We consider the (possibly) improperly posed final value problem

\[ u'(t) + Au(t) = 0, \quad 0 < t < T \tag{E} \]

\[ u(T) = f, \]

where \( A \) is a maximal accretive (linear) operator in a complex Hilbert space \( H \). When the numerical range of \( A \) lies in the sector of those complex numbers \( z \) with \( |\arg(z)| \leq \pi/4 \), we show there is at most one solution of the problem and we give a quasi-reversibility method which converges uniformly on compact subsets of \((0,T]\) if and only if there exists a solution.

The plan is as follows:

I is a discussion of the relation between solutions of (E) and the semigroup generated by \(-A\).

II introduces the QR-semigroups which describe our quasi-reversibility method.

III contains applications to certain parabolic evolution equations of second order in time.

I. THE SEMIGROUP AND SOLUTIONS

We shall assume that the linear operator \( A \) is maximal accretive and \( D(A) \) is dense in \( H \). This is equivalent to each of the following [4,5,10]:

(a) \( \text{Re}(Ax,x) \geq 0, \quad x \in D(A), \text{ and } I + A \text{ is onto } H; \)

(b) \( J_\alpha = (I + \alpha A)^{-1} \) is a contraction (defined everywhere) on \( H \) for each \( \alpha > 0; \)

(c) \(-A \) generates a strongly-continuous semigroup \( \{S(t): t \geq 0\} \) of contractions on \( H; \)

(i) \( S(\cdot)x \) is continuous for each \( x \in H, \)

76
(ii) \( S(t + s) = S(t)S(s), \) \( S(0) = I, \)

(iii) \( \| S(t) \| \leq 1, \)

\( \| \xi(H) \| \)

(iv) \( D(A) = \{ x : \lim_{h \to 0} h^{-1} (S(h)x - x) \text{ exists} \}, \) and the limit in \( H \) is \( -Ax. \)

**Definition.** A solution of \( (E) \) on \([a, b]\) is a function \( u \in C([a, b], H) \cap C^1((a, b), H) \) for which \( u(t) \in D(A) \) and \( (E) \) holds for all \( t \in (a, b). \) It follows that \( u \) is a solution of \( (E) \) on \([a, b]\) if and only if \( u(t) = S(t-a)u(a), \) \( a \leq t \leq b, \) and \( u(t) \in D(A), \) \( a < t < b. \)

**Definition.** A weak solution of \( (E) \) on \([a, b]\) is a function of the form \( u(t) = S(t-a)\xi \) for some \( \xi \in H. \)

Thus, the semigroup \( S \) generated by \(-A\) is precisely the operational representation of (weak) solutions of \( (E) \) in terms of initial values.

**Remark 1.** There exists a weak solution of the final value problem if and only if \( f = S(T)\xi \) for some \( \xi \in H. \)

Eventually we shall restrict our attention to those operators \( A \) as above which also satisfy

\[ \text{Re}(Ax, x) \geq |\text{Im}(Ax, x)|, \quad x \in D(A). \]

Then \( A \) is \( m \)-sectorial with angle \( \pi/4 \) [5] and the semigroup \( \{S(t)\} \) is holomorphic. This implies that \( S(t)x \) is (infinitely) differentiable at each \( t > 0, \) so every weak solution is a solution.

**Remark 2.** If \( S(\cdot) \) is holomorphic and if the final value problem is properly posed, then \( a \) is bounded [2]. Thus in "most" situations to which our results apply the final value problem is necessarily improperly posed.

We sketch Yosida's elegant proof of the generation theorem [10]. Define the bounded operators \( A_\alpha = AJ_\alpha, \) \( \alpha > 0, \) and note that \( A_\alpha = \alpha^{-1}(I-J_\alpha). \) Since \( J_\alpha \) is a contraction, \( \|A_\alpha x\| \leq \|Ax\| \)
and \( \| J_{\alpha} x - x \| \leq \alpha \| A x \|, \) \( x \in D(A) \), and hence \( \| A_{\alpha} x - A x \| \leq \alpha \| A^2 x \|, \) \( x \in D(A^2) \). These show \( A_{\alpha} \) approximates \( A \) and \( J_{\alpha} \) approximates \( I \) for small \( \alpha \). Each \( A_{\alpha} \) is accretive so the group \( S_{\alpha}(t) = \exp(-A_{\alpha} t) \) consists of contractions for \( t \geq 0 \). These facts are used to prove the existence of the strong limit \( S(t) = \lim_{\alpha \to 0} S_{\alpha}(t), t \geq 0 \), thereby defining the semigroup \( S(\cdot) \).

II. QUASIREVERSIBILITY

In hopes of obtaining an approximate solution of the final value problem, we first solve the properly posed problem

\[
\begin{align*}
&v_{\alpha}'(t) + \alpha A v_{\alpha}'(t) + A v_{\alpha}(t) = 0 \quad (\text{E}_\alpha) \\
v_{\alpha}(T) = f
\end{align*}
\]

for small \( \alpha > 0 \). Then we use \( v_{\alpha}(0) \) as the initial value to determine a solution \( u_{\alpha} \) of \((\text{E})\) with \( u_{\alpha}(0) = v_{\alpha}(0) \). We expect to have \( u_{\alpha}(T) \) close to \( f \) for sufficiently small \( \alpha > 0 \).

Note that \((\text{E}_\alpha)\) is equivalent to \((\text{E})\) with \( A \) replaced by the bounded operator \( A_{\alpha} \). Thus, we have \( v_{\alpha}(t) = S_{\alpha}(t-T)f \), a representation by the group \( S_{\alpha} \), and our approximate solution to the final value problem is given by

\[
u_{\alpha}(t) = S(t)S_{\alpha}(-T)f, \quad 0 \leq t \leq T.
\]

Our goal above is to show that \( S(T)S_{\alpha}(-T)f \) is close to \( f \). This suggests a Definition. For \( \alpha > 0 \), let \( E_{\alpha}(\cdot) = S(t)S_{\alpha}(-t), t \geq 0 \). \{\( E_{\alpha}(\cdot)\}\) is the collection of QR-semigroups for the operator \( A \). The QR-semigroups are stable if they are all contractions.

Lemma 1. \( E_{\alpha}(\cdot) \) is generated by \(- (A - A_{\alpha})\).

Lemma 2. The following are equivalent:

(a) \{\( E_{\alpha}(\cdot)\}\) is stable;

(b) \( A - A_{\alpha} \) is accretive for every \( \alpha > 0 \);

(c) \( A^2 \) is accretive;

(d) \( \text{Re}(A x, x) \geq |\text{Im}(A x, x)|, \) \( x \in D(A) \)
Consider \( \lim_{\alpha \to 0} E_{\alpha}(t)x \) for \( x \in D(A) \). The Fundamental Theorem of Calculus gives

\[
E_{\alpha}(t)x - E_{\beta}(t)x = \int_0^t \frac{d}{ds} \left\{ E_{\alpha}(s) E_{\beta}(t-s)x \right\} ds
= \int_0^t E_{\alpha}(s) E_{\beta}(t-s) (A_{\beta} x - A_{\alpha} x) \, ds,
\]

so when \( \{E_{\alpha}(\cdot)\} \) are stable we obtain

\[
\| E_{\alpha}(t)x - E_{\beta}(t)x \| \leq t \| A_{\beta} x - A_{\alpha} x \|, \quad x \in D(A).
\]

Hence we can define \( E(t)x \) as the limit of \( E_{\alpha}(t)x \) for \( \alpha \to 0 \) for \( x \in D(A) \) and extend by continuity to \( x \in H \). The convergence is uniform on bounded intervals, so we can take the limit in the integral

\[
\int_0^t E_{\alpha}(s)(Ax - A_{\alpha} x) ds = x - E_{\alpha}(x), \quad x \in D(A)
\]

to obtain \( E(t)x = x \), hence \( E(t) = I \). The preceding remarks indicate a proof of our

**Theorem 1.** In the situation of Lemma 2, \( E_{\alpha}(t)x \rightarrow x \) (strongly) as \( \alpha \to 0 \) for \( x \in H, t > 0 \); the convergence is uniform on bounded intervals, and

\[
\| E_{\alpha}(t)x - x \| \leq t \| Ax - A_{\alpha} x \|, \quad x \in D(A).
\]

**Corollary 1** (Backward Uniqueness). There is at most one solution of the final value problem.

**Proof.** By Remark 1 and linearity, this is equivalent to showing that the kernel of \( S(T) \) is \( \{0\} \). This is equivalent to showing the range of the adjoint \( S^*(T) \) is dense in \( H \). But the adjoint QR-semigroups \( \{E_{\alpha}^*(t)\} \) are stable exactly when \( \{E_{\alpha}(t)\} \) are stable, so Theorem 1 shows \( S^*(T)S_{\alpha}^*(-T)x = x \), hence the range of \( S^*(T) \) is dense, and we are done.

Suppose \( f = S(\delta)\xi \). Theorem 1 shows \( \xi = \lim_{\alpha \to 0} E_{\alpha}(\delta)\xi = \lim_{\alpha \to 0} S_{\alpha}(-\delta)f \).

Conversely if \( \xi = \lim_{\alpha \to 0} S_{\alpha}(-\delta)f \), then each \( S_{\alpha}(\delta) \) being a
contraction implies \( \lim_{\alpha \to 0} S_\alpha(\delta) S_\alpha(-\delta) f = S(\delta) \xi \). But this limit is \( f \) by Theorem 1. These observations and Remark 1 give us

**Corollary 2 (Existence).** Let \( 0 \leq \delta \leq T \). There is a solution \( u \) of (E) on \([T-\delta, T]\) with \( u(T) = f \) if and only if \( \lim_{\alpha \to 0} S_\alpha(-\delta) f \)

exists in \( H \). (In that case, the limit is precisely \( u(T-\delta) \).

In the situation of Corollary 2, we have the representations

\[
\begin{align*}
  u(t) & = S(t + \delta - T) \xi, \quad t \geq T - \delta, \\
  u_\alpha(t) & = S(t + \delta - T) E_\alpha(t) \xi, \quad t \geq T - \delta, \alpha > 0
\end{align*}
\]

and hence derivatives of the difference are given by

\[
  u_\alpha^{(m)}(t) - u^{(m)}(t) = (-A)^m S(t+\delta-T)(E_\alpha(T) \xi - \xi),
\]

\[ \alpha > 0, \ t > T - \delta, \ m \geq 0. \]

Since \( S(\cdot) \) is holomorphic, we obtain

**Corollary 3 (Estimates).** If there is a solution \( u \) of the final value problem on \([T-\delta, T]\), then

\[
  \| u_\alpha^{(m)}(t) - u^{(m)}(t) \| \leq |M/(t+\delta-T)|^m \| E_\alpha(T) \xi - \xi \|,
\]

\[ \alpha > 0, \ t > T - \delta, \ m \geq 0 \]

and

\[
  \| u_\alpha^{(m)}(t) - u^{(m)}(t) \| \leq |M/(t+\delta-T-\epsilon)|^m T \alpha \| A^2 S(\epsilon) \xi \|,
\]

\[ \alpha > 0, \ 0 < \epsilon \leq \delta, \ t > T - \delta + \epsilon. \]

**Remark 3.** The quasi-reversibility method was introduced by Lattes and Lions [6]. They approximated (E) by the equation

\[
  w'(t) + Aw(t) - \alpha A^2 w(t) = 0
\]

where \( A \) is self-adjoint and positive. See [7] for additional results and references.
Remark 4. When $A$ is a realization of an elliptic partial differential operator, $(E^\alpha)$ is a pseudo-parabolic or Sobolev partial differential equation [8]. Such equations arise in various applications in which $\alpha$ corresponds to viscosity. This writer and Ting [9] observed that Yosida's proof shows that such equations approximate the corresponding parabolic equation (E). We may regard this approximation as a method of vanishing viscosity.

Remark 5 By considering solutions which satisfy a prescribed global bound, one can use the logarithmic convexity of solutions of (E) to stabilize the final value problem [1].

III. A SECOND ORDER EVOLUTION EQUATION

We attempt to apply our preceding results to the equation

$$v''(t) + C v'(t) + B v(t) = 0$$

where (for simplicity) $B$ is self-adjoint and accretive in a Hilbert space $H$, and $C$ is accretive. (If $-C$ is accretive, the final value problem is properly posed.) The change of variable $w(t) = e^{-\lambda t} v(t)$ gives the equivalent equation

$$\frac{d}{dt} \begin{pmatrix} w \\ w' \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ \lambda^2 + \lambda C + B & 2 \lambda + C \end{pmatrix} \begin{pmatrix} w \\ w' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ 

Setting $\bar{w}(t) = e^{-\mu t} \begin{pmatrix} w \\ w' \end{pmatrix}$ gives us the equation (E) on the product space $H = H \times H$ with the operator

$$A = \begin{pmatrix} \mu & -1 \\ \lambda^2 + \lambda C + B & \mu + 2 \lambda + C \end{pmatrix}$$

whose square is given by

$$A^2 = \begin{pmatrix} \mu^2 - \lambda^2 - (\lambda C + B) & -(2 \mu + 2 \lambda + C) \\ (\lambda^2 + \lambda C + B)(2 \mu + 2 \lambda + C) & (\mu + 2 \lambda + C)^2 - (\lambda^2 + \lambda C + B) \end{pmatrix}$$

The difficulty with the preceding formalities is that it may be impossible, in general, to choose $\mu$ and $\lambda$ so as to make $A$
and $A^2$ accretive. (Consider $A^2$ with $\mu = \lambda = 0$ to appreciate the difficulty). In any event, such matrix operators almost always lead to technical difficulties.

We consider the extremely special case $C=B$, and take comfort in the fortunate fact that such examples do occur, e.g. in hydrodynamics and visco-elasticity [3,6]. Setting $\lambda=-1$ and $\mu=3$ in the above gives

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 1+B \end{pmatrix}, \quad A^2 = \begin{pmatrix} 8 & -(4+B) \\ 4+B & B(2+B) \end{pmatrix}.$$  

Then, $A$ satisfies the hypotheses of Theorem 1 and we have our final result.

**Theorem 2.** Let $B$ be self-adjoint and accretive on the Hilbert space $\mathcal{H}$, and $f, g \in \mathcal{H}$. There is at most one solution $v \in C^1([0,T],\mathcal{H}) \cap C^2((0,T),\mathcal{H})$ of

$$v''(t) + Bv'(t) + Bv(t) = 0,$$

with $v'(t) + v(t) \in D(B)$ for $0 < t < T$ and

$$v(T) = f, \quad v'(T) = g.$$

This problem is equivalent to the final value problem for (E) with $A$ given above, $\tilde{u}(t) = e^{-2t} \begin{pmatrix} v \\ v' + v \end{pmatrix}$, and $\tilde{f} = e^{-2T} \begin{pmatrix} f \\ f + g \end{pmatrix}$.

Thus, for $0 \leq \delta \leq T$, there exists a solution $v$ as above on the interval $[T-\delta, T]$ if and only if $\lim_{\alpha \to 0} \tilde{u}_\alpha^\ast(-\delta)$ exists in $\mathcal{H} \times \mathcal{H}$ where $\tilde{u}_\alpha^\ast$ is the solution of the approximating system

$$(1+3\alpha)u_1' + 3u_1 - \alpha u_2' - u_2 = 0,$$

$$\alpha u_1' + u_1 + [1+\alpha(1+B)]u_2' + (1+B)u_2 = 0,$$

$$u_1(T) = e^{-2T}f, \quad u_2(T) = e^{-2T}(f+g).$$

**Remark 6** Similar results should hold in more general situations. The above proof technique might extend to the case where $C$
Remark 7. The above procedure approximates the equation by one of the form

\[ B_1(\alpha)v''(t) + B_2(\alpha)v'(t) + B_3(\alpha)v(t) = 0 \]

in which each \( B_j(\alpha) \) is a polynomial in \( \alpha \) and \( B \) of second degree in \( \alpha \) and first degree in \( B \). A simpler approximation is the Sobolev regularization.

\[(1+\alpha B)v''(t) + C v'(t) + Bv(t) = 0\]

where we can assume without loss of generality that \( B \) dominates \( C \). Such examples appear in fluid mechanics where \( B = -\Delta \) and \( \alpha > 0 \) corresponds to inertia. (C.f., Remark 4.)

Remark 8. The techniques of this section require that we not make self-adjointness assumptions in Theorem 1: our matrix operator \( A \) is never self-adjoint.

References


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