsilon}_N \cdot \boldsymbol{\nabla}_T \mathbf{v}^{2,\epsilon}_N + \int_{\Omega_2} \mu\, \partial_z \mathbf{v}^{2,\epsilon}_N\, \partial_z \mathbf{v}^{2,\epsilon}_N \\ &\qquad + \alpha \int_\Gamma \left(\mathbf{v}^{1,\epsilon} \cdot \mathbf{n}\right)\left(\mathbf{v}^{1,\epsilon} \cdot \mathbf{n}\right) dS + \int_\Gamma \epsilon^2\, \beta \sqrt{\mathcal{Q}}\, \mathbf{v}^{2,\epsilon}_T \cdot \mathbf{v}^{2,\epsilon}_T\, dS = \epsilon \int_{\Omega_2} \mathbf{f}^{2,\epsilon} \cdot \mathbf{v}^{2,\epsilon} + \int_{\Omega_1} h^{1,\epsilon} p^{1,\epsilon}. \end{aligned} \tag{29}$$

The mixed terms were canceled out on the diagonal. Applying the Cauchy–Schwartz inequality to the right hand side and recalling the Hypothesis 1, we get

$$\begin{aligned} &\left\|\mathbf{v}^{1,\epsilon}\right\|^2_{0,\Omega_1} + \left\|\, \boldsymbol{\nabla}_T\left(\epsilon\, \mathbf{v}^{2,\epsilon}_T\right)\right\|^2_{0,\Omega_2} + \left\|\partial_z \mathbf{v}^{2,\epsilon}_T\right\|^2_{0,\Omega_2} + \left\|\epsilon\, \boldsymbol{\nabla}_T \mathbf{v}^{2,\epsilon}_N\right\|^2_{0,\Omega_2} + \left\|\partial_z \mathbf{v}^{2,\epsilon}_N\right\|^2_{0,\Omega_2} \\ &\qquad + \left\|\mathbf{v}^{2,\epsilon}_N\right\|^2_{0,\Gamma} + \left\|\epsilon\, \mathbf{v}^{2,\epsilon}_T\right\|^2_{0,\Gamma} \leq \frac{1}{k}\Big(\left\|\mathbf{f}^{2,\epsilon}\right\|_{0,\Omega_2} \left\|\, \epsilon\, \mathbf{v}^{2,\epsilon}\right\|_{0,\Omega_2} + \int_{\Omega_1} h^{1,\,\epsilon}\, p^{1,\epsilon}\Big). \end{aligned} \tag{30}$$

The summand involving an integral needs a special treatment in order to attain the a priori estimate.

$$\begin{aligned} \int_{\Omega_1} h^{1,\,\epsilon}\, p^{1,\epsilon} &\leq \left\|p^{1,\epsilon}\right\|_{0,\Omega_1} \left\|h^{1,\epsilon}\right\|_{0,\Omega_1} \\ &\leq C \left\|\boldsymbol{\nabla}\, p^{1,\epsilon}\right\|_{0,\Omega_1} \left\|h^{1,\epsilon}\right\|_{0,\Omega_1} = C \left\|\mathcal{Q}\, \mathbf{v}^{1,\epsilon}\right\|_{0,\Omega_1} \left\|h^{1,\,\epsilon}\right\|_{0,\Omega_1} \leq \widetilde{C} \left\|\mathbf{v}^{1,\epsilon}\right\|_{0,\Omega_1}. \end{aligned} \tag{31}$$

The second inequality holds due to Poincaré's inequality given that $p^{1,\epsilon} = 0$ on $\partial\Omega_1 - \Gamma$, as stated in Equation (24i). The equality holds due to (24a). The third inequality holds because the tensor $\mathcal{Q}$ and the family of sources $\{h^{1,\epsilon} : \epsilon > 0\} \subset L^2(\Omega_1)$ are bounded as stated in Hypothesis 1 and (27), Hypothesis 2 respectively. Next, we control the $L^2(\Omega_2)$-norm of $\mathbf{v}^{2,\epsilon}$. Recalling that $\mathbf{v}^{2,\epsilon} \in \left[H(\partial_z, \Omega_2)\right]^N$ then, a direct application of Estimate (26b) implies

$$\left\|\mathbf{v}^{2,\epsilon}\right\|_{0,\Omega_2} \leq \sqrt{2}\left(\left\|\partial_z\, \mathbf{v}^{2,\epsilon}\right\|_{0,\Omega_2} + \|\mathbf{v}^{2,\epsilon}\|_{0,\Gamma}\right). \tag{32}$$

Combining (31), (32) and the bound (27) from Hypothesis 2 in (30) we have

$$\begin{aligned}\left\|\mathbf{v}^{1,\epsilon}\right\|_{0,\Omega_1}^2 + \left\|\boldsymbol{\nabla}_T\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right)\right\|_{0,\Omega_2}^2 + \left\|\partial_z\mathbf{v}_T^{2,\epsilon}\right\|_{0,\Omega_2}^2 + \left\|\epsilon\,\boldsymbol{\nabla}_T\mathbf{v}_N^{2,\epsilon}\right\|_{0,\Omega_2}^2 + \left\|\partial_z\mathbf{v}_N^{2,\epsilon}\right\|_{0,\Omega_2}^2 + \left\|\mathbf{v}_N^{2,\epsilon}\right\|_{0,\Gamma}^2 + \left\|\epsilon\,\mathbf{v}_T^{2,\epsilon}\right\|_{0,\Gamma}^2 \\ \leq C\Big[\left\|\mathbf{f}^{2,\,\epsilon}\right\|_{0,\Omega_2}\sqrt{2}\left(\left\|\partial_z\left(\epsilon\,\mathbf{v}^{2,\epsilon}\right)\right\|_{0,\Omega_2} + \left\|\left(\epsilon\,\mathbf{v}^{2,\epsilon}\right)\right\|_{0,\Gamma}\right) + \widetilde{C}\left\|\,\mathbf{v}^{1,\epsilon}\right\|_{0,\Omega_1}\Big] \\ \leq \widehat{C}\Big(\left\|\partial_z\,\mathbf{v}_T^{2,\epsilon}\right\|_{0,\Omega_2} + \left\|\partial_z\,\mathbf{v}^{2,\epsilon}\right\|_{0,\Omega_2} + \left\|\epsilon\,\mathbf{v}_T^{2,\epsilon}\right\|_{0,\Gamma} + \left\|\mathbf{v}_N^{2,\epsilon}\right\|_{0,\Gamma} + \left\|\mathbf{v}^{1,\epsilon}\right\|_{0,\Omega_1}\Big).\end{aligned}$$

Using the equivalence of norms $\|\cdot\|_1, \|\cdot\|_2$ for 5-D vectors yields

$$\begin{aligned}\left\|\mathbf{v}^{1,\epsilon}\right\|_{0,\Omega_1}^2 + \left\|\boldsymbol{\nabla}_T\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right)\right\|_{0,\Omega_2}^2 + \left\|\partial_z\mathbf{v}_T^{2,\epsilon}\right\|_{0,\Omega_2}^2 + \left\|\epsilon\,\boldsymbol{\nabla}_T\mathbf{v}_N^{2,\epsilon}\right\|_{0,\Omega_2}^2 + \left\|\partial_z\mathbf{v}_N^{2,\epsilon}\right\|_{0,\Omega_2}^2 + \left\|\mathbf{v}_N^{2,\epsilon}\right\|_{0,\Gamma}^2 + \left\|\epsilon\,\mathbf{v}_T^{2,\epsilon}\right\|_{0,\Gamma}^2 \\ \leq C'\Big\{\left\|\partial_z\,\mathbf{v}_T^{2,\epsilon}\right\|_{0,\,\Omega_2}^2 + \left\|\partial_z\,\mathbf{v}^{2,\epsilon}\right\|_{0,\,\Omega_2}^2 + \left\|\epsilon\,\mathbf{v}_T^{2,\epsilon}\right\|_{0,\,\Gamma}^2 + \left\|\mathbf{v}_N^{2,\epsilon}\right\|_{0,\,\Gamma}^2 + \left\|\mathbf{v}^{1,\epsilon}\right\|_{0,\,\Omega_1}^2\Big\}^{1/2} \\ \leq C\Big\{\left\|\mathbf{v}^{1,\epsilon}\right\|_{0,\Omega_1}^2 + \left\|\,\boldsymbol{\nabla}_T\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right)\right\|_{0,\Omega_2}^2 + \left\|\partial_z\mathbf{v}_T^{2,\epsilon}\right\|_{0,\Omega_2}^2 \\ + \left\|\epsilon\,\boldsymbol{\nabla}_T\mathbf{v}_N^{2,\epsilon}\right\|_{0,\Omega_2}^2 + \left\|\partial_z\mathbf{v}_N^{2,\epsilon}\right\|_{0,\Omega_2}^2 + \left\|\mathbf{v}_N^{2,\epsilon}\right\|_{0,\Gamma}^2 + \left\|\epsilon\,\mathbf{v}_T^{2,\epsilon}\right\|_{0,\Gamma}^2\Big\}^{1/2}.\end{aligned}$$

The expression above implies the existence of a constant $K > 0$ satisfying the global Estimate (28). □

In the next subsections we use weak convergence arguments to derive the functional setting of the limiting problem.

3.1. Weak convergence of velocity and pressure

We begin this part with a direct consequence of Theorem 10.

Corollary 11. *Let $[\mathbf{v}^\epsilon, p^\epsilon] \in \mathbf{X} \times \mathbf{Y}$ be the solution to the Problem (23). There exists a subsequence, still denoted $\{\mathbf{v}^\epsilon : \epsilon > 0\}$ for which the following hold.*

(i) *There exist $\mathbf{v}^1 \in \mathbf{H}_{\mathrm{div}}(\Omega_1)$ and $\mathbf{v}_T^2 \in \left[H^1(\Omega_2)\right]^{N-1}$ such that*

$$\mathbf{v}^{1,\epsilon} \to \mathbf{v}^1 \quad \textit{weakly in} \quad \mathbf{H}_{\mathrm{div}}(\Omega_1), \tag{33a}$$

$$\epsilon\,\mathbf{v}_T^{2,\epsilon} \to \mathbf{v}_T^2 \quad \textit{weakly in} \quad \left[H^1(\Omega_2)\right]^{N-1}, \quad \textit{strongly in} \quad \left[L^2(\Omega_2)\right]^{N-1}. \tag{33b}$$

(ii) *There exist $\xi \in H(\partial_z, \Omega_2)$ and $\eta \in \left[L^2(\Omega_2)\right]^{N-1}$ such that*

$$\partial_z\mathbf{v}_T^{2,\epsilon} \to \eta \quad \textit{weakly in} \quad \left[L^2(\Omega_2)\right]^{N-1}, \quad \partial_z\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right) \to 0 \quad \textit{strongly in} \quad \left[L^2(\Omega_2)\right]^{N-1}, \tag{34a}$$

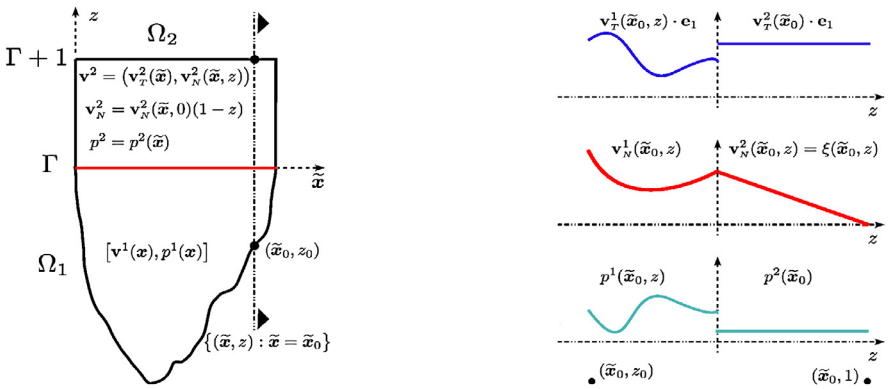
$$\mathbf{v}_N^{2,\epsilon} \to \xi \quad \textit{weakly in } L^2(\Omega_2), \quad \left(\epsilon\,\mathbf{v}_N^{2,\epsilon}\right) \to 0 \quad \textit{strongly in } H(\partial_z, \Omega_2), \tag{34b}$$

moreover, ξ satisfies the interface and boundary conditions

$$\xi|_\Gamma = \mathbf{v}^1 \cdot \mathbf{n}|_\Gamma, \qquad \xi\left(\widetilde{x}, 1\right) = 0. \tag{34c}$$

(iii) *The limit function $\mathbf{v}_T^2$ satisfies that (see Fig. 3)*

$$\mathbf{v}_T^2 = \mathbf{v}_T^2\left(\widetilde{x}\right). \tag{35}$$



(a) Limit Solutions in the Domain of Reference.

(b) Velocity and Pressure Schematic Traces for the Solution on the hyperplane $\{(\widetilde{x}, z) : \widetilde{x} = \widetilde{x}_0\}$.

Fig. 3. Figure (a) shows the dependence of the limit solution $[\mathbf{u}, p]$ according to the respective region. Figure (b) depicts some plausible schematics of the velocity and pressure traces on the hyperplane $\{(\widetilde{x}, z) : \widetilde{x} = \widetilde{x}_0\}$.

Proof.

(i) Due to the global a priori Estimate (28) there must exist a weakly convergent subsequence and $\mathbf{v}^2_T \in \left[H^1(\Omega_2)\right]^{N-1}$ such that (33b) holds; together with $\mathbf{v}^1 \in \mathbf{H}_{\mathrm{div}}(\Omega_1)$ such that (33a) holds only in the weak $\mathbf{L}^2(\Omega_1)$-sense. Because of the Hypothesis 2 and (24b), the sequence $\{\boldsymbol{\nabla} \cdot \mathbf{v}^{1,\epsilon} : \epsilon > 0\} \subset L^2(\Omega_1)$ is bounded. Then, there must exist yet another subsequence, still denoted the same, such that (33a) holds in the weak $\mathbf{H}_{\mathrm{div}}(\Omega_1)$-sense and the first part is complete.

(ii) For the higher order terms $\partial_z \mathbf{v}^{2,\epsilon}_N$, $\partial_z \mathbf{v}^{2,\epsilon}_T$, in view of the estimate (28), there must exist $\eta \in \left[L^2(\Omega_2)\right]^{N-1}$ for which (34a) holds. Next, the estimate (28) combined with (26b) implies that $\left\{\mathbf{v}^{2,\epsilon}_N : \epsilon > 0\right\}$ is a bounded sequence in $H(\partial_z, \Omega_2)$, so (34b) holds. Moreover, since the trace applications $\mathbf{v}^{2,\epsilon}_N \mapsto \mathbf{v}^{2,\epsilon}_N\big|_{\Gamma}$ and $\mathbf{v}^{2,\epsilon}_N \mapsto \mathbf{v}^{2,\epsilon}_N\big|_{\Gamma+1}$ are continuous in $H(\partial_z, \Omega_2)$ and $\mathbf{v}^{2,\epsilon}_N(\widetilde{x}, 1) = 0$, the properties (34c) follow. This concludes the second part.

(iii) The property (35), is a direct consequence of (34a). Hence, the proof is complete. □

Lemma 12. *Let $\left[\mathbf{v}^\epsilon, p^\epsilon\right] \in \mathbf{X} \times \mathbf{Y}$ be the solution of (23). There exists a subsequence, still denoted $\left\{p^\epsilon : \epsilon > 0\right\}$ verifying the following.*

(i) *There exists $p^1 \in H^1(\Omega_1)$ such that*

$$p^{1,\epsilon} \to p^1 \quad \textit{weakly in } H^1(\Omega_1), \ \textit{strongly in } L^2(\Omega_1). \tag{36}$$

(ii) *There exists $p^2 \in L^2(\Omega_2)$ such that*

$$p^{2,\epsilon} \to p^2 \quad \textit{weakly in } L^2(\Omega_2). \tag{37}$$

(iii) *The pressure $p = \left[p^1, p^2\right]$ belongs to $L^2(\Omega)$.*

Proof.

(i) Due to (24a) and (30) it follows that

$$\left\|\boldsymbol{\nabla}\, p^{1,\epsilon}\right\|_{0,\Omega_1} = \left\|\sqrt{\mathcal{Q}}\, \mathbf{v}^{1,\epsilon}\right\|_{0,\Omega_1} \leq C,$$

with $C > 0$ an adequate positive constant. From (24i), the Poincaré inequality implies there exists a constant $\widetilde{C} > 0$ satisfying

$$\left\|p^{1,\epsilon}\right\|_{1,\Omega_1} \leq \widetilde{C}\left\|\boldsymbol{\nabla} p^{1,\epsilon}\right\|_{0,\Omega_1}, \qquad \text{for all } \epsilon > 0. \tag{38}$$

Therefore, the sequence $\{p^{1,\epsilon} : \epsilon > 0\}$ is bounded in $H^1(\Omega_1)$ and the Statement (36) follows directly.

(ii) In order to show that the sequence $\{p^{2,\epsilon} : \epsilon > 0\}$ is bounded in $L^2(\Omega_2)$, take any $\phi \in C_0^\infty(\Omega_2)$ and define the auxiliary function

$$\varsigma(\widetilde{x}, z) \stackrel{\textbf{def}}{=} \int_z^1 \phi(\widetilde{x}, t)\, dt. \tag{39}$$

By construction it is clear that $\|\varsigma\|_{1,\Omega_2} \leq C\|\phi\|_{0,\Omega_2}$. Since $\varsigma\, \mathbb{1}_\Gamma \in L^2(\Gamma) \subseteq H^{-1/2}(\partial\Omega_1)$, due to Lemma 1, there must exist a function $\mathbf{w}^1 \in \mathbf{H}_{\text{div}}(\Omega_1)$ such that $\mathbf{w}^1 \cdot \mathbf{n} = \mathbf{w}^2 \cdot \mathbf{n} = \varsigma(\widetilde{x}, 0) = \int_0^1 \phi(\widetilde{x}, t)\, dt$ on Γ, $\mathbf{w}^1 \cdot \mathbf{n} = 0$ on $\partial\Omega_1 - \Gamma$ and $\|\mathbf{w}^1\|_{\mathbf{H}_{\text{div}}(\Omega_1)} \leq \|\varsigma\|_{0,\Gamma} \leq C\|\phi\|_{0,\Omega_2}$. Hence, the function $\mathbf{w}^2 = \big(\mathbf{0}_T, \varsigma(\widetilde{x}, z)\big)$ is such that $\mathbf{w} \stackrel{\textbf{def}}{=} [\mathbf{w}^1, \mathbf{w}^2] \in \mathbf{X}$; testing (23a) with $\mathbf{w}$ yields

$$\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^{1,\epsilon} \cdot \mathbf{w}^1 - \int_{\Omega_1} p^{1,\epsilon}\, \boldsymbol{\nabla} \cdot \mathbf{w}^1 + \alpha \int_\Gamma \big(\mathbf{v}^{1,\epsilon} \cdot \mathbf{n}\big)\big(\mathbf{w}^1 \cdot \mathbf{n}\big) dS + \int_{\Omega_2} p^{2,\epsilon}\, \phi$$
$$+ \epsilon^2 \int_{\Omega_2} \mu\, \boldsymbol{\nabla}_T\, \mathbf{v}_N^{2,\epsilon} \cdot \boldsymbol{\nabla}_T\, \varsigma - \int_{\Omega_2} \mu\, \partial_z\, \mathbf{v}_N^{2,\epsilon}\, \phi = \epsilon \int_{\Omega_2} \mathbf{f}_N^{2,\epsilon}\, \varsigma. \tag{40}$$

Applying the Cauchy–Schwarz inequality to the integrals and reordering we get

$$\Big|\int_{\Omega_2} p^{2,\epsilon}\, \phi\Big| \leq C_1 \|\mathbf{v}^{1,\epsilon}\|_{0,\Omega_1} \|\mathbf{w}^1\|_{0,\Omega_1} + \|p^{1,\epsilon}\|_{0,\Omega_1} \|\boldsymbol{\nabla} \cdot \mathbf{w}^1\|_{0,\Omega_1} + C_2 \|\mathbf{v}^{1,\epsilon} \cdot \mathbf{n}\|_{0,\Gamma} \|\varsigma\|_{0,\Gamma}$$
$$+ \epsilon\, C_3 \left\|\boldsymbol{\nabla}_T \left(\epsilon\, \mathbf{v}_N^{2,\epsilon}\right)\right\|_{0,\Omega_2} \left\|\boldsymbol{\nabla}_T\, \varsigma(\widetilde{x}, z)\right\|_{0,\Omega_2} + C_4 \left\|\partial_z\, \mathbf{v}_N^{2,\epsilon}\right\|_{0,\Omega_2} \|\phi\|_{0,\Omega_2} + \left\|\epsilon\, \mathbf{f}_N^{2,\epsilon}\right\|_{0,\Omega_2} \|\varsigma\|_{0,\Omega_2}.$$

Notice that due to the construction, all the norms depending on $\mathbf{w}^1$ and ς, with the exception of $\boldsymbol{\nabla}_T\varsigma$, are controlled by the norm $\|\phi\|_{0,\Omega_2}$. Therefore, the above expression can be reduced to

$$\Big|\int_{\Omega_2} p^{2,\epsilon}\, \phi\Big| \leq C\Big(\|\mathbf{v}^{1,\epsilon}\|_{0,\Omega_1} + \|p^{1,\epsilon}\|_{0,\Omega_1} + \|\mathbf{v}^{1,\epsilon} \cdot \mathbf{n}\|_{0,\Gamma} + \|\partial_z\, \mathbf{v}_N^{2,\epsilon}\|_{0,\Omega_2} + \|\epsilon\, \mathbf{f}_N^{2,\epsilon}\|_{0,\Omega_2}\Big) \|\phi\|_{0,\Omega_2}$$
$$+ \epsilon \left\|\boldsymbol{\nabla}_T \left(\epsilon\, \mathbf{v}_N^{2,\epsilon}\right)\right\|_{0,\Omega_2} \left\|\boldsymbol{\nabla}_T\, \varsigma(\widetilde{x}, z)\right\|_{0,\Omega_2} \leq \widetilde{C}\left(\|\phi\|_{0,\Omega_2} + \epsilon \left\|\boldsymbol{\nabla}_T\, \varsigma(\widetilde{x}, z)\right\|_{0,\Omega_2}\right).$$

The last inequality holds since all the summands in the parenthesis are bounded due to the estimates (28), (38) and the Hypothesis 2. Taking upper limit when $\epsilon \to 0$, in the previous expression we get

$$\limsup_{\epsilon \downarrow 0} \Big|\int_{\Omega_2} p^{2,\epsilon}\, \phi\Big| \leq \widetilde{C}\, \|\phi\|_{0,\Omega_2}. \tag{41}$$

Since the above holds for any $\phi \in C_0^\infty(\Omega_2)$, we conclude that the sequence $\big\{p^{2,\epsilon} : \epsilon > 0\big\} \subset L^2(\Omega_2)$ is bounded and consequently (37) follows.

(iii) From the previous part it is clear that the sequence $\big\{[p^{1,\epsilon}, p^{2,\epsilon}] : \epsilon > 0\big\}$ is bounded in $L^2(\Omega)$, so the proof is complete. □

Finally, we identify the dependence of p^2 and ξ.

Theorem 13. *Let ξ, p^2 be the higher order limiting term in Corollary 11 (ii) and the limit pressure in Ω_2 in Lemma 12 (ii), respectively. Then (see Fig. 3) we have*

$$\partial_z \xi = \partial_z \xi(\widetilde{x}), \tag{42a}$$

$$p^2 = p^2(\widetilde{x}). \tag{42b}$$

Proof. Testing (23b) with $\varphi = [0, \varphi^2] \in \mathbf{Y}$ and letting $\epsilon \to 0$ together with (33b) and (34b), we get

$$\int_{\Omega_2} \boldsymbol{\nabla}_T \cdot \mathbf{v}_T^2 \, \varphi^2 + \int_{\Omega_2} \partial_z \xi \, \varphi^2 = 0,$$

for all $\varphi^2 \in L^2(\Omega_2)$, consequently

$$\boldsymbol{\nabla}_T \cdot \mathbf{v}_T^2 + \partial_z \xi = 0.$$

Now, due to the dependence of $\mathbf{v}_T^2$ from Corollary 11 (iii) the Identity (42a) follows.

For the Identity (42b), take the limit as $\epsilon \downarrow 0$ in (40); since the sequence $\{[p^{1,\epsilon}, p^{2,\epsilon}] : \epsilon > 0\}$ is weakly convergent as seen in Lemma 12, this yields

$$\int_{\Omega_1} \mathcal{Q}\, \mathbf{v}^1 \cdot \mathbf{w}^1 - \int_{\Omega_1} p^1 \, \boldsymbol{\nabla} \cdot \mathbf{w}^1 + \alpha \int_{\Gamma} \xi \big(\mathbf{w}^1 \cdot \mathbf{n}\big) dS + \int_{\Omega_2} p^2 \, \phi - \int_{\Omega_2} \mu \, \partial_z \xi \, \phi = 0.$$

Integrating by parts the second summand and using (24a) we get

$$-\int_{\Gamma} p^1 \big(\mathbf{w}^1 \cdot \mathbf{n}\big) dS + \alpha \int_{\Gamma} \xi \big(\mathbf{w}^1 \cdot \mathbf{n}\big) dS + \int_{\Omega_2} p^2 \, \phi - \int_{\Omega_2} \mu \, \partial_z \xi \, \phi = 0.$$

Recalling that $\mathbf{w}^1 \cdot \mathbf{n}\,|_\Gamma = \int_0^1 \phi\,(\widetilde{x}, z)\, dz$, we see the above expression transforms into

$$-\int_{\Gamma} p^1 \,|_\Gamma \, \Big(\int_0^1 \phi(\widetilde{x}, t)\, dt\Big) d\widetilde{x} + \alpha \int_{\Gamma} \xi \,|_\Gamma \, \Big(\int_0^1 \phi(\widetilde{x}, t)\, dt\Big) d\widetilde{x}$$
$$+ \int_{\Omega_2} p^2 \, \phi(\widetilde{x}, z)\, d\widetilde{x}\, dz - \int_{\Omega_2} \mu \, \partial_z \xi \, \phi(\widetilde{x}, z)\, d\widetilde{x}\, dz = 0.$$

The above holds for any $\phi \in C_0^\infty(\Omega_2)$ and $\xi\,|_\Gamma$, $p^1\,|_\Gamma$ can be embedded in Ω_2 with the extension, constant with respect to z, to the whole domain, so we conclude that

$$-p^1\,|_\Gamma + \alpha\, \xi\,|_\Gamma + p^2 - \mu\, \partial_z \xi = 0 \;\text{ in }\; L^2(\Omega_2).$$

Together with (34c) this shows

$$p^2 = \mu\, \partial_z \xi - \alpha\, \mathbf{v}^1 \cdot \mathbf{n}|_\Gamma + p^1\,|_\Gamma \;\text{ in }\; L^2(\Omega_2)\,, \tag{43}$$

and then with (42a) we obtain (42b). □

We close this section with an equivalent form for (23) which will be useful in characterizing the limiting problem.

Proposition 14. *The problem (23) is equivalent to*

$\left[\mathbf{v}^\epsilon, p^\epsilon\right] \in \mathbf{X} \times \mathbf{Y}$:

$$\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^{1,\epsilon}\cdot\mathbf{w}^1 - \int_{\Omega_1} p^{1,\epsilon}\,\boldsymbol{\nabla}\cdot\mathbf{w}^1 - \int_{\Omega_2} p^{2,\epsilon}\,\boldsymbol{\nabla}_T\cdot\mathbf{w}^2_T - \int_{\Omega_2} p^{2,\epsilon}\,\partial_z\mathbf{w}^2_N$$
$$+ \int_{\Omega_2} \mu\boldsymbol{\nabla}_T\left(\epsilon\,\mathbf{v}^{2,\epsilon}_T\right) : \boldsymbol{\nabla}_T\,\mathbf{w}^2_T + \frac{1}{\epsilon^2}\int_{\Omega_2} \mu\,\partial_z\left(\epsilon\,\mathbf{v}^{2,\epsilon}_T\right)\cdot\,\partial_z\mathbf{w}^2_T$$
$$+ \epsilon\int_{\Omega_2} \mu\,\boldsymbol{\nabla}_T\left(\epsilon\,\mathbf{v}^{2,\epsilon}_N\right)\cdot\boldsymbol{\nabla}_T\mathbf{w}^2_N + \int_{\Omega_2} \mu\,\partial_z\mathbf{v}^{2,\epsilon}_N\,\partial_z\mathbf{w}^2_N$$
$$+ \alpha\int_{\Gamma}\left(\mathbf{v}^{1,\epsilon}\cdot\mathbf{n}\right)\left(\mathbf{w}^1\cdot\mathbf{n}\right)dS + \int_{\Gamma}\beta\sqrt{\mathcal{Q}}\left(\epsilon\,\mathbf{v}^{2,\epsilon}_T\right)\cdot\mathbf{w}^2_T\,dS = \epsilon\int_{\Omega_2}\mathbf{f}^{2,\,\epsilon}\cdot\mathbf{w}^2, \tag{44a}$$

$$\int_{\Omega_1}\boldsymbol{\nabla}\cdot\mathbf{v}^{1,\epsilon}\varphi^1 + \epsilon\int_{\Omega_2}\boldsymbol{\nabla}_T\cdot\mathbf{v}^{2,\epsilon}_T\,\varphi^2 + \int_{\Omega_2}\partial_z\mathbf{v}^{2,\epsilon}_N\,\varphi^2 = \int_{\Omega_1} h^{1,\,\epsilon}\,\varphi^1, \tag{44b}$$

for all $[\mathbf{w},\varphi] \in \mathbf{X}\times\mathbf{Y}$.

Proof. It is enough to observe that in the quantifier $\mathbf{w} = [\mathbf{w}^1, \mathbf{w}^2] \in \mathbf{X}$, the tangential and normal components of $\mathbf{w}^2$ are decoupled. Therefore, the satisfaction of the Statement (23a) for every $\left[\mathbf{w}^1, (\mathbf{w}^2_T, \mathbf{w}^2_N)\right] \in \mathbf{X}$ or for every $\left[\mathbf{w}^1, (\epsilon^{-1}\,\mathbf{w}^2_T, \mathbf{w}^2_N)\right] \in \mathbf{X}$ are equivalent logical statements; this proves the result. □

4. The limiting problem

In order to characterize the limiting problem, we introduce appropriate spaces. The limiting pressure space is given by

$$\mathbf{Y}^0 \stackrel{\mathbf{def}}{=} \left\{(\varphi^1,\varphi^2) \in \mathbf{Y} : \varphi^2 = \varphi^2(\widetilde{x})\right\}. \tag{45}$$

We shall exploit below the equivalence $\mathbf{Y}^0 \cong L^2(\Omega_1)\times L^2(\Gamma)$. The construction of the velocities limiting space is more sophisticated. First define

$$\mathbf{X}^0_2 \stackrel{\mathbf{def}}{=} \Big\{\mathbf{w}^2 = [\mathbf{w}^2_T, \mathbf{w}^2_N] : \mathbf{w}^2_T \in \left(H^1(\Omega_2)\right)^{N-1},\ \mathbf{w}^2_T = \mathbf{w}^2_T(\widetilde{x}),$$
$$\mathbf{w}^2_T = \mathbf{0}\,\text{on}\,\partial\Gamma,\ \mathbf{w}^2_N \in H(\partial_z,\Omega_2)\,,\ \partial_z\mathbf{w}^2_N = \partial_z\,\mathbf{w}^2_N(\widetilde{x}\,)\,,\ \mathbf{w}^2_N(\widetilde{x},1) = 0\Big\} \tag{46a}$$

endowed with its natural norm

$$\|\mathbf{w}^2\|_{\mathbf{X}^0_2} = \left(\|\mathbf{w}^2_T\|^2_{1,\,\Omega_2} + \|\mathbf{w}^2_N\|^2_{H(\partial_z,\Omega_2)}\right)^{1/2}. \tag{46b}$$

Next we introduce a subspace of $\mathbf{X}$ fitting the limiting process together with its closure,

$$\mathbf{W} \stackrel{\mathbf{def}}{=} \left\{\left(\mathbf{w}^1,\mathbf{w}^2\right) \in \mathbf{X} : \mathbf{w}^2_T = \mathbf{w}^2_T(\widetilde{x}),\ \partial_z\,\mathbf{w}^2_N = \partial_z\,\mathbf{w}^2_N(\widetilde{x})\right\}, \tag{47a}$$

$$\mathbf{X}^0 \stackrel{\mathbf{def}}{=} \left\{(\mathbf{w}^1,\mathbf{w}^2) \in \mathbf{H}_{\text{div}}(\Omega_1)\times\mathbf{X}^0_2 : \mathbf{w}^1\cdot\mathbf{n} = \mathbf{w}^2_N = \mathbf{w}^2\cdot\mathbf{n}\,\text{on}\,\Gamma\right\}. \tag{47b}$$

Clearly $\mathbf{W} \subseteq \mathbf{X}^0 \cap \mathbf{X}$; before presenting the limiting problem, we verify the density.

Lemma 15. *The subspace* $\mathbf{W} \subseteq \mathbf{X}$ *is dense in* $\mathbf{X}^0$.

Proof. Consider an element $\mathbf{w} = (\mathbf{w}^1, \mathbf{w}^2) \in \mathbf{X}^0$, then $\mathbf{w}^2 = (\mathbf{w}^2_T, \mathbf{w}^2_N) \in \mathbf{X}^0_2$, where $\mathbf{w}^2_N \in H(\partial_z, \Omega_2)$ is completely defined by its trace on the interface Γ. Given $\epsilon > 0$ take $\varpi \in H^1_0(\Gamma)$ such that $\|\varpi - \mathbf{w}^2_N|_\Gamma\|_{L^2(\Gamma)} \leq \epsilon$. Now extend the function to the whole domain by $\varrho(\widetilde{x}, z) \stackrel{\mathbf{def}}{=} \varpi(\widetilde{x})(1-z)$, then $\|\varrho - \mathbf{w}^2_N\|_{H(\partial_z,\Omega_2)} \leq \epsilon$. The function $(\mathbf{w}^2_T, \varrho)$ clearly belongs to $\mathbf{W}$. From the construction of ϱ we know that $\|\varrho\,|_\Gamma - \mathbf{w}^2_N\,|_\Gamma\|_{0,\Gamma} = \|\varpi - \mathbf{w}^2_N\,|_\Gamma\|_{0,\Gamma} \leq \epsilon$. Define $g = \varrho\,|_\Gamma - \mathbf{w}^2_N\,|_\Gamma \in L^2(\Gamma)$, due to Lemma 7 there exists $\mathbf{u} \in \mathbf{H}_{\mathrm{div}}(\Omega_1)$ such that $\mathbf{u}\cdot\mathbf{n} = g$ on Γ, $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega_1 - \Gamma$ and $\|\mathbf{u}\|_{\mathbf{H}_{\mathrm{div}}(\Omega_1)} \leq C_1\|g\|_{0,\Gamma}$ with C_1 depending only on Ω_1. Then, the function $\mathbf{w}^1 + \mathbf{u}$ is such that $(\mathbf{w}^1 + \mathbf{u})\cdot\mathbf{n} = \mathbf{w}^1\cdot\mathbf{n} + \varpi - \mathbf{w}^2_N = \varpi$ and $\|\mathbf{w}^1 + \mathbf{u} - \mathbf{w}^1\|_{\mathbf{H}_{\mathrm{div}}(\Omega_1)} = \|\mathbf{u}\|_{\mathbf{H}_{\mathrm{div}}(\Omega_1)} \leq C_1\|g\|_{0,\Gamma} \leq C_1\,\epsilon$. Moreover, we notice that the function $(\mathbf{w}^1 + \mathbf{u}, [\mathbf{w}^2_T, \varrho])$ belongs to $\mathbf{W}$, and due to the previous observations we have

$$\big\|\mathbf{w} - (\mathbf{w}^1 + \mathbf{u}\,, [\mathbf{w}^2_T, \varrho])\big\|_{\mathbf{X}^0} = \big\|(\mathbf{w}^1, \mathbf{w}^2) - (\mathbf{w}^1 + \mathbf{u}\,, [\mathbf{w}^2_T, \varrho])\big\|_{\mathbf{X}^0} \leq \sqrt{C_1 + 1}\,\epsilon.$$

Since the constants depend only on the domains Ω_1 and Ω_2, it follows that $\mathbf{W}$ is dense in $\mathbf{X}^0$. □

Now we are ready to give the variational formulation of the limiting problem.

Theorem 16. *Let* $[\mathbf{v}, p]$, *with* $\mathbf{v}^2 = \big[\mathbf{v}^2_T, \xi\big]$, *be the weak limits obtained in Corollary 11 and Lemma 12. Then* $[\mathbf{v}, p]$ *satisfies the variational statement*

$[\mathbf{v}, p] \in \mathbf{X}^0 \times \mathbf{Y}^0$:

$$\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^1\cdot\mathbf{w}^1 - \int_{\Omega_1} p^1\,\boldsymbol{\nabla}\cdot\mathbf{w}^1 - \int_{\Omega_2} p^2\,\boldsymbol{\nabla}\cdot\big[\mathbf{w}^2_T, \mathbf{w}^2_N\big]$$
$$+ \int_{\Omega_2} \mu\,\boldsymbol{\nabla}_T\,\mathbf{v}^2_T : \boldsymbol{\nabla}_T\,\mathbf{w}^2_T + \int_{\Omega_2} \mu\,\big(\partial_z\,\xi\big)\,\big(\partial_z\mathbf{w}^2_N\big)$$
$$+ \alpha \int_\Gamma \big(\mathbf{v}^1\cdot\mathbf{n}\big)\,\big(\mathbf{w}^1\cdot\mathbf{n}\big)\,dS + \int_\Gamma \beta\,\sqrt{\mathcal{Q}}\;\mathbf{v}^2_T\;\cdot\mathbf{w}^2_T\,dS = \int_{\Omega_2} \mathbf{f}^2_T\cdot\mathbf{w}^2_T, \tag{48a}$$

$$\int_{\Omega_1} \boldsymbol{\nabla}\cdot\mathbf{v}^1\varphi^1 + \int_{\Omega_2} \boldsymbol{\nabla}\cdot[\mathbf{v}^2_T,\,\xi]\;\varphi^2 = \int_{\Omega_1} h^1\,\varphi^1, \tag{48b}$$

for all $[\mathbf{w}, \varphi] \in \mathbf{X}^0 \times \mathbf{Y}^0$.

Moreover, the mixed variational formulation of the problem above is given by

$$\begin{aligned} [\mathbf{v}, p] \in \mathbf{X}^0 \times \mathbf{Y}^0 : A\,\mathbf{v} - B'\,p &= \mathbf{f}, \\ B\,\mathbf{v} &= h, \end{aligned} \tag{49}$$

where the forms $A : \mathbf{X}^0 \to (\mathbf{X}^0)'$ *and* $B : \mathbf{X}^0 \to (\mathbf{Y}^0)'$ *are defined by*

$$A \stackrel{\mathbf{def}}{=} \begin{pmatrix} \mathcal{Q} + \gamma'_{\mathbf{n}}\,\alpha\,\gamma_{\mathbf{n}} & \mathbf{0} \\ \mathbf{0} & \big[\gamma'_T\,\beta\,\sqrt{\mathcal{Q}}\,\gamma_T + (\boldsymbol{\nabla}_T)'\,\mu\,\boldsymbol{\nabla}_T, (\partial_z)'\,\mu\,\partial_z\big] \end{pmatrix}, \tag{50a}$$

$$B \stackrel{\mathbf{def}}{=} \begin{pmatrix} \boldsymbol{\nabla}\cdot & 0 \\ 0 & \boldsymbol{\nabla}\cdot \end{pmatrix} = \begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix}. \tag{50b}$$

Proof. First, test the Problem (44) with a function of the form $[\mathbf{w}, \varphi] \in \mathbf{W} \times \mathbf{Y}^0$. This gives

$$\int_{\Omega_1} \mathcal{Q}\, \mathbf{v}^{1,\epsilon} \cdot \mathbf{w}^1 - \int_{\Omega_1} p^{1,\epsilon}\, \boldsymbol{\nabla} \cdot \mathbf{w}^1 - \int_{\Omega_2} p^{2,\epsilon}\, \boldsymbol{\nabla}_T \cdot \mathbf{w}_T^2 - \int_{\Omega_2} p^{2,\epsilon}\, \partial_z\, \mathbf{w}_N^2$$
$$+ \int_{\Omega_2} \mu\, \boldsymbol{\nabla}_T \big(\epsilon\, \mathbf{v}_T^{2,\epsilon}\big) : \boldsymbol{\nabla}_T\, \mathbf{w}_T^2 + \epsilon \int_{\Omega_2} \mu\, \boldsymbol{\nabla}_T \left(\epsilon\, \mathbf{v}_N^{2,\epsilon}\right) \cdot \boldsymbol{\nabla}_T\, \mathbf{v}_N^2 + \int_{\Omega_2} \mu\, \partial_z\, \mathbf{v}_N^{2,\epsilon}\, \partial_z \mathbf{w}_N^2$$
$$+ \alpha \int_{\Gamma} \left(\mathbf{v}^{1,\epsilon} \cdot \mathbf{n}\right) \left(\mathbf{w}^1 \cdot \mathbf{n}\right) dS + \int_{\Gamma} \beta\, \sqrt{\mathcal{Q}}\, \left(\epsilon\, \mathbf{v}_T^{2,\epsilon}\right) \cdot \mathbf{w}_T^2\, dS = \int_{\Omega_2} \mathbf{f}_T^{2,\epsilon} \cdot \mathbf{w}_T^2 + \epsilon \int_{\Omega_2} \mathbf{f}_N^{2,\epsilon} \cdot \mathbf{w}_N^2,$$

$$\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{v}^{1,\epsilon}\, \varphi^1 + \int_{\Omega_2} \boldsymbol{\nabla}_T \cdot \left(\epsilon\, \mathbf{v}_T^{2,\epsilon}\right) \varphi^2 + \int_{\Omega_2} \partial_z \mathbf{v}_N^{2,\epsilon}\, \varphi^2 = \int_{\Omega_1} h^{1,\,\epsilon}\, \varphi^1,$$

and then letting $\epsilon \downarrow 0$ yields

$$\int_{\Omega_1} \mathcal{Q}\, \mathbf{v}^1 \cdot \mathbf{w}^1 - \int_{\Omega_1} p^1\, \boldsymbol{\nabla} \cdot \mathbf{w}^1 - \int_{\Omega_2} p^2\, \boldsymbol{\nabla}_T \cdot \mathbf{w}_T^2 - \int_{\Omega_2} p^2\, \partial_z\, \mathbf{w}_N^2$$
$$+ \int_{\Omega_2} \mu\, \boldsymbol{\nabla}_T\, \mathbf{v}_T^2 : \boldsymbol{\nabla}_T\, \mathbf{w}_T^2 + \int_{\Omega_2} \mu\, \partial_z\, \xi\, \partial_z \mathbf{w}_N^2$$
$$+ \alpha \int_{\Gamma} \left(\mathbf{v}^1 \cdot \mathbf{n}\right) \left(\mathbf{w}^1 \cdot \mathbf{n}\right) dS + \int_{\Gamma} \beta\, \sqrt{\mathcal{Q}}\, \mathbf{v}_T^2 \cdot \mathbf{w}_T^2\, dS = \int_{\Omega_2} \mathbf{f}_T^2 \cdot \mathbf{w}_T^2, \tag{52a}$$

$$\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{v}^1 \varphi^1 + \int_{\Omega_2} \boldsymbol{\nabla}_T \cdot \left(\mathbf{v}_T^2\right) \varphi^2 + \int_{\Omega_2} \partial_z \xi\, \varphi^2 = \int_{\Omega_1} h^1\, \varphi^1. \tag{52b}$$

Since the variational statements above hold for all $[\mathbf{w}, \varphi] \in \mathbf{W} \times \mathbf{Y}^0$ and the bilinear forms are continuous with respect to the space $\mathbf{X}^0 \times \mathbf{Y}^0$, we can extend them by density to all test functions $[\mathbf{w}, \varphi] \in \mathbf{X}^0 \times \mathbf{Y}^0$; these yield (48). Finally, the mixed variational characterization (49) follows immediately from the definition of the bilinear forms A and B given in (50a) and (50b), respectively. □

The existence of a solution of Problem (49) follows from that of the limits above. For an independent proof of the well-posedness of Problem (49) we prepare the following intermediate results.

Lemma 17. *The operator A is $\mathbf{X}^0$-coercive over $\mathbf{X}^0 \cap \ker(B)$.*

Proof. The form $A\mathbf{v}\,(\mathbf{v}) + \int_{\Omega_1} \left(\boldsymbol{\nabla} \cdot \mathbf{v}\right)^2$ is $\mathbf{X}^0$-coercive, and $\boldsymbol{\nabla} \cdot \mathbf{v}\,|_{\Omega_1} = 0$ whenever $\mathbf{v} \in \ker(B)$. □

Lemma 18. *The operator B has closed range.*

Proof. Fix $\varphi = [\varphi^1, \varphi^2] \in \mathbf{Y}^0$. With $\varphi^2 = \varphi^2(\widetilde{x}) \in L^2(\Gamma)$, solve the auxiliary problem

$$-\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \phi = \varphi^1 \quad \text{in } \Omega_1, \qquad \boldsymbol{\nabla} \phi \cdot \mathbf{n} = \varphi^2 \quad \text{on } \Gamma, \qquad \phi = 0 \quad \text{on } \partial\Omega_1 - \Gamma. \tag{53}$$

Then $\mathbf{u}^1 = -\boldsymbol{\nabla}\phi$ satisfies $\boldsymbol{\nabla}\cdot\mathbf{u}^1 = \varphi^1$, $\mathbf{u}^1\cdot\mathbf{n} = -\varphi^2$ and

$$\|\mathbf{u}^1\|_{\mathbf{H}_{\mathrm{div}}(\Omega_1)} \le C_1\left(\|\varphi^1\|^2_{0,\Omega_1} + \|\varphi^2\|^2_{0,\Omega_2}\right)^{1/2}, \tag{54}$$

because $\|\varphi^2\|_{L^2(\Gamma)} = \|\varphi^2\|_{L^2(\Omega_2)}$. Next, define $\mathbf{u}^2_N(\widetilde{x}, z) \stackrel{\mathbf{def}}{=} -\varphi^2(\widetilde{x})(1-z)$. The function $\mathbf{u} = \left(\mathbf{u}^1, [\mathbf{0}_T, \mathbf{u}^2_N]\right)$ belongs to the space $\mathbf{X}^0$ (see Fig. 3), and

$$\|\mathbf{u}\|_{\mathbf{X}^0} \le C\left(\|\mathbf{u}^1\|^2_{\mathbf{H}_{\mathrm{div}}(\Omega_1)} + \|\mathbf{u}^2_N\|^2_{H(\partial_z,\Omega_2)}\right)^{1/2} \le \widetilde{C}\left(\|\varphi^1\|^2_{0,\Omega_1} + \|\varphi^2\|^2_{0,\Omega_2}\right)^{1/2}. \tag{55}$$

Here $\widetilde{C}$ depends on the domains Ω_1, Ω_2 as well as the equivalence of norms for 2-D vectors, but it is independent of $\varphi \in \mathbf{Y}^0$. Moreover, notice that $\boldsymbol{\nabla}\cdot[\mathbf{0}_T, \mathbf{u}^2_N] = \varphi^2$. Hence, we have the inequalities

$$\begin{aligned}
\sup_{\mathbf{w}\in\mathbf{X}^0} \frac{\int_\Omega \varphi\, \boldsymbol{\nabla}\cdot\mathbf{w}\, dx}{\|\mathbf{w}\|_{\mathbf{X}^0}} &\ge \frac{\int_{\Omega_1} \varphi^1\, \boldsymbol{\nabla}\cdot\mathbf{u}^1\, dx + \int_{\Omega_2} \varphi^2\, \boldsymbol{\nabla}\cdot[\mathbf{0}_T, \mathbf{u}^2_N]\, d\widetilde{x}\, dz}{\|\mathbf{u}\|_{\mathbf{X}^0}} \\
&\ge \frac{1}{\widetilde{C}} \frac{\|\varphi^1\|^2_{0,\Omega_1} + \|\varphi^2\|^2_{0,\Omega_2}}{(\|\varphi^1\|^2_{0,\Omega_1} + \|\varphi^2\|^2_{0,\Omega_2})^{1/2}} = \frac{1}{\widetilde{C}}\left(\|\varphi^1\|^2_{0,\Omega_1} + \|\varphi^2\|^2_{0,\Omega_2}\right)^{1/2} \\
&= \frac{1}{\widetilde{C}}\|\varphi\|_{0,\Omega}, \quad \forall\, \varphi \in \mathbf{Y}^0. \qquad \square
\end{aligned} \tag{56}$$

Theorem 19. *The Problem (49) is well-posed.*

Proof. Due to Lemmas 17 and 18 above, the operators A and B satisfy the hypotheses of Theorem 2 and the result follows. $\square$

Remark 4. Note that the proof of Lemma 18 for the limit problem (49) is substantially different from the corresponding Lemma 5 for the ϵ-problem (16). This is due to the respective spaces $\mathbf{X}^0$ and $\mathbf{X}$; the condition $\mathbf{w}^1\cdot\mathbf{n} = \mathbf{w}^2\cdot\mathbf{n}$ on Γ is significantly different in terms of regularity, from one case to the other. Specifically, in the case of the limit problem $\mathbf{w}^1\cdot\mathbf{n}\big|_\Gamma \in L^2(\Gamma)$, while in the ϵ-problem $\mathbf{w}^1\cdot\mathbf{n}\big|_\Gamma \in H^{1/2}(\Gamma)$. The demands of normal trace regularity on Γ are weakened in the limit as a consequence of the upscaling process.

Corollary 20. *Let $\left\{\left[\mathbf{v}^\epsilon, p^\epsilon\right] : \epsilon > 0\right\} \subseteq \mathbf{X}\times\mathbf{Y}$ be the sequence of solutions to the family of problems (23), then the whole sequence converges weakly to $[\mathbf{v}, p] \in \mathbf{X}^0\times\mathbf{Y}^0$, the solution of Problem (48).*

Proof. Due to the well-posedness shown in Theorem 19, the solution $[\mathbf{v}, p] \in \mathbf{X}^0\times\mathbf{Y}^0$ of Problem (49) is unique. On the other hand, all the reasoning from Section 3.1 on, is applicable to any subsequence of $\left\{\left[\mathbf{v}^\epsilon, p^\epsilon\right] : \epsilon > 0\right\}$; which yields a further subsequence weakly convergent to $[\mathbf{v}, p]$. Hence, the result follows. $\square$

4.1. Dimension reduction

The limit tangential velocity and pressure in Ω_2 are independent of x_N (see (35) and (42)). Consequently, the spaces $\mathbf{X}^0$, $\mathbf{Y}^0$ and the problem (48) can be dimensionally reduced to yield a coupled problem on $\Omega_1\times\Gamma$. To that end, we first modify the function spaces. For the pressures we define the space

$$\mathbf{Y}^{00} \stackrel{\mathbf{def}}{=} \left\{\left[\varphi^1, \varphi^2\right] \in L^2(\Omega_1)\times L^2(\Gamma) : \int_{\Omega_1} \varphi^1 + \int_\Gamma \varphi^2 d\widetilde{x} = 0\right\}, \tag{57}$$

endowed with its natural norm. For the velocities we define the space

$$\mathbf{X}^{00} \stackrel{\mathbf{def}}{=} \left\{ \left(\mathbf{w}^1, \mathbf{w}^2\right) \in \mathbf{H}_{\mathrm{div}}(\Omega_1) \times \left(H_0^1(\Gamma)\right)^2 : \mathbf{w}^1 \cdot \mathbf{n}\Big|_\Gamma \in L^2(\Gamma) \right\}, \tag{58a}$$

with the norm

$$\left\| \left[\mathbf{w}^1, \mathbf{w}^2\right] \right\|_{\mathbf{X}^{00}} \stackrel{\mathbf{def}}{=} \left(\left\|\mathbf{w}^1\right\|^2_{\mathbf{H}_{\mathrm{div}}} + \left\|\mathbf{w}^1 \cdot \mathbf{n}\right\|^2_{0,\Gamma} + \left\|\mathbf{w}^2\right\|_{1,\Gamma} \right)^{1/2}. \tag{58b}$$

Remark 5. Clearly the pressure spaces $\mathbf{Y}^{00}$ and $\mathbf{Y}^0$ are isomorphic (see Fig. 3). It is also direct to see that the application $\iota : \mathbf{X}^0 \to \mathbf{X}^{00}$, given by $\left[\mathbf{w}^1, \mathbf{w}^2\right] \mapsto \left[\mathbf{w}^1, \mathbf{w}^2_T\right]$, is an isomorphism, because $\mathbf{w}^2_N$ is entirely determined by its trace on Γ and $\mathbf{w}^2_N(\widetilde{x}, 0) = \mathbf{w}^1 \cdot \mathbf{n}$ (see Fig. 3).

Theorem 21. *Let $[\mathbf{v}, p]$, with $\mathbf{v}^2 = \left[\mathbf{v}^2_T, \xi\right]$, be the weak limits found in Corollary 11 and Lemma 12. Then, with the corresponding identifications, $[\mathbf{v}, p]$ satisfies the following variational statement*

$[\mathbf{v}, p] \in \mathbf{X}^{00} \times \mathbf{Y}^{00}$:

$$\begin{aligned} &\int_{\Omega_1} \mathcal{Q}\, \mathbf{v}^1 \cdot \mathbf{w}^1 - \int_{\Omega_1} p^1\, \boldsymbol{\nabla} \cdot \mathbf{w}^1 + (\mu + \alpha) \int_\Gamma \left(\mathbf{v}^1 \cdot \mathbf{n}\right) \left(\mathbf{w}^1 \cdot \mathbf{n}\right) d\widetilde{x}, \\ &\quad + \int_\Gamma p^2 \left(\mathbf{w}^1 \cdot \mathbf{n}\right) d\widetilde{x} - \int_\Gamma p^2\, \boldsymbol{\nabla}_T \cdot \mathbf{w}^2_T\, d\widetilde{x} + \int_\Gamma \beta \sqrt{\mathcal{Q}}\, \mathbf{v}^2_T \cdot \mathbf{w}^2_T\, d\widetilde{x} + \int_\Gamma \mu\, \boldsymbol{\nabla}_T\, \mathbf{v}^2_T : \boldsymbol{\nabla}_T\, \mathbf{w}^2_T\, d\widetilde{x} = \int_\Gamma \mathbf{f}^2_T \cdot \mathbf{w}^2_T\, d\widetilde{x}, \end{aligned} \tag{59a}$$

$$\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{v}^1\, \varphi^1 - \int_\Gamma \left(\mathbf{v}^1 \cdot \mathbf{n}\right) \varphi^2\, d\widetilde{x} + \int_\Gamma \boldsymbol{\nabla}_T \cdot \mathbf{v}^2_T\, \varphi^2\, d\widetilde{x} = \int_{\Omega_1} h^1\, \varphi^1, \tag{59b}$$

for all $[\mathbf{w}, \varphi] \in \mathbf{X}^{00} \times \mathbf{Y}^{00}$.

Furthermore, the mixed formulation of the Problem above is given by

$$\begin{aligned} [\mathbf{v}, p] \in \mathbf{X}^{00} \times \mathbf{Y}^{00} : \mathcal{A}\, \mathbf{v} - \mathcal{B}' p &= \mathbf{f}, \\ \mathcal{B}\, \mathbf{v} &= h, \end{aligned} \tag{60}$$

with the forms $\mathcal{A} : \mathbf{X}^{00} \to (\mathbf{X}^{00})'$ and $\mathcal{B} : \mathbf{X}^{00} \to (\mathbf{Y}^{00})'$ defined by

$$\mathcal{A} \stackrel{\mathbf{def}}{=} \begin{pmatrix} \mathcal{Q} + \gamma'_{\mathbf{n}}\, (\mu + \alpha)\, \gamma_{\mathbf{n}} & \mathbf{0} \\ \mathbf{0} & \beta \sqrt{\mathcal{Q}} + (\boldsymbol{\nabla}_T)'\, \mu\, \boldsymbol{\nabla}_T \end{pmatrix}, \tag{61a}$$

$$\mathcal{B} \stackrel{\mathbf{def}}{=} \begin{pmatrix} \boldsymbol{\nabla} \cdot & 0 \\ -\gamma_{\mathbf{n}} & \boldsymbol{\nabla}_T \cdot \end{pmatrix} = \begin{pmatrix} \mathrm{div} & 0 \\ -\gamma_{\mathbf{n}} & \mathrm{div}_T \end{pmatrix}. \tag{61b}$$

Proof. Notice that if $\mathbf{w}^2 \in \mathbf{X}^0$ then $\partial_z\, \mathbf{w}^2_N = \partial_z\, \mathbf{w}^2_N\, (\widetilde{x}) = -\mathbf{w}^2_N(\widetilde{x}, 0) = -\mathbf{w}^2 \cdot \mathbf{n} = -\mathbf{w}^1 \cdot \mathbf{n}$ (see Fig. 3). Next, we introduce this observation in the Statement (48) above, together with the Identity (34c); this gives

$[\mathbf{v}, p] \in \mathbf{X}^0 \times \mathbf{Y}^0$:

$$\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^1 \cdot \mathbf{w}^1 - \int_{\Omega_1} p^1\,\boldsymbol{\nabla} \cdot \mathbf{w}^1 + (\mu+\alpha) \int_{\Gamma} \left(\mathbf{v}^1 \cdot \mathbf{n}\right)\left(\mathbf{w}^1 \cdot \mathbf{n}\right)\, d\widetilde{x}$$
$$+ \int_{\Gamma} p^2 \left(\mathbf{w}^1 \cdot \mathbf{n}\right)\, d\widetilde{x} - \int_{\Gamma} p^2\,\boldsymbol{\nabla}_T \cdot \mathbf{w}_T^2\, d\widetilde{x} + \int_{\Gamma} \mu\,\boldsymbol{\nabla}_T\,\mathbf{v}_T^2 : \boldsymbol{\nabla}_T\,\mathbf{w}_T^2\, d\widetilde{x} + \int_{\Gamma} \beta\,\sqrt{\mathcal{Q}}\,\mathbf{v}_T^2 \cdot \mathbf{w}_T^2\, dS = \int_{\Gamma} \mathbf{f}_T^2 \cdot \mathbf{w}_T^2\, d\widetilde{x}, \tag{62a}$$

$$\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{v}^1\,\varphi^1 - \int_{\Gamma} \left(\mathbf{v}^1 \cdot \mathbf{n}\right)\varphi^2\, d\widetilde{x} + \int_{\Gamma} \boldsymbol{\nabla}_T \cdot\, \mathbf{v}_T^2\,\varphi^2\, d\widetilde{x} = \int_{\Omega_1} h^1\,\varphi^1, \tag{62b}$$

for all $[\mathbf{w}, \varphi] \in \mathbf{X}^0 \times \mathbf{Y}^0$.

Due to the isomorphism between spaces as highlighted in Remark 5, the result follows. □

Now we outline an independent proof that Problem (60) is well-posed.

Theorem 22. *The problem (60) is well-posed.*

Proof. First, showing that the form $\mathbf{v} \mapsto \mathcal{A}(\mathbf{v})\mathbf{v}$ is $\mathbf{X}^{00}$-elliptic, is identical to the proof of Lemma 17. Next, proving that $\mathcal{B}$ has closed range follows exactly as the proof of Lemma 18 with only one extra observation. Notice that defining $\mathbf{u}^1 \overset{\mathbf{def}}{=} -\boldsymbol{\nabla}\phi$, where ϕ is the solution of the auxiliary Problem (53), satisfies $\mathbf{u} \cdot \mathbf{n}\big|_{\Gamma} = \varphi^2 \in L^2(\Gamma)$ then, recalling the Estimate (54) we get

$$\left\|\mathbf{u}^1\right\|^2_{\mathbf{H}_{\mathrm{div}}(\Omega_1)} + \left\|\mathbf{u}^1 \cdot \mathbf{n}\right\|^2_{0,\Gamma} \le C_1^2\left(\|\varphi^1\|^2_{0,\Omega_1} + \|\varphi^2\|^2_{0,\Omega_2}\right) + \|\varphi^2\|^2_{0,\Omega_2} \le C\big(\|\varphi^1\|^2_{0,\Omega_1} + \|\varphi^2\|^2_{0,\Omega_2}\big).$$

From here, it follows trivially that the function $\mathbf{u} = \left(\mathbf{u}^1, \mathbf{0}_T\right)$ satisfies an estimate of the type (55) as well as a chain of inequalities analogous to (56). Therefore, the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the hypotheses of Theorem 2 and the Problem (60) is well-posed. □

We close this section presenting the strong form of Problem (59).

Proposition 23. *The problem (59) is the weak formulation of the boundary-value problem*

$$\mathcal{Q}\,\mathbf{v}^1 + \boldsymbol{\nabla} p^1 = 0\,, \tag{63a}$$
$$\boldsymbol{\nabla} \cdot \mathbf{v}^1 = h^1 \quad \textit{in} \quad \Omega_1, \tag{63b}$$
$$\boldsymbol{\nabla}_T\, p^2 + \beta\,\sqrt{\mathcal{Q}}\,\mathbf{v}_T^2 - \left(\boldsymbol{\nabla}_T\right)'\mu\,\boldsymbol{\nabla}_T\left(\mathbf{v}_T^2\right) = \mathbf{f}_T^2\,, \tag{63c}$$
$$\boldsymbol{\nabla}_T \cdot \mathbf{v}_T^2 - \mathbf{v}^1 \cdot \mathbf{n} = 0\,, \tag{63d}$$
$$p^1 - p^2 = (\mu+\alpha)\,\mathbf{v}^1 \cdot \mathbf{n} \quad \textit{in} \quad \Gamma, \tag{63e}$$
$$p^1 = 0 \quad \textit{on} \quad \partial\Omega_1 - \Gamma, \tag{63f}$$
$$\mathbf{v}_T^2 = \mathbf{0} \quad \textit{on} \quad \partial\Gamma. \tag{63g}$$

The problem (63) is obtained using the standard decomposition of weak derivatives in Problem (59) to get the differential equations in the interior and then the boundary and interface conditions.

5. Strong convergence of the solutions

In this section we show the strong convergence of the velocities and pressures to that of the limiting Problem (48). The strategy is the standard approach in Hilbert spaces: given that the weak convergence of the solutions $\left[\mathbf{v}^\epsilon, p^\epsilon\right] \xrightarrow[\epsilon\to 0]{} [\mathbf{v}, p]$ holds, it is enough to show the convergence of the norms in order to conclude strong convergence statements. Before showing these results a further hypothesis needs to be accepted.

Hypothesis 3. In the following, it will be assumed that the sequence of forcing terms $\{\mathbf{f}^{2,\epsilon} : \epsilon > 0\} \subseteq \mathbf{L}^2(\Omega_2)$ and $\{h^{1,\epsilon} : \epsilon > 0\} \subseteq L^2(\Omega_1)$ is strongly convergent i.e., there exist $\mathbf{f}^2 \in \mathbf{L}^2(\Omega_2)$ and $h^1 \in L^2(\Omega_1)$ such that

$$\left\|\mathbf{f}^{2,\epsilon} - \mathbf{f}^2\right\|_{0,\Omega_2} \xrightarrow[\epsilon\to 0]{} 0, \qquad \left\|h^{1,\epsilon} - h^1\right\|_{0,\Omega_1} \xrightarrow[\epsilon\to 0]{} 0\,. \tag{64}$$

Theorem 24. *Let $\left\{\left[\mathbf{v}^\epsilon, p^\epsilon\right] : \epsilon > 0\right\} \subseteq \mathbf{X} \times \mathbf{Y}$ be the sequence of solutions to the family of Problems (23) and let $[\mathbf{v}, p] \in \mathbf{X}^0 \times \mathbf{Y}^0$, with $\mathbf{v}^2 = [\mathbf{v}_T^2, \xi]$, be the solution of Problem (48), then*

$$\left\|\mathbf{v}_T^{2,\epsilon} - \mathbf{v}_T^2\right\|_{0,\Omega_2} \to 0, \qquad \left\|\boldsymbol{\nabla}_T\, \mathbf{v}_T^{2,\epsilon} - \boldsymbol{\nabla}_T\, \mathbf{v}_T^2\right\|_{0,\Omega_2} \to 0, \tag{65a}$$

$$\left\|\mathbf{v}_N^{2,\epsilon} - \mathbf{v}_N^2\right\|_{H(\partial_z,\Omega_2)} \to 0, \tag{65b}$$

$$\left\|\mathbf{v}^{1,\epsilon} - \mathbf{v}^1\right\|_{\mathbf{H}_{\mathrm{div}}(\Omega_1)} \to 0. \tag{65c}$$

Proof. In order to prove the convergence of norms, a new norm on the space $\mathbf{X}_2^0$, defined in (46a), must be introduced

$$\mathbf{w} \mapsto \left\{\|\sqrt{\mu}\,\boldsymbol{\nabla}_T\,(\mathbf{w}_T^2)\|_{0,\Omega_2}^2 + \|\sqrt{\mu}\,\partial_z\,\mathbf{w}_T^2\|_{0,\Omega_2}^2 + \|\sqrt{\mu}\,\partial_z\,\mathbf{w}_N^2\|_{0,\Omega_2}^2 + \|\sqrt{\alpha}\,\mathbf{w}_N^2\|_{0,\Gamma}^2 + \|\sqrt{\beta}\,\sqrt[4]{\mathcal{Q}}\,\mathbf{w}_T^2\|_{0,\Gamma}^2\right\}^{1/2} \stackrel{\mathbf{def}}{=} \left\|\mathbf{w}\right\|_{\mathbf{X}_2^0}. \tag{66}$$

Clearly, this norm is equivalent to the $\|\cdot\|_{\mathbf{X}_2^0}$-norm defined in (46b). Now consider

$$\begin{aligned}
&\limsup_{\epsilon\downarrow 0}\left\{\|\sqrt{\mathcal{Q}}\,\mathbf{v}^{1,\epsilon}\|_{0,\Omega_1}^2 + \left\|\left[\mathbf{v}_T^2,\,\xi\right]\right\|_{\mathbf{X}_2^0}^2\right\} \\
&\quad\le \limsup_{\epsilon\downarrow 0}\left\{\|\sqrt{\mathcal{Q}}\,\mathbf{v}^{1,\epsilon}\|_{0,\Omega_1}^2 + \|\sqrt{\mu}\,\boldsymbol{\nabla}_T\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right)\|_{0,\Omega_2}^2 + \|\sqrt{\mu}\,\partial_z\,\mathbf{v}_N^{2,\epsilon}\|_{0,\Omega_2}^2 + \|\sqrt{\mu}\left(\epsilon\,\boldsymbol{\nabla}_T\,\mathbf{v}_N^{2,\epsilon}\right)\|_{0,\Omega_2}^2\right. \\
&\qquad\left. + \|\sqrt{\mu}\left(\partial_z\,\mathbf{v}_T^{2,\epsilon}\right)\|_{0,\Omega_2}^2 + \|\sqrt{\alpha}\,\mathbf{v}^{1,\epsilon}\cdot\mathbf{n}\|_{0,\Gamma}^2 + \|\sqrt{\beta}\,\sqrt[4]{\mathcal{Q}}\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right)\|_{0,\Gamma}^2\right\} \le \int_{\Omega_2}\mathbf{f}^2\cdot\mathbf{v}^2 + \int_{\Omega_1} h^1\,p^1.
\end{aligned} \tag{67}$$

On the other hand, testing the Equations (48) on the solution $[\mathbf{v}, p]$ and adding them together, gives

$$\begin{aligned}
&\|\sqrt{\mathcal{Q}}\,\mathbf{v}^1\|_{0,\Omega_1}^2 + \|\sqrt{\mu}\,\boldsymbol{\nabla}_T\,\mathbf{v}_T^2\|_{0,\Omega_2}^2 + \|\sqrt{\mu}\,\partial_z\,\xi\|_{0,\Omega_2}^2 \\
&\qquad + \|\sqrt{\alpha}\,\mathbf{v}^1\cdot\mathbf{n}\|_{0,\Gamma}^2 + \|\sqrt{\beta}\,\sqrt[4]{\mathcal{Q}}\,\mathbf{v}_T^2\|_{0,\Gamma}^2 = \int_{\Omega_2}\mathbf{f}^2\cdot\mathbf{v}^2\,d\widetilde{x}\,dz + \int_{\Omega_1} h^1\,p^1\,dx.
\end{aligned} \tag{68}$$

Comparing the left hand side of (67) and (68) we conclude one inequality.

Next, due to the weak convergence of the sequence $\{[\epsilon\,\mathbf{v}_T^{2,\epsilon}, \mathbf{v}_N^{2,\epsilon}] : \epsilon > 0\} \subseteq \mathbf{X}_2^0$, it must hold that

$$\big\|\big[\mathbf{v}_T^2,\, \xi\big]\big\|_{\mathbf{X}_2^0}^2 \leq \liminf_{\epsilon\downarrow 0} \big\|\,\big[\epsilon\,\mathbf{v}_T^{2,\epsilon}, \mathbf{v}_N^{2,\epsilon}\big]\big\|_{\mathbf{X}_2^0}^2 = \liminf_{\epsilon\downarrow 0} \Big\{\|\,\sqrt{\mu}\,\boldsymbol{\nabla}_T\,(\epsilon\,\mathbf{v}_T^{2,\epsilon})\|_{0,\Omega_2}^2 + \|\sqrt{\mu}\,\partial_z\,\mathbf{v}_T^{2,\epsilon}\|_{0,\Omega_2}^2 + \|\sqrt{\mu}\,\partial_z\,\mathbf{v}_N^{2,\epsilon}\|_{0,\Omega_2}^2 + \|\sqrt{\alpha}\,\mathbf{v}_N^{2,\epsilon}\|_{0,\Gamma}^2 + \|\sqrt{\beta}\,\sqrt[4]{\mathcal{Q}}\,\mathbf{v}_T^{2,\epsilon}\|_{0,\Gamma}^2\Big\}. \tag{69}$$

In addition, due to the weak convergence discussed in Corollary 20, in particular it holds that

$$\big\|\sqrt{\mathcal{Q}}\,\mathbf{v}^1\big\|_{0,\Omega_1}^2 \leq \liminf_{\epsilon\downarrow 0} \big\|\sqrt{\mathcal{Q}}\,\mathbf{v}^{1,\epsilon}\big\|_{0,\Omega_1}^2. \tag{70}$$

Putting together (69), (70), (67) and (68) we conclude that the norms are convergent, i.e.

$$\big\|(\mathbf{v}^1, [\mathbf{v}_T^2,\, \xi])\big\|_{\mathbf{L}^2(\Omega_1)\times\mathbf{X}_2^0}^2 = \lim_{\epsilon\downarrow 0} \big\|(\mathbf{v}^{1,\epsilon}, [\epsilon\,\mathbf{v}_T^{2,\epsilon},\, \mathbf{v}_N^{2,\epsilon}])\big\|_{\mathbf{L}^2(\Omega_1)\times\mathbf{X}_2^0}^2. \tag{71}$$

Since the norm $\mathbf{w} \mapsto \|\sqrt{\mathcal{Q}}\,\mathbf{w}\|_{0,\Omega_1}$ is equivalent to the standard $\mathbf{L}^2(\Omega_1)$-norm, we conclude the strong convergence of the sequence $\big\{\big(\mathbf{v}^{1,\epsilon}, \big[\epsilon\,\mathbf{v}_T^{2,\epsilon},\, \mathbf{v}_N^{2,\epsilon}\big]\big) : \epsilon > 0\big\}$ to $[\mathbf{v}, p]$ as elements of $\mathbf{L}^2(\Omega_1)\times\mathbf{X}_2^0$. In particular, the Statements (65a) and (65b) follow. Finally, recalling the Equality (24b) and the strong convergence of the forcing terms $\{h^{1,\epsilon} : \epsilon > 0\}$, the Statement (65c) follows and the proof is complete. □

Remark 6. Notice that (71) together with (67) implies

$$c \lim_{\epsilon\downarrow 0} \Big\{\big\|\boldsymbol{\nabla}_T\big(\epsilon\,\mathbf{v}_N^{2,\epsilon}\big)\big\|_{0,\Omega_2}^2 + \big\|\partial_z\,\mathbf{v}_T^{2,\epsilon}\big\|_{0,\Omega_2}^2\Big\} \leq \lim_{\epsilon\downarrow 0} \Big\{\big\|\mu\,\boldsymbol{\nabla}_T\big(\epsilon\,\mathbf{v}_N^{2,\epsilon}\big)\big\|_{0,\Omega_2}^2 + \big\|\mu\,\partial_z\,\mathbf{v}_T^{2,\epsilon}\big\|_{0,\Omega_2}^2\Big\} = 0, \tag{72}$$

where $c > 0$ is an ellipticity constant coming from μ.

Next, we show the strong convergence of the pressures.

Theorem 25. *Let $\{\big[\mathbf{v}^\epsilon, p^\epsilon\big] : \epsilon > 0\} \subseteq \mathbf{X}\times\mathbf{Y}$ be the sequence of solutions to the family of Problems (23) and let $[\mathbf{v}, p] \in \mathbf{X}^0 \times \mathbf{Y}^0$ be the solution of Problem (48), then*

$$\big\|p^{1,\epsilon} - p^1\big\|_{1,\Omega_1} \to 0, \tag{73a}$$

$$\big\|p^{2,\epsilon} - p^2\big\|_{0,\Omega_2} \to 0. \tag{73b}$$

Proof. For the Statement (73a) first observe that (65c) together with (24a) implies $\big\|\boldsymbol{\nabla}p^{1,\epsilon} - \boldsymbol{\nabla}p^1\big\|_{0,\Omega_1} \to 0$. Again, since $p^{1,\epsilon} = 0$ on $\partial\Omega_1 - \Gamma$ then $p^1 = 0$ on $\partial\Omega_1 - \Gamma$ and, due to Poincaré's inequality, the gradient controls the $H^1(\Omega_1)$-norm, of p^1 and $p^{1,\epsilon}$ for all $\epsilon > 0$. Consequently, the convergence (73a) follows.

Proving the Statement (73b) is significantly more technical. We start taking a previous localization step for the function $p^{2,\epsilon}$. Let $\phi_\epsilon \in C_0^\infty(\Omega_2)$ be a function such that

$$\big\|p^{2,\epsilon} - \phi_\epsilon\big\|_{0,\Omega_2} < \epsilon.$$

Observe that

$$\Big|\int_{\Omega_2} p^{2,\epsilon}\,p^{2,\epsilon} - \int_{\Omega_2} p^{2,\epsilon}\,\phi_\epsilon\Big| = \Big|\int_{\Omega_2} p^{2,\epsilon}\,\big(p^{2,\epsilon} - \phi_\epsilon\big)\Big| \leq \big\|p^{2,\epsilon}\big\|_{0,\Omega_2}\,\big\|p^{2,\epsilon} - \phi_\epsilon\big\|_{0,\Omega_2} < \widetilde{C}\,\epsilon, \tag{74}$$

where the last inequality in the expression above holds due to the Statement (37). In addition $\phi_\epsilon \to p^2$ weakly in $L^2(\Omega_2)$ because, for any $w \in L^2(\Omega_2)$ it holds that

$$\int_{\Omega_2} \phi_\epsilon\, w = \int_{\Omega_2} \left(\phi_\epsilon - p^{2,\epsilon}\right) w + \int_{\Omega_2} p^{2,\epsilon}\, w \to 0 + \int_{\Omega_2} p^2\, w.$$

In particular, taking $w = w(\widetilde{x})$ in the expression above, we conclude that $\int_0^1 \phi_\epsilon\, dz \to p^2$ weakly in $L^2(\Gamma)$.

Now, for ϕ_ϵ define the function ς_ϵ using the rule presented in the Identity (39), therefore $\varsigma_\epsilon\,|_\Gamma = \varsigma_\epsilon(\widetilde{x}, 0) = \int_0^1 \phi_\epsilon\, dz$ belongs to $L^2(\Gamma)$ and, by construction, $\varsigma_\epsilon\,|_\Gamma$ is bounded in $L^2(\Gamma)$. Then, due to Lemma 1, there must exist $\mathbf{w}_\epsilon^1 \in \mathbf{H}_{\mathrm{div}}(\Omega_1)$ such that $\mathbf{w}_\epsilon^1 \cdot \mathbf{n} = \int_0^1 \phi_\epsilon\, dz$ on Γ, $\mathbf{w}_\epsilon^1 \cdot \mathbf{n} = 0$ on $\partial\Omega_1 - \Gamma$ and $\left\|\mathbf{w}_\epsilon^1\right\|_{\mathbf{H}_{\mathrm{div}}(\Omega_1)} \leq C_1 \left\|\varsigma_\epsilon(\widetilde{x}, 0)\right\|_\Gamma < C$. Where C_1 depends only on the domain. Hence, the function $\mathbf{w}_\epsilon \overset{\mathbf{def}}{=} [\mathbf{w}_\epsilon^1, \mathbf{w}_\epsilon^2]$, with $\mathbf{w}_\epsilon^2 \overset{\mathbf{def}}{=} \left(\mathbf{0}_T,\, \varsigma_\epsilon(\widetilde{x}, z)\right)$, belongs to $\mathbf{X}$. Test, (23a) with $\mathbf{w}_\epsilon$ and get

$$\begin{aligned}\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^{1,\epsilon} \cdot \mathbf{w}_\epsilon^1 - \int_{\Omega_1} p^{1,\epsilon}\, \boldsymbol{\nabla} \cdot \mathbf{w}_\epsilon^1 + \alpha \int_\Gamma \left(\mathbf{v}^{1,\epsilon} \cdot \mathbf{n}\right)\left(\mathbf{w}_\epsilon^1 \cdot \mathbf{n}\right) d\widetilde{x} \\ + \int_{\Omega_2} p^{2,\epsilon}\, \phi_\epsilon(\widetilde{x}, z)\, d\widetilde{x}\, dz + \epsilon \int_{\Omega_2} \mu\, \boldsymbol{\nabla}_T \left(\epsilon\, \mathbf{v}_N^{2,\epsilon}\right) \cdot\, \boldsymbol{\nabla}_T\, \varsigma_\epsilon(\widetilde{x}, z)\, d\widetilde{x}\, dz \\ - \int_{\Omega_2} \mu\, \partial_z\, \mathbf{v}_N^{2,\epsilon}\, \phi_\epsilon(\widetilde{x}, z)\, d\widetilde{x}\, dz = \epsilon \int_{\Omega_2} \mathbf{f}_N^{2,\,\epsilon}\, \varsigma_\epsilon\, d\widetilde{x}\, dz. \quad (75)\end{aligned}$$

In the Identity (75) all the summands but the fourth, are known to be convergent due to the previous strong convergence statements, therefore, this last summand must converge too. The first two summands satisfy

$$\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^{1,\epsilon} \cdot \mathbf{w}_\epsilon^1 - \int_{\Omega_1} p^{1,\epsilon} \boldsymbol{\nabla} \cdot \mathbf{w}_\epsilon^1 = -\int_\Gamma p^{1,\epsilon} \left(\mathbf{w}_\epsilon^1 \cdot \mathbf{n}\right) d\widetilde{x} = -\int_\Gamma p^{1,\epsilon} \Big(\int_0^1 \phi_\epsilon\, dz \Big)\, d\widetilde{x} \to -\int_\Gamma p^1\, p^2\, d\widetilde{x}.$$

The limit above holds due to the strong convergence of the pressure in $H^1(\Omega_1)$ and the weak convergence of $\int_{[0,1]} \phi_\epsilon\, dz$. The third summand in (75) behaves as

$$\alpha \int_\Gamma \left(\mathbf{v}^{1,\epsilon} \cdot \mathbf{n}\right)\left(\mathbf{w}_\epsilon^1 \cdot \mathbf{n}\right) d\widetilde{x} = \int_\Gamma \alpha \left(\mathbf{v}^{1,\epsilon} \cdot \mathbf{n}\right) \Big(\int_0^1 \phi_\epsilon\, dz \Big)\, d\widetilde{x} \to \int_\Gamma \alpha \left(\mathbf{v}^1 \cdot \mathbf{n}\right) p^2\, d\widetilde{x},$$

because of the Statement (65b). Next, the fifth summand in (75) vanishes due to the Estimates (72) and the sixth summand behaves in the following way

$$\begin{aligned}-\int_{\Omega_2} \mu\, \partial_z\, \mathbf{v}_N^{2,\epsilon}\, \phi_\epsilon(\widetilde{x}, z)\, d\widetilde{x}\, dz \to -\int_{\Omega_2} \mu\, \partial_z\, \xi \left(\operatorname*{wk-lim}_{\epsilon \downarrow 0} \phi_\epsilon(\widetilde{x}, z)\right) d\widetilde{x}\, dz \\ = -\int_\Gamma \mu\, \partial_z\, \xi \Big(\int_0^1 \operatorname*{wk-lim}_{\epsilon \downarrow 0} \phi_\epsilon(\widetilde{x}, z)\, dz \Big)\, d\widetilde{x} = -\int_\Gamma \mu\, \partial_z\, \xi\, p^2\, d\widetilde{x}.\end{aligned}$$

The first equality above holds since $\partial_z\, \xi = \partial_z\, \xi(\widetilde{x})$, while the second holds, because $\int_0^1 \operatorname*{wk-lim}_{\epsilon \downarrow 0} \phi_\epsilon(\widetilde{x}, z)\, dz = \operatorname*{wk-lim}_{\epsilon \downarrow 0} \int_0^1 \phi_\epsilon(\widetilde{x}, z)\, dz = p^2$. Finally, the right hand side on (75) vanishes. Putting together all these observations we conclude that

$$\int_{\Omega_2} p^{2,\epsilon}\, \phi_\epsilon(\widetilde{x}, z)\, d\widetilde{x}\, dz \to \int_{\Omega_2} \left(\mu\, \partial_z\, \xi - \alpha\, \mathbf{v}^1 \cdot \mathbf{n}\,|_\Gamma + p^1\,|_\Gamma\right) p^2\, d\widetilde{x}.$$

The latter, together with (74) and (43) implies

$$\left\|p^{2,\epsilon}\right\|_{0,\Omega_2}^2 \to \int_{\Omega_2} \left(\mu\,\partial_z\,\xi - \alpha\,\mathbf{v}^1\cdot\mathbf{n}\,|_\Gamma + p^1\,|_\Gamma\right) p^2\,d\widetilde{x} = \int_\Gamma p^2\,p^2\,d\widetilde{x} = \left\|p^2\right\|_{0,\Omega_2}^2.$$

Again, the convergence of norms together with the weak convergence of the solutions stated in Corollary 20, implies the strong convergence Statement (73b). □

5.1. Comments on the ratio of velocities

The ratio of velocity magnitudes in the tangential and the normal directions is very high and tends to infinity as expected. Since $\{\|\mathbf{v}_N^{2,\epsilon}\|_{0,\Omega_2} : \epsilon > 0\}$ is bounded, it follows that $\|\epsilon\,\mathbf{v}_N^{2,\epsilon}\|_{0,\Omega_2} = \epsilon\|\mathbf{v}_N^{2,\epsilon}\|_{0,\Omega_2} \to 0$. Suppose first that $\mathbf{v}_T^2 \neq 0$ and consider the following quotients

$$\frac{\|\mathbf{v}_T^{2,\epsilon}\|_{0,\Omega_2}}{\|\mathbf{v}_N^{2,\epsilon}\|_{0,\Omega_2}} = \frac{\|\epsilon\,\mathbf{v}_T^{2,\epsilon}\|_{0,\Omega_2}}{\|\epsilon\,\mathbf{v}_N^{2,\epsilon}\|_{0,\Omega_2}} > \frac{\|\mathbf{v}_T^2\|_{0,\,\Omega_2} - \delta}{\|\epsilon\,\mathbf{v}_N^{2,\epsilon}\|_{0,\Omega_2}} > 0.$$

The lower bound holds true for $\epsilon > 0$ small enough and adequate $\delta > 0$. Then, we conclude that the ratio of tangent component over normal component L^2-norms, blows-up to infinity i.e., the tangential velocity is much faster than the normal one in the thin channel.

If $\mathbf{v}_T^2 = 0$ we can not use the same reasoning, so a further analysis has to be made. Suppose then that the solution $[(\mathbf{v}^1, \mathbf{v}^2), (p^1, p^2)]$ of Problem (59) is such that $\mathbf{v}_T^2 = 0$. Then, the Equation (63d) implies that $\mathbf{v}^1 \cdot \mathbf{n} = 0$ on Γ i.e., the activity on the region Ω_1 is independent from the activity on the interface Γ. The pressure on Γ becomes subordinate and it must satisfy the following conditions:

$$p^2 = \left(p^1 - (\mu+\alpha)\,\mathbf{v}^1\cdot\mathbf{n}\right)\big|_\Gamma = \left(p^1 - (\mu+\alpha)\,\mathcal{Q}^{-1}\,\boldsymbol{\nabla}\,p^1\cdot\mathbf{n}\right)\big|_\Gamma,$$
$$\boldsymbol{\nabla}_T\,p^2 = \mathbf{f}_T^2.$$

On the other hand, the values of p^1 are defined by h^1. Hence, if we impose the condition that the forcing term $\mathbf{f}_T^2$ does not have potential, then

$$\mathbf{f}_T^2 \neq \boldsymbol{\nabla}_T\left(p^1 - (\mu+\alpha)\,\mathbf{v}^1\cdot\mathbf{n}\right)\big|_\Gamma,$$

and we obtain a contradiction. Consequently, restrictions on the forcing terms $\mathbf{f}^2$ and h^1 can be given, so that $\mathbf{v}_T^2 \neq 0$ and the magnitudes relation $\|\mathbf{v}_T^{2,\epsilon}\|_{0,\Omega_2} >> \|\mathbf{v}_N^{2,\epsilon}\|_{0,\Omega_2}$ holds for $\epsilon > 0$ small enough, as discussed above. Two aspects must be stressed. Up to the authors' best knowledge, is not possible to conclude the latter velocities ratio without requiring adequate constraints on the forcing terms, in the same way that it is not possible to attain strong convergence or dimensional reduction without the corresponding conditions on the forcing terms. On the other hand, the well-posedness of the system, shown in Theorem 22, is assured independently from the constraints suggested in this section.

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