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Quasi-Reversibility of First and Second Order Parabolic Evolution Equations

INTRODUCTION

We consider the (possibly) improperly posed <u>final value problem</u>

$$u'(t) + Au(t) = 0, \quad 0 < t < T \qquad (E)$$

$$u(T) = f,$$

where A is a maximal accretive (linear) operator in a complex Hilbert space H. When the numerical range of A lies in the sector of those complex numbers z with $|\arg(z)| \leq \pi/4$, we show there is at most one solution of the problem and we give a quasi-reversibility method which converges uniformly on compact subsets of $(0,T]$ if and only if there exists a solution.

The plan is as follows:

I is a discussion of the relation between solutions of (E) and the semigroup generated by -A.

II introduces the QR-semigroups which describe our quasi-reversibility method.

III contains applications to certain parabolic evolution equations of second order in time.

I. THE SEMIGROUP AND SOLUTIONS

We shall assume that the linear operator A is maximal accretive and D(A) is dense in H. This is equivalent to each of the following [4,5,10]:

(a) $\mathrm{Re}(Ax,x) \geq 0$, $x \in D(A)$, and $I + A$ is onto H;

(b) $J_\alpha = (I + \alpha A)^{-1}$ is a contraction (defined everywhere) on H for each $\alpha > 0$;

(c) -A generates a strongly-continuous semigroup $\{S(t): t \geq 0\}$ of contractions on H:

(i) $S(\cdot)x$ is continuous for each $x \in H$,

(ii) $S(t+s)=S(t)S(s)$, $S(0)=I$,

(iii) $\|S(t)\|_{\mathcal{L}(H)} \le 1$,

(iv) $D(A) = \{x\colon \lim_{h\to 0} h^{-1}(S(h)x-x) \text{ exists}\}$, and the limit in H is $-Ax$.

<u>Definition</u>. A <u>solution</u> of (E) on $[a,b]$ is a function $u \in C([a,b],H) \cap C^1((a,b),H)$ for which $u(t) \in D(A)$ and (E) holds for all $t \in (a,b)$. It follows that u is a solution of (E) on $[a,b]$ if and only if $u(t)=S(t-a)u(a)$, $a \le t \le b$, and $u(t) \in D(A)$, $a < t < b$.

<u>Definition</u>. A <u>weak solution</u> of (E) on $[a,b]$ is a function of the form $u(t) = S(t-a)\xi$ for some $\xi \in H$.
Thus, the semigroup S generated by $-A$ is precisely the operational representation of (weak) solutions of (E) in terms of initial values.

<u>Remark 1</u>. There exists a weak solution of the final value problem if and only if $f=S(T)\xi$ for some $\xi \in H$.

Eventually we shall restrict our attention to those operators A as above which also satisfy

$$\mathrm{Re}(Ax,x) \ge |\mathrm{Im}(Ax,x)|, \qquad x \in D(A).$$

Then A is <u>m-sectorial</u> with angle $\pi/4$ [5] and the semigroup $\{S(t)\}$ is <u>holomorphic</u>. This implies that $S(t)x$ is (infinitely) differentiable at each $t > 0$, so every weak solution is a solution.

<u>Remark 2</u>. If $S(\cdot)$ is holomorphic and if the final value problem is properly posed, then a is bounded [2]. Thus in "most" situations to which our results apply the final value problem is necessarily improperly posed.

We sketch Yosida's elegant proof of the generation theorem [10]. Define the bounded operators $A_\alpha = AJ_\alpha$, $\alpha > 0$, and note that $A_\alpha = \alpha^{-1}(I-J_\alpha)$. Since J_α is a contraction, $\|A_\alpha x\| \le \|Ax\|$

and $\|J_\alpha x - x\| \le \alpha\|Ax\|$, $x \in D(A)$, and hence $\|A_\alpha x - Ax\| \le \alpha\|A^2x\|$, $x \in D(A^2)$. These show A_α approximates A and J_α approximates I for small α. Each A_α is accretive so the group $S_\alpha(t) \equiv \exp(-A_\alpha t)$ consists of contractions for $t \ge 0$. These facts are used to prove the existence of the strong limit $S(t) \equiv \lim_{a\to 0} S_\alpha(t)$, $t \ge 0$, thereby defining the semigroup $S(\cdot)$.

II. QUASIREVERSIBILITY

In hopes of obtaining an approximate solution of the final value problem, we first solve the properly posed problem

$$v_\alpha'(t) + \alpha A v_\alpha'(t) + Av_\alpha(t) = 0 \qquad (E^\alpha)$$

$$v_\alpha(T) = f$$

for small $\alpha > 0$. Then we use $v_\alpha(0)$ as the initial value to determine a solution u_α of (E) with $u_\alpha(0) = v_\alpha(0)$. We expect to have $u_\alpha(T)$ close to f for sufficiently small $\alpha > 0$.

Note that (E^α) is equivalent to (E) with A replaced by the bounded operator A_α. Thus, we have $v_\alpha(t) = S_\alpha(t-T)f$, a representation by the group S_α, and our approximate solution to the final value problem is given by

$$u_\alpha(t) = S(t)S_\alpha(-T)f, \qquad 0 \le t \le T.$$

Our goal above is to show that $S(T)S_\alpha(-T)f$ is close to f. This suggests a Definition. For $\alpha > 0$, let $E_\alpha(t) = S(t)S_\alpha(-t)$, $t \ge 0$. $\{E_\alpha(\cdot)\}$ is the collection of QR-semigroups for the operator A. The QR-semigroups are stable if they are all contractions.

Lemma 1. $E_\alpha(\cdot)$ is generated by $-(A-A_\alpha)$.

Lemma 2. The following are equivalent:

(a) $\{E_\alpha(\cdot)\}$ is stable;

(b) $A-A_\alpha$ is accretive for every $\alpha > 0$;

(c) A^2 is accretive;

(d) $\mathrm{Re}(Ax,x) \ge |\mathrm{Im}(Ax,x)|$, $x \in D(A)$

Consider $\lim_{\alpha \to 0} E_\alpha(t)x$ for $x \in D(A)$. The Fundamental Theorem Calculus gives

$$E_\alpha(t)x - E_\beta(t)x = \int_0^t \frac{d}{ds}\left\{E_\alpha(s)E_\beta(t-s)x\right\}ds$$
$$= \int_0^t E_\alpha(s)E_\beta(t-s)(A_\beta x - A_\alpha x)\,ds,$$

so when $\{E_\alpha(\cdot)\}$ are stable we obtain

$$\| E_\alpha(t)x - E_\beta(t)x \| \leq t \| A_\beta x - A_\alpha x \|, \quad x \in D(A).$$

Hence we can define $E(t)x$ as the limit of $E_\alpha(t)x$ for $\alpha \to 0$ for $x \in D(A)$ and extend by continuity to $x \in H$. The convergence is uniform on bounded intervals, so we can take the limit in the integral

$$\int_0^t E_\alpha(s)(Ax - A_\alpha x)ds = x - E_\alpha(x), \quad x \in D(A)$$

to obtain $E(t)x = x$, hence $E(t) = I$. The preceding remarks indicate a proof of our

<u>Theorem 1</u>. In the situation of Lemma 2, $E_\alpha(t)x \to x$ (strongly) as $\alpha \to 0$ for $x \in H$, $t > 0$; the convergence is uniform on bounded intervals, and

$$\| E_\alpha(t)x - x \| \leq t \| Ax - A_\alpha x \|, \quad x \in D(A).$$

<u>Corollary 1</u> (Backward Uniqueness). There is at most one solution of the final value problem.

<u>Proof</u>. By Remark 1 and linearity, this is equivalent to showing that the kernel of $S(T)$ is $\{0\}$. This is equivalent to showing the range of the adjoint $S^*(T)$ is dense in H. But the adjoint QR-semigroups $\{E^*_\alpha(t)\}$ are stable exactly when $\{E_\alpha(t)\}$ are stable, so Theorem 1 shows $S^*(T)S^*_\alpha(-T)x \to x$, hence the range of $S^*(T)$ is dense, and we are done.

Suppose $f = S(\delta)\xi$. Theorem 1 shows $\xi = \lim_{\alpha \to 0} E_\alpha(\delta)\xi = \lim_{\alpha \to 0} S_\alpha(-\delta)f$. Conversely if $\xi = \lim_{\alpha \to 0} S_\alpha(-\delta)f$, then each $S_\alpha(\delta)$ being a contraction implies $\lim_{\alpha\to 0} S_\alpha(\delta)S_\alpha(-\delta)f = S(\delta)\xi$. But this limit is f by Theorem 1. These observations and Remark 1 give us

Corollary 2 (Existence). Let $0 \le \delta \le T$. There is a solution u of (E) on $[T-\delta, T]$ with $u(T) = f$ if and only if $\lim_{\alpha\to 0} S_\alpha(-\delta)f$ exists in H. (In that case, the limit is precisely $u(T-\delta)$.)

In the situation of Corollary 2, we have the representations

$$u(t) = S(t+\delta-T)\xi, \qquad t \ge T-\delta,$$

$$u_\alpha(t) = S(t+\delta-T)E_\alpha(t)\xi, \qquad t \ge T-\delta,\ \alpha > 0$$

and hence derivatives of the difference are given by

$$u_\alpha^{(m)}(t) - u^{(m)}(t) = (-A)^m S(t+\delta-T)(E_\alpha(T)\xi - \xi),$$

$$\alpha > 0,\ t > T-\delta,\ m \ge 0.$$

Since $S(\cdot)$ is holomorphic, we obtain

Corollary 3 (Estimates). If there is a solution u of the final value problem on $[T-\delta, T]$, then

$$\|u_\alpha^{(m)}(t) - u^{(m)}(t)\| \le [M/(t+\delta-T)]^m \|E_\alpha(T)\xi - \xi\|,$$

$$\alpha > 0,\ t > T-\delta,\ m \ge 0$$

and

$$\|u_\alpha^{(m)}(t) - u^{(m)}(t)\| \le [M/(t+\delta-T-\epsilon)]^m T\alpha \|A^2 S(\epsilon)\xi\|,$$

$$\alpha > 0,\ 0 < \epsilon \le \delta,\ t > T-\delta+\epsilon.$$

Remark 3. The quasi-reversibility method was introduced by Lattes and Lions [6]. They approximated (E) by the equation

$$w'(t) + Aw(t) - \alpha A^2 w(t) = 0$$

where A is self-adjoint and positive. See [7] for additional results and references.

Remark 4. When A is a realization of an elliptic partial differential operator, (E^{α}) is a pseudo-parabolic or Sobolev partial differential equation [8]. Such equations arise in various applications in which α corresponds to viscosity. This writer and Ting [9] observed that Yosida's proof shows that such equations approximate the corresponding parabolic equation (E). We may regard this approximation as a method of vanishing viscosity.

Remark 5 By considering solutions which satisfy a prescribed global bound, one can use the logarithmic convexity of solutions of (E) to stabilize the final value problem [1].

III. A SECOND ORDER EVOLUTION EQUATION

We attempt to apply our preceding results to the equation

$$v''(t) + Cv'(t) + Bv(t) = 0$$

where (for simplicity) B is self-adjoint and accretive in a Hilbert space $\mathcal{H}$, and C is accretive. (If -C is accretive, the final value problem is properly posed.) The change of variable $w(t) = e^{-\lambda t}\cdot v(t)$ gives the equivalent equation

$$\frac{d}{dt}\begin{pmatrix} w \\ w' \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ \lambda^2+\lambda C+B & 2\lambda+C \end{pmatrix}\begin{pmatrix} w \\ w' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Setting $\bar{u}(t) = e^{-\mu t}\begin{pmatrix} w \\ w' \end{pmatrix}$ gives us the equation (E) on the product space $H = \mathcal{H}\times\mathcal{H}$ with the operator

$$A = \begin{pmatrix} \mu & -1 \\ \lambda^2+\lambda C+B & \mu+2\lambda+C \end{pmatrix}$$

whose square is given by

$$A^2 = \begin{pmatrix} \mu^2-\lambda^2-(\lambda C+B) & -(2\mu+2\lambda+C) \\ (\lambda^2+\lambda C+B)(2\mu+2\lambda+C) & (\mu+2\lambda+C)^2-(\lambda^2+\lambda C+B) \end{pmatrix}$$

The difficulty with the preceding formalities is that it may be impossible, in general, to choose μ and λ so as to make A and A^2 accretive. (Consider A^2 with $\mu = \lambda = 0$ to appreciate the difficulty). In any event, such matrix operators almost always lead to technical difficulties.

We consider the extremely special case $C=B$, and take comfort in the fortunate fact that such examples do occur, e.g. in hydrodynamics and visco-elasticity [3,6]. Setting $\lambda=-1$ and $\mu=3$ in the above gives

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 1+B \end{pmatrix} \qquad A^2 = \begin{pmatrix} 8 & -(4+B) \\ 4+B & B(2+B) \end{pmatrix}$$

Then, A satisfies the hypotheses of Theorem 1 and we have our final result.

<u>Theorem 2</u>. Let B be self-adjoint and accretive on the Hilbert space $\mathcal{H}$, and $f, g \in \mathcal{H}$. There is at most one solution $v \in C^1([0,T],\mathcal{H}) \cap C^2((0,T),\mathcal{H})$ of

$$v''(t) + Bv'(t) + Bv(t) = 0,$$

with $v'(t) + v(t) \in D(B)$ for $0 < t < T$ and

$$v(T) = f,\ v'(T) = g.$$

This problem is equivalent to the final value problem for (E) with A given above, $\bar{u}(t) = e^{-2t}\begin{pmatrix} v \\ v'+v \end{pmatrix}$, and $\bar{f} = e^{-2T}\begin{pmatrix} f \\ f+g \end{pmatrix}$.

Thus, for $0 \leq \delta \leq T$, there exists a solution v as above on the interval $[T-\delta,T]$ if and only if $\lim_{\alpha\to 0} \bar{u}_\alpha(-\delta)$ exists in $\mathcal{H}\times\mathcal{H}$ where $\bar{u}_\alpha$ is the solution of the approximating system

$$(1+3\alpha)u_1' + 3u_1 - \alpha u_2' - u_2 = 0,$$

$$\alpha u_1' + u_1 + [1+\alpha(1+B)]u_2' + (1+B)u_2 = 0,$$

$$u_1(T) = e^{-2T}f,\ u_2(T) = e^{-2T}(f+g).$$

<u>Remark 6</u> Similar results should hold in more general situations. The above proof technique might extend to the case where C

dominates B.

Remark 7. The above procedure approximates the equation by one of the form

$$B_1(\alpha)v''(t) + B_2(\alpha)v'(t) + B_3(\alpha)v(t) = 0$$

in which each $B_j(\alpha)$ is a polynomial in α and B of second degree in α and first degree in B. A simpler approximation is the Sobolev regularization.

$$(1+\alpha B)v''(t) + Cv'(t) + Bv(t) = 0$$

where we can assume without loss of generality that B dominates C. Such examples appear in fluid mechanics where $B = -\Delta$ and $\alpha > 0$ corresponds to inertia. (C.f., Remark 4.)

Remark 8. The techniques of this section require that we not make self-adjointness assumptions in Theorem 1: our matrix operator A is never self-adjoint.

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