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Plasticity Models and Nonlinear Semigroups*

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The evolution of an elastic-plastic material is modeled as an initial boundary value problem consisting of the dynamic momentum equation coupled with a constitutive law for which the hysteretic dependence between stress and strain is described by a system of variational inequalities. This system is posed as an evolution equation in Hilbert space for which is proved the existence and uniqueness of three classes of solutions which are distinguished by their regularity. Weak solutions are obtained in a very general situation, strong solutions arise in the presence of kinematic work-hardening or viscosity, and the solution is even more regular under a stability assumption connecting the constraint set with the divergence operator.

1. INTRODUCTION

We shall consider the problem of coupling the dynamic equations

$$u_{tt} + D^*\sigma = f(x, t) \tag{1.1.a}$$

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with a special class of constitutive laws

$$\sigma = F(\varepsilon) \tag{1.1.b}$$

for small strain plasticity. Here u is the displacement vector, σ is the tensor of internal stress, f is the volume density of body force, and ε is the strain tensor

$$\varepsilon = Du. \tag{1.1.c}$$

The strain is given by the symmetric gradient

$$(Du)_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

of displacement, and the corresponding dual operator takes the divergence form

$$(D^*\sigma)i = -\sum_{j=1}^{3} \sigma_{ij,j}$$

in (1.1.a). The constitutive laws (1.1.b) considered here permit a variety of well-known hysteresis models of elastic-plastic materials with multi-yield surfaces.

The existence and uniqueness of solutions for the fundamental Prandtl–Reuss model with a single yield surface was given by Duvaut and Lions [6]. The *weak* solution for this model is obtained as the limit of *strong* solutions of corresponding problems which are regularized with viscosity. For these strong solutions, the constitutive law is characterized as a variational equation of evolution type whose input, the strain-rate

$$\frac{\partial \varepsilon}{\partial t} = Dv$$

corresponding to the displacement-rate $v = u_t$, is in L^2. The general approach is to express the constitutive relation (1.1.b) as a variational equation or inequality

$$\sigma_t + \partial\varphi(\sigma) \ni Dv \tag{1.2}$$

which is coupled to the dynamic equation (1.1.a). Here $\varphi(\cdot)$ denotes either the indicator function $I_K(\cdot)$ of a given closed convex set K characterizing the particular plasticity model or a smooth convex function for the viscosity models, and $\partial\varphi$ is the corresponding subgradient or derivative, respectively. For a weak solution, the strain-rate is *not* in L^2, so it must be

understood in a weak form by means of the dual operator, D^*. The sense in which the weak solution satisfies a "nearly strong" form of the constitutive law is substantially developed in [1] where the existence of the weak solution is proved by taking limits of the strong solutions of a different "viscous regularized" equation. The dynamic problem with a very general Prandtl–Ishlinski model for multi-yield surfaces was addressed by Visintin [18]. There the existence and uniqueness of the weak solution was obtained directly by monotonicity methods. In these models, the total stress is given as the (generalized) sum of a collection of stress components, i.e., $\sigma = \sum_j \sigma_j$, where the collection of these components $\boldsymbol{\sigma} \equiv \{\sigma_j\}$ satisfies a system of the form (1.2). Then $D^*\sigma$ will belong to L^2 but the individual σ_j's need not be smooth. An alternative approach is taken in the work of Krejci [13], where a large class of such general multiple component models is considered. There the problem (1.1) is written as a quasilinear wave equation

$$u_{tt} + D^*F(Du) = f \tag{1.3}$$

for which the dissipation properties of the hysteresis functional are developed and exploited. Existence and uniqueness of a strong solution are obtained by the monotonicity method; there the strain-rate Dv is L^2. For the one-dimensional case, the existence of a strong solution is independently proved by a compactness method.

A predominate theme in the above is that the weak solution of a rate-independent perfectly plastic model can be obtained as a limit by penalty method which corresponds to an approximation by the strong solution of a rate-dependent visco-plasticity model. The regularizing effects of viscosity are well known in many contexts, and these approximations are a natural application of the strong solutions obtained. The quasi-static case, in which the dynamic equation (1.1.a) is replaced by the corresponding static equation, was developed in Johnson [9, 10]. There appears a regularizing effect due to work-hardening of the material, and both weak and strong forms of solutions are obtained. The existence and uniqueness of weak solutions of a single-yield Prandtl–Reuss material was further developed by Suquet [17], where the dynamic and quasi-static problems lead to evolution equations with time-dependent monotone operators [5]

$$\frac{d}{dt}A(t)u(t) + B(t, u(t)) = f(t).$$

In this work we write the system (1.1) in the form

$$v_t + D^*\sigma = f(x,t), \qquad \sigma = \sum_j \sigma_j \tag{1.4.a}$$

$$\boldsymbol{\sigma}_t + \partial\varphi(\boldsymbol{\sigma}) - Dv \ni g(x,t), \tag{1.4.b}$$

for which we show the dynamics is governed by a nonlinear semigroup of contractions in L^2-type spaces. That is, the spatial part of this system is the realization of an m-*accretive* operator in Hilbert space. From this representation of the solution of (1.4) via semigroup theory, we shall obtain three classes of solutions which we call *weak*, *strong*, and *regular*, respectively. In this configuration, the smoother strong solution with Dv in L^2 results from a boundedness assumption on a non-trivial measurable subset of the subgradients $\partial\varphi_j$ in the system (1.4.b). In the plasticity examples, this assumption corresponds to the existence of a *kinematic work hardening* component in the stress, and it is also satisfied in the presence of *viscosity*. This shows that each of these characteristics has a regularizing effect. Also see [12]. With an additional stability condition relating the convex sets of the plasticity model to the divergence operator, D^*, we obtain the regular solution for which each component of $\boldsymbol{\sigma}$ is smooth.

Although we have provided all details of our results here only for the one-dimensional case, it is clear how to extend most of them to the realistic three-dimensional case. In particular, the included proofs of existence and uniqueness of weak solutions of the dynamic problem with multiple-yield surfaces, already known from the work of Visintin, as well as the existence and uniqueness of strong solutions of such problems given by Krejci, extend directly to the higher dimensional case where our abstract hypotheses are easy to verify. Our results on the regular solutions are easy to obtain from the abstract framework for one dimension, but we have not been able to verify them for a three-dimensional model of plasticity, so these appear limited to the one-dimensional case.

Our plan is as follows. We first recall below some topics from convex analysis and evolution equations in Hilbert space. Section 2 consists of some elementary examples of systems of differential equations or related variational inequalities which illustrate a variety of models of plasticity. These examples are used to motivate the general construction to follow, and we indicate briefly for each both the corresponding results that we shall obtain and the method of proof that we shall employ in the abstract setting. We introduce in Section 3 an abstract setting for these examples and show that each such model is described by a corresponding nonlinear semigroup of contractions generated by an *m-accretive* operator in Hilbert space. Specifically, we recover the above mentioned well-known theorems as *weak* solutions, and additionally we give sufficient general conditions

under which these solutions are *strong*. An even more *regular* solution is obtained in Section 4 for the one-dimensional case.

A (possibly multi-valued) *operator* or *relation* $\mathbb{C}$ in a real Hilbert space H is a collection of related pairs $[x, y] \in H \times H$ denoted by $y \in \mathbb{C}(x)$; the *domain* $\mathrm{Dom}(\mathbb{C})$ is the set of all such x and the *range* $\mathrm{Rg}(\mathbb{C})$ consists of all such y. The operator $\mathbb{C}$ is called *accretive* if for all $y_1 \in \mathbb{C}(x_1)$, $y_2 \in \mathbb{C}(x_2)$, and $\varepsilon > 0$, we have

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + \varepsilon(y_1 - y_2)\|.$$

This is equivalent to requiring that $(I + \varepsilon\mathbb{C})^{-1}$ be a contraction on $\mathrm{Rg}(I + \varepsilon\mathbb{C})$ for every $\varepsilon > 0$. This is also equivalent to requiring

$$(y_1 - y_2, x_1 - x_2)_H \geq 0 \qquad \text{for all } [x_1, y_1], [x_2, y_2] \in \mathbb{C}.$$

If, additionally, $\mathrm{Rg}(I + \varepsilon\mathbb{C}) = H$ for some (equivalently, for all) $\varepsilon > 0$, then we say $\mathbb{C}$ is *m-accretive*. For such an operator, the *Cauchy problem* is known to be well-posed, and we shall realize each of our initial-boundary-value problems as such a problem in an appropriate function space.

THEOREM A.. *Let $\mathbb{C}$ be m-accretive in the Hilbert space H. If $T > 0, \mathbf{x}_0 \in \mathrm{Dom}(\mathbb{C})$ and $\mathbf{f} \in W^{1,1}(0, T; H)$, then there exists a unique solution $\mathbf{x} \in W^{1,\infty}(0, T; H)$ of the Cauchy problem*

$$\begin{aligned} \mathbf{x}'(t) + \mathbb{C}(\mathbf{x}(t)) &\ni \mathbf{f}(t), t > 0 \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned} \tag{1.5}$$

with $x(t) \in \mathrm{Dom}(\mathbb{C})$ for all $0 \leq t \leq T$.

We will use some techniques of convex analysis to construct the operators below. For details, see [7, 2, 3]. Let W be a Hilbert space, and let $\varphi: W \to (-\infty, +\infty]$ be convex, proper, and lower-semi-continuous. Then the functional $f \in W'$, the dual space, is a *subgradient* of φ at $u \in W$ if

$$f(v - u) \leq \varphi(v) - \varphi(u) \qquad \text{for all } v \in W.$$

The set of all subgradients of φ at u is denoted by $\partial\varphi(u)$. The subgradient is a generalized notion of the derivative, comparable to a directional derivative. We regard $\partial\varphi$ as a multivalued operator from W to W'; it is easily shown to be *monotone*. That is, if $f_1 \in \partial\varphi(u_1)$, $f_2 \in \partial\varphi(u_2)$, then $(f_1 - f_2)(u_1 - u_2) \geq 0$.

If K is a closed, convex, nonempty subset of W, then the *indicator function* $I_K(\cdot)$ of K, given by $I_K(w) = 0$ if $x \in K$ and $I_K(w) = +\infty$ otherwise, is convex, proper, and lower-semi-continuous. Its subgradient is characterized by a *variational inequality*: $f \in \partial I_K(w)$ means

$$f \in W', w \in K: f(y - w) \leq 0 \text{ for all } y \in K.$$

As an example we consider first the *indicator function* $I_1(\cdot)$ of the interval $[-1, 1]$. Thus, $I_1: \mathbb{R} \to +\mathbb{R}_\infty$ is convex, proper, and lower-semi-continuous, and its subgradient is characterized as follows: $f \in \partial I_1(x)$ means

$$|x| \leq 1 \text{ and } \begin{cases} f \geq 0, & \text{for } x = 1, \\ f = 0, & \text{for } -1 < x < 1, \\ f \leq 0, & \text{for } x = -1. \end{cases}$$

Thus, ∂I_1 is just the inverse of the *sign graph*,

$$\operatorname{sgn}(x) = \begin{cases} \{1\}, & \text{if } x > 0, \\ [-1, 1], & \text{if } x = 0, \\ \{-1\} & \text{if } x < 0. \end{cases}$$

A second example is the corresponding realization on the Hilbert space $W = L^2(0, 1)$ given by

$$\varphi_1(\sigma) = \int_0^1 I_1(\sigma(x))\, dx, \qquad \sigma \in W, \tag{1.6}$$

and here we have $f \in \partial\varphi_1(\sigma)$ if $f, \sigma \in W = W'$ and $f(x) \in \partial I_1(\sigma(x))$ at a.e. $x \in (0, 1)$. For a third example, let φ_1 be given by (1.6) on the Sobolev space $W = H^1(0, 1)$. Then the inclusion $f \in \partial\varphi_1(\sigma)$ implies that σ is smoother, but it permits f to be a distribution, so the pointwise characterization above does not necessarily hold.

2. EXAMPLES

We shall describe a variety of models of plasticity in very simple form. These are given here in one spatial dimension for the ease of exposition, and they are intended only to illustrate the theorems which will follow. The full 3-dimensional models can be developed similarly by using the appropriate Sobolev spaces and operators that are so well known and described in the literature. For each of these examples, we shall describe the operator in L_2 that realizes the corresponding initial-boundary value problem, and we give a brief indication in each case of what results will follow from the general theory to be given in the next section.

1. *Elastic-perfectly Plastic*. Consider a 1-dimensional elastic-plastic deformation. The momentum and constitutive equations are, respectively,

$$v_t - \sigma_x = f, \qquad \sigma_t + \operatorname{sgn}^{-1}(\sigma) \ni \varepsilon_t.$$

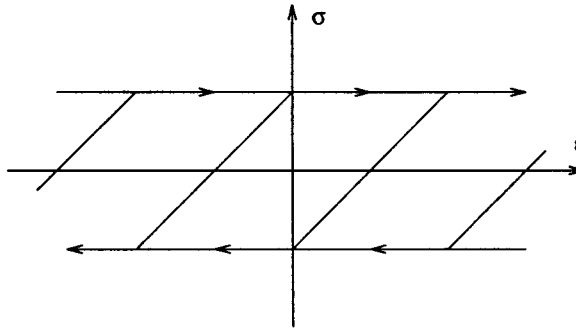

FIGURE 1

The phase diagram showing the relationship between stress σ and strain ε is given in Fig. 1. Since the value of σ depends on the history of ε, this relationship is a *hysteresis* functional.

This model results from the *series* addition of an elastic element, $\sigma_t = \varepsilon_t$, and a perfectly-plastic element, $\operatorname{sgn}^{-1}(\sigma) \ni \varepsilon_t$. By equality of mixed derivatives, $u_{xt} = u_{tx}$, the resulting dynamical system is given by

$$v_t - \sigma_x = f, \qquad 0 < x < 1, 0 < t, \qquad v(0,t) = 0 \tag{2.1.a}$$

$$\sigma_t - v_x + \operatorname{sgn}^{-1}(\sigma) \ni 0, \qquad \sigma(1,t) = 0 \tag{2.1.b}$$

with appropriate initial conditions on v and σ. We shall write this as an evolution equation

$$\frac{d}{dt}[v,\sigma] + \mathbb{C}([v,\sigma]) \ni [f,0] \tag{2.2}$$

in the appropriate product space.

Define the Hilbert space

$$W = \{\sigma \in H^1(0,1) \colon \sigma(1) = 0\}.$$

Let the function φ_1 be defined on this space W by (1.6).

DEFINITION. The operator $\mathbb{C}$ is determined as follows: $[f,g] \in \mathbb{C}[v,\sigma]$ if

$$[f,g] \in L^2(0,1) \times L^2(0,1), \qquad [v,\sigma] \in L^2(0,1) \times H^1(0,1),$$

and there exists a $c \in W'$ for which

$$-\sigma_x(x) = f(x), \qquad 0 < x < 1, \sigma(1) = 0 \tag{2.3.a}$$

$$-v_x + c = g, \qquad c \in \partial\varphi_1(\sigma). \tag{2.3.b}$$

Remarks. The first term in (2.3.b) is defined in W' by

$$-v_x(\rho) \equiv \int_0^1 v(x)\rho'(x)\,dx, \qquad \rho \in W.$$

Formally (2.3.b) means

$$-v_x(x) + c(x) = g(x), \qquad c(x) \in \partial\varphi_1(\sigma(x)), v(0) = 0,$$

but this holds only if c is sufficiently regular, e.g., if $c \in L^2(0,1)$. Then $v \in H^1(0,1)$ and the boundary condition is meaningful. The range, $\mathrm{Rg}(\mathbb{C})$, is easily seen by a direct calculation to be the set of pairs $[f,g] \in L^2(0,1) \times L^2(0,1)$ for which $|\int_x^1 f dx| \le 1$ for each $x \in [0,1]$. Neither $\mathbb{C}$ nor $\mathbb{C}^{-1}$ is a function.

For $\varepsilon > 0$, the corresponding resolvent equation, $(I + \varepsilon\mathbb{C})[v,\sigma] \ni [f,g]$, is given by

$$v \in L^2 : v - \varepsilon\sigma_x = f, \qquad 0 < x < 1, \sigma(1) = 0,$$

$$\sigma \in W : \sigma - \varepsilon v_x + \varepsilon\partial\varphi_1(\sigma) \ni g.$$

Note that εv_x and $\varepsilon\partial\varphi_1(\sigma)$ are in W', so this is a *weak* solution in our notation below, and there is no boundary value assigned to $v(0)$. If we eliminate v, we can write this as a single equation or *variational inequality*, formally of the form

$$\sigma - \varepsilon^2\sigma_{xx} + \varepsilon\partial\varphi_1(\sigma) \ni g + \varepsilon f_x.$$

To be precise, this problem has the following variational form: Find a pair of functions

$$\sigma \in W, \qquad c \in W' \text{ such that } c \in \partial\varphi_1(\sigma),$$

and

$$\int_0^1 \left(\sigma\varphi + \varepsilon(\varepsilon\sigma_x + f)\varphi_x\right) dx + \varepsilon c(\varphi) = \int_0^1 g\varphi\,dx \text{ for all } \varphi \in W.$$

In particular, this is the characterization of the solution to the problem of minimizing the convex function

$$\Phi(\sigma) = \int_0^1 \left(\frac{1}{2}\sigma^2 + \frac{\varepsilon^2}{2}\sigma_x^2 + \varepsilon I_1(\sigma)\right) dx - \int_0^1 (g\sigma - f\sigma_x)\,dx$$

over the space W, so it is known to have a solution and, hence, $\mathrm{Rg}(I + \varepsilon\mathbb{C}) = L^2(0,1) \times L^2(0,1)$. It will follow from an explicit computation that the map $[f,g] \to [v,\sigma]$ is a contraction on $L^2(0,1) \times L^2(0,1)$, so $\mathbb{C}$ is *m-accretive*, and Theorem A above from nonlinear semigroup theory will show directly that there is a unique *weak* solution of (2.1) with

$$v, \frac{\partial v}{\partial t}, \frac{\partial\sigma}{\partial t} \in L^\infty(0,T;L^2(0,1)), \qquad \sigma \in L^\infty(0,T;W).$$

This is the content of Theorem W in the next section. This *weak* solution was already obtained in Theorem 4.2 of [6] and Theorem 1 of [18]. See [1] for regularity of the solution and the interpretation of (2.1.b).

Remark. The corresponding equation for v is degenerate in the gradient, hence, not coercive.

2. *Isotropic Hardening*. Assume that the material work-hardens each time the yield stress is reached. That is, after reaching the yield limit, the stress continues to increase with increasing strain, but at a much lower rate. In this case, the minimum (negative) yield stress is lowered by the same amount that the maximum (positive) yield stress is raised, so the *length* of the stress interval is non-decreasing, and the *position* of the stress interval is constant. We introduce an internal variable, s, to keep track of the "size" of the non-yielding stresses. In the preceding examples this was scaled to unity. Instead of using the graph sgn^{-1} the subgradient of the indicator function of the interval $[-1, 1]$ in $\mathbb{R}^1$ we introduce the set in $\mathbb{R}^2$ given by

$$K = \{(\sigma, s) \in \mathbb{R}^2 \colon As + 1 \geq |\sigma|\},$$

where $A \geq 0$ is given. Then I_K is the indicator function of K and its subgradient is denoted by ∂I_K. If the strain-rate is given by $\varepsilon_t = v_x$ as above, then the stress is determined by the evolution system

$$\sigma_t + c = \epsilon_t, \qquad s_t + b = 0, \qquad (c, b) \in \partial I_K(\sigma, s).$$

Note that if we set $A = 0$, then $\partial I_K(\sigma, s) = (\operatorname{sgn}^{-1}(\sigma), 0)$, and this system decouples and reduces to (2.1.b), i.e., the elastic-plastic element with constant b. The isotropic hardening system is given by

$$\begin{aligned} &\frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial x} = f \\ &\frac{\partial \sigma}{\partial t} + c \ni \frac{\partial v}{\partial x}, \qquad (c, b) \in \partial I_K(\sigma, s), \\ &\frac{\partial s}{\partial t} + b = 0. \end{aligned}$$

The existence and uniqueness of a *weak* solution of this system will be obtained below.

We illustrate the relation between total stress σ and strain ε in Fig. 2. (Take $A = 1$ for the set K.) If we impose a strain which drives the stress as indicated in Fig. 2, the stress is first driven to its initial yield limit, $\sigma = 1$, and then this is driven beyond this yield limit to $\sigma = 1.5$. The stress reverses and then goes down to $\sigma = -1.5$ where the yield limit is reached

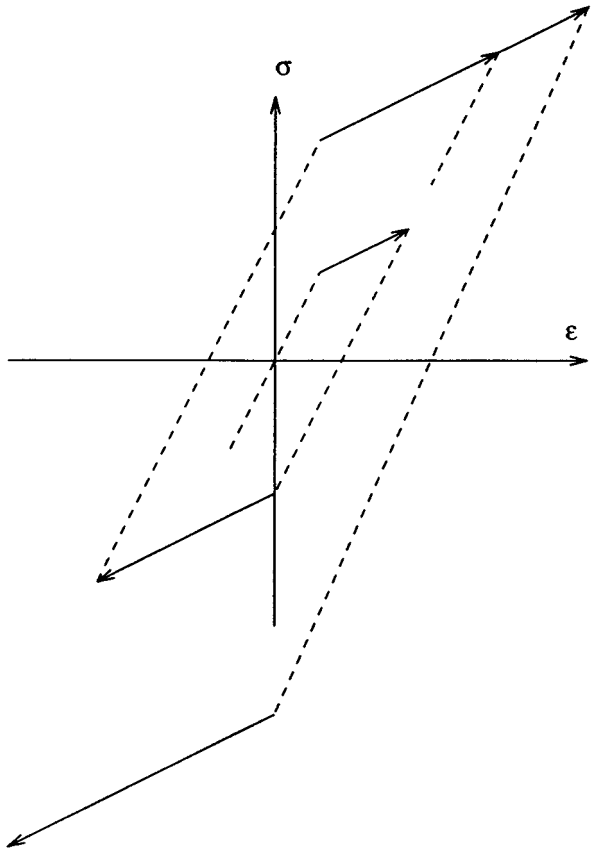

FIGURE 2

and then driven beyond to $\sigma = -2.5$ before it reverses direction, etc. The size of the yield set can be followed on the set K as indicated in Fig. 3.

Note that the yield limit began at 1, then was driven upward to 1.5, then 2.5, then 4, then 5.5. That is, the length of the yield stress interval increased from 2 to 3 to 5 to 8 to 11.

3. *Kinematic Hardening*. Here we again assume that the material work-hardens each time the yield stress is reached. However in this case the *length* of the interval of stress, i.e., the interval between the maximum yield stress and the minimum yield stress, remains constant. Only the *position* of this stress interval is moved upward or downward. Momentum

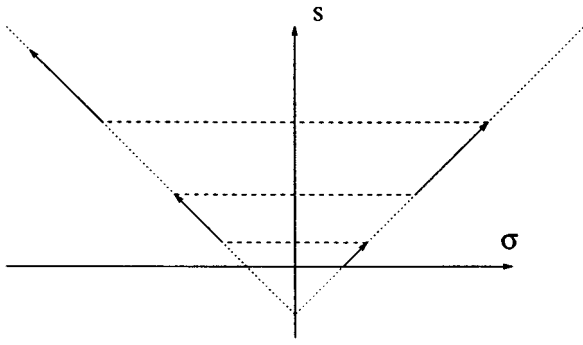

FIGURE 3

and constitutive equations are, respectively,

$$\frac{\partial}{\partial t} v - \sigma_x = f, \qquad \sigma = \beta_1 \sigma_1 + \beta_2 \sigma_2,$$

$$\frac{\partial}{\partial t} \sigma_1 + \partial \varphi_1(\sigma_1) \ni \beta_1 \frac{\partial}{\partial x} v, \qquad \frac{\partial}{\partial t} \sigma_2 = \beta_2 \frac{\partial}{\partial x} v. \tag{2.4}$$

This model results from the *parallel* addition of the elastic-plastic stress from Section 1 (corresponding to σ_1) with a purely elastic stress (corresponding to σ_2) which records the position of the center of the yield stress interval. Thus the lines in Fig. 4. representing the upper and lower yield surfaces are at a vertical distance apart of 2, and they have slope β_2^2.

We shall write the system (2.4) as an evolution equation in the appropriate product space.

DEFINITION. The operator $\mathbb{C}$ is determined as follows: $[f, g^1, g^2] \in \mathbb{C}[v, \sigma_1, \sigma_2]$ if

$$[f, g_1, g_2] \in L^2(0,1)^3, \qquad [v, \sigma_1, \sigma_2] \in H^1(0,1) \times L^2(0,1)^2,$$

$$\beta_1 \sigma_1 + \beta_2 \sigma_2 \in H^1(0,1),$$

and there exists a $c \in L^2(0,1)$ for which

$$-\frac{d}{dx}(\beta_1 \sigma_1 + \beta_2 \sigma_2)(x) = f(x), \qquad 0 < x < 1,$$

$$(\beta_1 \sigma_1 + \beta_2 \sigma_2)(1) = 0 \qquad -\beta_1 \frac{d}{dx} v(x) + c(x) = g_1(x),$$

$$c(x) \in \partial I_1(\sigma_1(x)), \qquad v(0) = 0, \qquad -\beta_2 \frac{d}{dx} v(x) = g_2(x).$$

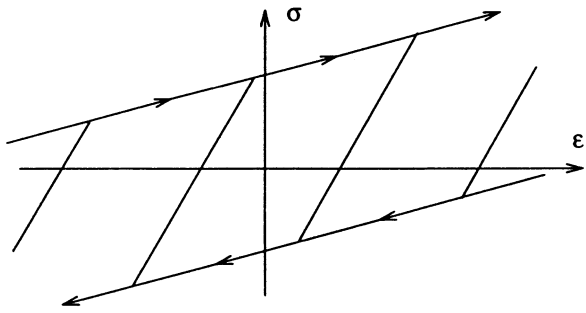

FIGURE 4

Note that since $c \in L^2(0,1)$, the inclusion $c(x) \in \partial I_1(\sigma_1(x))$ is a pointwise variational inequality in $\mathbb{R}$ for a.e. $x \in [0,1]$. Namely, it is equivalent to

$$c(x) \in \mathbb{R}, |\sigma_1(x)| \leq 1 : c(x)(p - \sigma_1(x)) \leq 0 \text{ for all } \rho \in \mathbb{R} \text{ with } |\rho| \leq 1.$$

We shall show that the operator $\mathbb{C}$ is *m-accretive* in the space $H \equiv L^2(0,1)^3$ and, since $v_x \in L^2(0,1)$, that it leads to a *strong* solution. This solution agrees with that of Theorem 1.2 of Chapter III in [13] where much more general situations are obtained. To this end, as well as to motivate our notation in the next section, we introduce

$$V = \{v \in H^1(0,1) : v(0) = 0\}, \qquad D = \frac{d}{dx} : V \to L^2(0,1)$$

$$D^* = -\frac{d}{dx} : L^2(0,1) \to V' \text{ is the } \textit{continuous} \text{ dual operator}$$

$$\beta = [\beta_1 I, \beta_2 I] : L^2(0,1) \to L^2(0,1)^2, \qquad \text{where } \beta_1, \beta_2 \in \mathbb{R} \text{ are given,}$$

$$\beta^*[\sigma] = \beta_1\sigma_1 + \beta_2\sigma_2, \qquad \beta^* : L^2(0,1)^2 \to L^2(0,1)$$

$$W_0 = \{\sigma = [\sigma_1, \sigma_2] \in L^2(0,1)^2 : \beta^*\sigma \in H^1(0,1), \beta^*\sigma(1) = 0\}.$$

Denote by D_* the $L^2(0,1)$-adjoint of the closed operator, D. That is,

$$D_* w = f \Leftrightarrow w, f \in L^2(0,1) \text{ and}$$
$$(Dv, w) = (v, f) \text{ for all } v \in \operatorname{Dom}(D) \equiv V.$$

Then $D_* : \operatorname{Dom}(D_*) \to L^2(0,1)$ is also closed and dense, and it can be characterized as follows.

LEMMA. *$D_* w = f \in L^2(0,1) \Leftrightarrow w \in L^2(0,1)$, $f = -dw/dx$ and $w(\cdot)v(\cdot)|_0^1 = 0$ for all $v \in$ dom (D).*

This shows how the boundary conditions imposed on D determine those associated with D_*. Then we set $W = \mathrm{Dom}\,(D_*)$ so that $(D_*): W \to L^2(0,1)$. Note that for any solution of (2.4), either weak or strong, we have $\beta^*\sigma \in W$ and D^* can be replaced by D_* in the momentum equation. In particular, $\beta^*\sigma$ satisfies the appropriate boundary condition.

First we check by a direct estimate that $\mathbb{C}$ is accretive. Second, the resolvent equation $(I + \mathbb{C})[v, \sigma] \ni [f, g]$ is equivalent to solving the system

$$v \in V: v + D^*\beta^*\sigma = f,$$

$$\sigma \in W_0: \sigma - \beta D v + \left[\partial\varphi_1(\sigma_1), 0\right] = g \in L^2(0,1)^2.$$

This is equivalent to solving for v the equation

$$\begin{aligned} v \in V: v + D^*\Big(\beta_1(I + \partial\varphi_1)^{-1}(\beta_1 D v + g_1) + \beta_2^2 D v + \beta_2 g_2\Big) \\ = f \qquad \text{in } V'. \end{aligned}$$

Since $\beta_2^2 > 0$, the form is coercive, and existence of a solution follows. The components of $[\sigma_1, \sigma_2] \in W_0$ are obtained directly from the second and third terms in this equation, respectively, and then we check that $\sigma \in W_0$. In particular, the boundary condition at $x = 1$ is satisfied. These remarks show that Theorem A applies directly to give existence and uniqueness of a *strong* solution of (2.4) with

$$v \in L^\infty(0,T;V), \qquad \sigma \in L^\infty(0,T;W_0), \qquad \frac{\partial v}{\partial t}, \frac{\partial \sigma}{\partial t} \in L^\infty\big(0,T;L^2(0,1)\big).$$

This is the content of Theorem S in Section 3.

Remark 1. Since v belongs to V instead of merely to $L^2(I)$, the solution here is *smoother* than that of Section 1. This is made possible here by the coercivity resulting from the β_2 term.

Remark 2. $\mathbb{C}(v, \sigma)$ is single valued only if $\sigma_1 \neq 0$ a.e., and $\mathbb{C}^{-1}(f, g)$ is single valued only if $\beta_2 g_1 \neq \beta_1 g_2$ a.e.

Remark 3. The isotropic hardening model, Example 2, can be put in a form similar to (2.4). We need only to identify the operators $\beta = [1, 0]$ and $\beta^*([\sigma, s]) = \sigma$ and to relate σ, s in that model with σ_1, σ_2 above. Of course, the subgradient there acts in $\mathbb{R} \times \mathbb{R}$ and is *not* in diagonal form.

Remark 4. We can include a viscous element in parallel to the above by adding a third equation of the form

$$\frac{1}{k}\frac{\partial}{\partial t}\sigma_3 + \frac{1}{\mu}\sigma_3 = \beta_3\frac{\partial}{\partial x}v.$$

More generally, we can include *visco-elastic* elements in the form

$$\frac{1}{k}\frac{\partial}{\partial t}\sigma_3 + J'(\sigma_3) = c\frac{\partial}{\partial x}v,$$

where J has a *bounded* derivative. This represents a series combination of elastic element and a purely viscous element, and one obtains *strong* solutions as above. See Theorem 3.1 of [6] for the case of a single stress component.

Remark 5. By setting $v = u_t$, eliminating $\sigma_2 = \beta_2\ \partial/\partial x\, u$, and replacing the term $\partial\varphi_1(\sigma_1)$ by $-\Delta\sigma_1$ in (2.4), we obtain the system

$$u_{tt} - \beta_1\frac{\partial}{\partial x}\sigma_1 - \beta_2^2 u_{xx} = f,$$

$$\frac{\partial}{\partial t}\sigma_1 - \Delta\sigma_1 = \beta_1\frac{\partial}{\partial x}u_t.$$

This is the classical problem of *thermoelasticity*, and its similarity to (2.4) motivated the regularity results in Section 4.

We next give a simple but important extension of the preceding example to a plasticity model built on four stress components. This will motivate the consideration of generalized sums or *integrals* of a collection or even a *continuum* of such components. The system is given by

$$\frac{\partial}{\partial t}v - \sigma_x = f,$$

$$\sigma = \sigma_1 + \frac{1}{2}\sigma_2 + \frac{1}{4}\sigma_3 + \frac{1}{4}\sigma_4,$$

$$\frac{\partial}{\partial t}\sigma_1 + \partial\varphi_1(\sigma_1) \ni \frac{\partial}{\partial x}v,$$

$$\frac{\partial}{\partial t}\sigma_2 + \partial\varphi_2(\sigma_2) \ni \frac{1}{2}\frac{\partial}{\partial x}v,$$

$$\frac{\partial}{\partial t}\sigma_3 + \partial\varphi_3(\sigma_3) \ni \frac{1}{4}\frac{\partial}{\partial x}v,$$

$$\frac{\partial}{\partial t}\sigma_4 = \frac{1}{4}\frac{\partial}{\partial x}v. \tag{2.5}$$

For each $j = 1, 2, 3$, φ_j is the indicator function of the interval $[-j, j]$, so the corresponding stress component σ_j is constrained to lie within that interval. The relation between total stress σ and strain ε is indicated by Fig. 5. (Recall that $\partial\varepsilon/\partial t = \partial v/\partial x$ is the *strain rate*.) Here we begin with all components at 0. We increase the strain, ε, from 0 to 5, decrease it to -5, then increase it to 2, and we follow the resulting stress, σ.

The slope of the stress starting upward from the origin is 2, then it decreases to 1 and to $1/2$ on successive intervals until only σ_4 with slope $1/4$ is active for $\varepsilon \geq 3$. When the curve begins to decrease from $\varepsilon = 5$, the slope is initially 2, and then the slope decreases successively to 1 and to $1/2$ on intervals of length 2 until only σ_4 with slope $1/4$ is active for $\varepsilon \leq -1$. The applied strain ε reverses direction again at -5, and the resulting stress begins to rise with slope 2 again. The limiting positive slope $1/4$ is the *work-hardening* component, and it is this component of the stress that will lead to a *strong* solution of (2.5) as before. Since the bounding lines in this hysteresis functional are straight lines, such models are called *multilinear*. By using a collection of such components, one can approximate a large class of convex bounding curves; with a continuum of such components, the corresponding class of convex functions can be matched. Most models of plasticity involve such multiple yield surfaces, and these provide an approximation of the observed smooth transitions

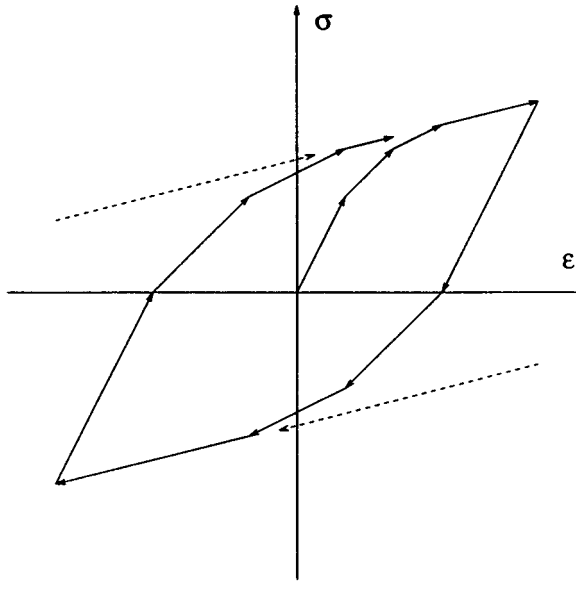

FIGURE 5

between elastic and plastic regimes. Such smooth transitions are best modeled by a continuum of elastic-plastic elements with varying yield surfaces.

3. A GENERAL PLASTICITY MODEL

Let $D : \mathrm{Dom}\,(D) \to L^2(0,1)$ be a closed operator with dense domain $\mathrm{Dom}\,(D)$ in $L^2(0,1)$. Let D_* be the $L^2(0,1)$-adjoint of this closed operator. That is,

$$D_* w = f \Leftrightarrow w, f \in L^2(0,1) \text{ and } (Dv, w) = (v, f) \text{ for all } v \in \mathrm{Dom}(D).$$

Therefore, $D_* : \mathrm{Dom}(D_*) \to L^2(0,1)$ is also closed and dense. We set $W \equiv \mathrm{Dom}\,(D_*)$ and give it the graph norm. Then $D_* : W \to L^2(0,1)$ is a *bounded* operator between Banach spaces. The *continuous* dual operator will be denoted by $D_*^* = (D_*)^* : L^2(0,1) \to \mathrm{Dom}(D_*)' = W'$. Note that

$$\begin{aligned} D_*^* v(w) &= (v, D_* w)_{L^2} \qquad \text{for all } w \in \mathrm{Dom}(D_*), v \in L^2, \\ &= (Dv, w)_{L^2} \qquad \text{for all } w \in \mathrm{Dom}(D_*), v \in \mathrm{dom}(D), \end{aligned}$$

so we have $D_*^* \supset D$ in the sense of graphs. Similarly, we put the graph norm on $\mathrm{Dom}(D)$ and denote the resulting space by V. We define the continuous dual $D^* : L^2(0,1) \to V'$ and note that $D^* \supset D_*$.

Let S, μ be a measure space. Define $\beta : L^2(0,1) \to L^2(S, d\mu; L^2(0,1)) = L^2(S \times (0,1))$ by $(\beta g)(s, x) \equiv \beta(x, s) g(x)$ where $\beta(\cdot,\cdot) \in L^\infty((0,1), L^2(S))$. Then the continuous dual is an operator $\beta^* : L^2(S, d\mu; L^2(0,1)$, and we have for all $\rho \in L^2(S, d\mu; L^2(0,1))$, $g \in L^2(0,1)$ the calculation

$$\begin{aligned} \int_0^1 \beta^* \rho(x) g(x)\, dx &= \int_0^1 \int_S \rho(s,x)(\beta g)(s,x)\, d\mu_s\, dx \\ &= \int_0^1 \left\{ \int_S \beta(x,s) \rho(s,x)\, d\mu_s \right\} g(x)\, dx, \end{aligned}$$

so we obtain

$$\beta^* \rho(x) = \int_S \beta(x,s) \rho(s,x)\, d\mu_s, \qquad \text{a.e. } x \in (0,1).$$

Define $W_S \equiv \{\sigma \in L^2(S, d\mu; L^2(0,1)) : \beta^* \sigma \in \mathrm{Dom}(D_*)\}$. Let $\beta_* : W_S \to W$ be the indicated restriction, which is bounded on W_S with the graph norm, and denote its *continuous* dual by $\beta_*^* : W' \to W_S'$. We shall fre-

quently hereafter denote the space $L_2(S, d\mu; L_2(0,1))$ by $L^2(S \times (0,1))$. The various operators are summarized in the diagram

$$\begin{array}{ccccccccccc} L^2(I) & \xrightarrow{D_*^*} & W' & \xrightarrow{\beta_*^*} & W_S' & \qquad & L^2(S\times I) & \xrightarrow{\beta^*} & L^2(I) & \xrightarrow{D^*} & V' \\ \cup & & \cup & & \cup & & \cup & & \cup & & \cup \\ V & \xrightarrow{D} & L^2(I) & \xrightarrow{\beta} & L^2(S\times I) & & W_S & \xrightarrow{\beta_*} & W & \xrightarrow{D_*} & L^2(I) \end{array}$$

Let $\varphi : W_S \to \mathbb{R}_\infty$ be proper, convex, and lower-semicontinuous, and denote its subgradient by $\partial\varphi : W_S \to W_S'$.

DEFINITION.. *The weak Cauchy Problem* is to find $v(t)$, $\sigma(t)$ for $0 < t < T$ such that

$$v, \frac{\partial v}{\partial t} \in L^\infty\big(0,T; L^2(0,1)\big), \qquad \sigma \in L^\infty(0,T; W_S),$$

$$\frac{\partial \sigma}{\partial t} \in L^\infty\big(0,T; L^2(S\times(0,1))\big),$$

and they satisfy the system

$$\frac{dv(t)}{dt} + D_*\beta_*\sigma(t) = f(t) \qquad \text{in } L^2(0,1), \tag{3.1.a}$$

$$\frac{d\sigma(t)}{dt} + \partial\varphi(\sigma(t)) - \beta_*^* D_*^* v(t) \ni g(t) \qquad \text{in } W_S', \tag{3.1.b}$$

$$v(0) = v_0 \text{ in } L^2(0,1), \qquad \sigma(0) = \sigma_0 \text{ in } L^2\big(S, d\mu; L^2(0,1)\big), \tag{3.1.c}$$

where the four functions $v_0 \in L^2(0,1)$, $\sigma_0 \in W_S$, $f \in L^\infty(0,T; L^2(0,1))$, and $g \in L^\infty(0,T; L^2(S, d\mu; L^2(0,1)))$ are given.

Note that the variational form of (3.1.b) is

$$\begin{aligned} \sigma(t) \in W_S : -\left(\frac{d\sigma(t)}{dt}, \rho - \sigma(t)\right)_{L^2(S\times(0,1))} & \\ + \big(v(t), D_*\beta_*(\rho - \sigma(t))\big)_{L^2(0,1)} & \\ + \big(g(t), \rho - \sigma(t)\big)_{L^2(S\times(0,1))} & \\ \le \varphi(\rho) - \varphi(\sigma(t)) \qquad \text{for all } \rho \in W_S. & \end{aligned}$$

THEOREM W. *Assume that the linear operator $D : V \to L^2(0,1)$, the function $\beta(\cdot,\cdot) \in L^\infty((0,1), L^2(S))$, and the convex functional $\varphi : W_S \to \mathbb{R}_\infty$ are given as above, and define the corresponding operators*

$$D_* : W \to L^2(0,1), \qquad D_*^* : L^2(0,1) \to W',$$

$$\beta : L^2(0,1) \to L^2\big(S, d\mu; L^2(0,1)\big), \qquad \beta_* : W_S \to W.$$

Let $v_0 \in L^2(0,1)$ *and* $\sigma_0 \in W_S$ *be given with* $(\partial\varphi(\sigma_0) - \beta_*^* D_*^* v_0) \cap L^2(S, d\mu; L^2(0,1))$ *non-empty. Let* $f \in W^{1,1}(0,T; L^2(0,1)$ *and* $g \in W^{1,1}(0,T; L^2(S, d\mu; L^2(0,1)))$ *be given. Then there is a unique weak solution of* (3.1) *with* $v(0) = v_0$, $\sigma(0) = \sigma_0$.

Proof. Define the operator $\mathbb{C}$ on the Hilbert space $H \equiv L^2(0,1) \times L^2(S \times (0,1))$ by $\mathbb{C}[v, \sigma] \ni [f, g]$ if

$$\begin{aligned} &v \in L^2(0,1) : D_* \beta_* \sigma = f \qquad \text{in } L^2(0,1), \\ &\sigma \in W_S : \partial\varphi(\sigma) \ni \beta_*^* D_*^* v + g \qquad \text{in } W_S', \end{aligned}$$

where $f \in L^2(0,1)$ and $g \in L^2(S \times (0,1))$.

In order to show that $\mathbb{C}$ is m-accretive, we first check that it is accretive. If $\mathbb{C}[v_1, \sigma_1] \ni [f_1, g_1]$ and $\mathbb{C}[v_2, \sigma_2] \ni [f_2, g_2]$, then we have

$$\begin{aligned} &(f_1 - f_2)(v_1 - v_2)_{L^2(0,1)} + (g_1 - g_2, \sigma_1 - \sigma_2)_{L^2(S\times(0,1))} \\ &\quad = (D_* \beta_*(\sigma_1 - \sigma_2), v_1 - v_2)_{L^2(0,1)} \\ &\qquad - (v_1 - v_2, D_* \beta_*(\sigma_1 - \sigma_2))_{L^2(S\times(0,1)} \\ &\qquad + (c_1 - c_2)(\sigma_1 - \sigma_2), \end{aligned}$$

where $c_1 \in \partial\varphi(\sigma_1)$, $c_2 \in \partial\varphi(\sigma_2)$. But the first two terms add to zero and the third is nonnegative by the monotonicity of the subgradient, so the indicated sum is nonnegative.

Next we consider the range condition. The (weak) *resolvent equation*, $(I + \mathbb{C})[v,\sigma] \ni [f, g]$, is to find a solution of the stationary system

$$v \in L^2(0,1) : v + D_* \beta_* \sigma = f \qquad \text{in } L^2(0,1), \tag{3.2.a}$$

$$\sigma \in W_S : \sigma + \partial\varphi(\sigma) - \beta_*^* D_*^* v \ni g \qquad \text{in } W_S'. \tag{3.2.b}$$

By eliminating v from (3.2) we obtain the single equation

$$\sigma \in W_S : \sigma + \partial\varphi(\sigma) + \beta_*^* D_*^* (D_* \beta_* \sigma - f) \ni g \qquad \text{in } W_S'. \tag{3.3}$$

This is a variational problem of the form

$$\begin{aligned} \sigma \in W_s : &-(\sigma, \rho - \sigma)_{L^2(S\times(0,1))} \\ &- (D_* \beta_* \sigma, D_* \beta_*(\rho - \sigma))_{L^2(0,1)} \\ &+ (g, \rho - \sigma)_{L^2(S\times(0,1))} + (f, D_* \beta_*(\rho - \sigma))_{L^2(0,1)} \\ \le\ &\varphi(\rho) - \varphi(\sigma) \qquad \text{for all } \rho \in W_S. \end{aligned}$$

Since W_S is complete with the norm $(\|\sigma\|^2_{L^2(S\times I)} + \|D_* \beta_* \sigma\|^2_{L^2(I)})^{1/2}$, (3.3) has a unique solution, that is, there exists a unique

$$\sigma \in W_S : \sigma + h + \beta_*^* D_*^* (D_* \beta_* \sigma - f) = g \qquad \text{in } L^2\big(S, d\mu; L^2(0,1)\big),$$

$$h \in \partial\varphi(\sigma) \qquad \text{with } h - \beta_*^* D_*^* v \in L^2\big(S, d\mu; L^2(0,1)\big),$$

and then we set $v \equiv -D_* \beta_* \sigma + f \in L^2(0,1))$, to get a solution of (3.2). (Here we do *not* get $\beta_*^* D_*^* v \in L^2(S \times (0,1))$.) Thus, $\mathbb{C}$ is m-accretive, and so Theorem W follows immediately from Theorem A. ∎

We shall show that under additional assumptions we can obtain $Dv \in L^2(0,1)$ and thus $v \in V$. Then the pair v, σ is a *strong* solution of the resolvent equation (3.2) corresponding to the *strong Cauchy Problem*. By this, we mean the weak Cauchy Problem (3.1) in which we additionally require that $v \in L^\infty(0,T;V)$. Hence, one can then replace D_*^* with D and β_*^* with β. This takes the form of a system

$$\frac{\partial}{\partial t} v(x,t) + D_* \int_S \beta(x,s)\sigma(s,x,t)\, d\mu_s = f(x,t), \tag{3.4.a}$$

$$\frac{\partial}{\partial t}\sigma(s,x,t) + \partial\varphi(\sigma(s,x,t)) - \beta(x,s) Dv(x,t)$$
$$\ni g(x,s,t), \qquad \text{a.e. } s \in S,$$
$$\text{for a.e. } x \in (0,1), t > 0 \tag{3.4.b}$$

$$v(0) = v_0 \text{ in } L^2(0,1), \qquad \sigma(0) = \sigma_0 \text{ in } L^2\big(S, d\mu; L^2(0,1)\big). \tag{3.4.c}$$

Assume that φ is given in the *diagonal* form

$$\varphi(\sigma) = \int_S \int_0^1 \varphi_s(\sigma(s,x))\, dx\, d\mu_s, \qquad \sigma \in L^2(S \times (0,1)), \tag{3.5}$$

with a *normal integrand* [15] for which each $\varphi_s : L^2(0,1) \to \mathbb{R}_\infty$ is convex, lower-semicontinuous, and takes its minimum at $\varphi_s(\mathbf{0}) = 0$. We shall require that *some* of the $\partial\varphi_s$'s be *regular*, i.e., that they are linearly bounded. In order to quantify this condition, we set $\alpha_s \equiv (I + \partial\varphi_s)^{-1}$. Note that each α_s is (uniformly) Lipschitz and that we have

$$\alpha_s(\xi)(\xi) \geq 0, \qquad s \in S, \xi \in \mathbb{R}.$$

We shall assume additionally that there is an $\varepsilon > 0$ and a measurable set $S_0 \subset S$ such that

$$\alpha_s(\xi)(\xi) \geq \varepsilon |\xi|^2, \qquad s \in S_0, \xi \in \mathbb{R},$$

and

$$\int_{S_0} (\beta(x,s))^2\, d\mu_s \geq \epsilon. \tag{3.6}$$

THEOREM S. *In the situation of Theorem W, assume in addition that the function* φ *is given on* $L^2(S \times (0,1))$ *by the formula* (3.5) *and the normal family* φ_s *of convex and lower-semicontinuous nonnegative functionals for which* $\varphi_s(\mathbf{0}) = 0, s \in S$, *and the estimates* (3.6) *hold. Also let* $v_0 \in V$. *Then the weak solution is a strong solution, i.e.,* $v \in L^\infty(0,T;V)$ *and the strong Cauchy Problem has a unique solution.*

Proof. We shall show that (3.2) has a strong solution. Eliminate σ from the system (3.2) to see that the first component of any such strong solution satisfies the single equation

$$v \in V : v + D_* \beta_* \alpha(\beta D v + g) = f \qquad \text{in } V', \tag{3.7}$$

where $\alpha : L^2(S \times (0,1)) \to L^2(S \times (0,1))$ is the Lipschitz substitution operator defined pointwise by $\alpha(\sigma)(s,x) = \alpha_s(\sigma(s,x)), (s,x) \in S \times (0,1)$. Conversely, if we can solve (3.7) and set $\sigma = \alpha(\beta D v + g)$, then we have $\sigma \in W_S$, and thereby we obtain a *strong* solution of (3.2). From the assumption (3.6), we obtain for each $v \in V$ the estimate

$$\begin{aligned}
&\int_0^1 \left\{ v(x)^2 + \int_S \alpha_s(\beta D v + g)\beta D v \, d\mu_s \right\} dx \\
&\quad \geq \int_0^1 \left\{ v(x)^2 + \int_{S_0} \alpha_s(\beta D v)\beta D v \, d\mu_s \right\} dx \\
&\qquad + \int_0^1 \int_S \left\{ (\alpha_s(\beta D v + g) - \alpha_s(\beta D v))\beta D v \, d\mu_s \right\} dx \\
&\quad \geq \int_0^1 \left\{ v(x)^2 + \varepsilon^2 |Dv(x)|^2 \right\} dx - \int_0^1 \int_S |g| \, |\beta D v| \, d\mu_s \, dx \\
&\quad \geq c_\varepsilon \|v\|_V^2 - C\|v\|_V,
\end{aligned} \tag{3.8}$$

and, hence, the convex functional which is minimized in order to solve (3.7) is *V-coercive*. It follows that $\mathrm{Dom}(\mathbb{C}) \subset V \times W_S$. If $[v(t), \sigma(t)]$ is the weak solution, then $(I + \mathbb{C})[v(t), \sigma(t)]$ is uniformly bounded in H for $0 \leq t \leq 1$, so it follows from (3.7) and (3.8) that $v \in L^\infty(0,T;V)$. Thus, the corresponding *strong* problem is well-posed. ∎

4. REGULAR SOLUTIONS

The momentum equation (3.1.a) requires only that the generalized sum, $\beta_* \sigma(t)$, belong to W at each $t > 0$. We would like to show that when $\beta(\cdot,\cdot)$ is independent of x one may obtain a solution for which *each component*, $\sigma(s,t)$, belongs to W at each $t > 0$.

Define the *distributed operator* $\mathbb{D}: L^2(S;V) \to L^2(S \times I)$ by

$$\mathbb{D}(v)(s) = Dv(s), \qquad s \in S, v \in L^2(S;V),$$

and denote its L^2-adjoint by $\mathbb{D}_* : L^2(S;W) \to L^2(S \times I)$. The corresponding continuous duals are $\mathbb{D}^*$ and D_*^* as before.

$$\begin{array}{ccccccccc}
L^2(I) & \xrightarrow{D_*^*} & W' & \xrightarrow{\beta_*^*} & W_S' & \qquad & L^2(S \times I) & \xrightarrow{D_*^*} & L^2(S;W') \\
\cup & & \cup & & \cup & & \cup & & \cup \\
V & \xrightarrow{D} & L^2(I) & \xrightarrow{\beta} & L^2(S \times I) & & L^2(S;V) & \xrightarrow{\mathbb{D}} & L^2(S \times I)
\end{array}$$

Assume that the function $\beta(\cdot,\cdot)$ is independent of x, so we have $\beta(\cdot) \in L^2(S)$. Let $\sigma \in L^2(S;W)$ so that $\mathbb{D}_* \sigma \in L^2(S \times I)$. Then for each $G \in V$ we have

$$\begin{aligned}
(\beta^*\sigma, DG) &= \int_S (\sigma(s), \beta(s)DG)\, d\mu_s = \int_S (\sigma(s), D\beta(s)G)\, d\mu_s \\
&= \left(\int_S \beta(s) D_* \sigma(s)\, d\mu_s, G\right).
\end{aligned}$$

This shows that

$$L^2(S;W) \subset W_S \qquad \text{and} \qquad \sigma \in L^2(S;W) \text{ implies } D_* \beta^*\sigma = \beta^* \mathbb{D}_* \sigma, \tag{4.1}$$

hence, $D_* \beta^*\sigma = D_* \beta_* \sigma$. We summarize the structure as

$$\begin{array}{ccccccccc}
L^2(S \times I) & \xrightarrow{\beta^*} & L^2(I) & \xrightarrow{D^*} & V' & \qquad & L^2(S \times I) & \xrightarrow{\mathbb{D}^*} & L^2(S;V') \\
\cup & & \cup & & \cup & & \cup & & \cup \\
W_S & \xrightarrow{\beta_*} & W & \xrightarrow{D_*} & L^2(I) & & L^2(S;W) & \xrightarrow{\mathbb{D}_*} & L^2(S \times I) \\
\cup & & & & \uparrow \beta^* & & & & \\
L^2(S;W) & & \xrightarrow{\mathbb{D}_*} & & L^2(S \times I) & & & &
\end{array}$$

Let $F \in L^2(S;V)$ so that $\mathbb{D}F \in L^2(S \times I)$. For each $g \in W$ we have similarly

$$(\beta^* \mathbb{D}F, g) = (\beta^* F, D_* g),$$

and this shows that

$$F \in L^2(S;V) \text{ implies } \beta^* F \in V \text{ and } D\beta^* F = \beta^* \mathbb{D}F. \tag{4.2}$$

Thus, (4.1) and (4.2) show that β^* commutes with both $\mathbb{D}_*$ and $\mathbb{D}$.

A *regular* solution of the resolvent equation (3.2) is a strong solution (with $v \in V$) for which, in addition, $\sigma \in L^2(S; W)$. We shall obtain such solutions as above by solving (3.3) for a $\sigma \in L^2(S; W)$. To this end, consider in $L^2(S \times I)$ the corresponding *regularized equation*

$$\varepsilon \mathbb{D}\mathbb{D}_* \sigma + \beta D D_* \beta_* \sigma + \sigma + \partial\varphi(\sigma) = g + \beta D f, \qquad \varepsilon > 0. \quad (4.3)$$

We shall regard this as the sum of three accretive operators,

$$A_1 \equiv \mathbb{D}\mathbb{D}_*, \qquad A_2 \equiv \beta D D_* \beta_*, \qquad \partial\varphi,$$

on $L^2(S \times I)$. It is easy to check that the first of these, A_1, is the subgradient on this space of the function Φ_1, where

$$\Phi_1(\cdot) = \frac{1}{2} \| \mathbb{D}_*(\cdot) \|^2_{L^2(S \times I)}.$$

The second, A_2 is likewise the subgradient of the function Φ_2 given by

$$\Phi_2(\sigma) = \begin{cases} \frac{1}{2} \| D_* \beta_*(\sigma) \|^2_{L^2(I)} & \text{if } \sigma \in W_S, \\ +\infty & \text{if } \sigma \in L^2(S \times I) \sim W_S. \end{cases}$$

To see this, note that if $F \in \partial\Phi_2(\sigma)$, then $\sigma \in W_S$ and

$$(F, g)_{L^2(S \times I)} = (D_* \beta_*(\sigma) D_* \beta_*(g))_{L^2(I)}, \qquad g \in W_S.$$

Since β is independent of x for each $w \in W$, the choice of $g = \beta w$ gives a $g \in W_S$ with

$$\beta_* g = \beta_* \beta w = \left(\int_S \beta(s)^2 \, d\mu \right) w,$$

and we have $\int_S \beta(s)^2 \, d\mu > 0$, so it follows that $D_* \beta_*(\sigma) \in V$ and that

$$\begin{aligned} (F, g)_{L^2(S \times I)} &= (D D_* \beta_*(\sigma), \beta_*(g))_{L^2(I)} \\ &= (\beta D D_* \beta_*(\sigma), g)_{L^2(S \times I)}, \qquad g \in W_S. \end{aligned}$$

Since W_S is dense in $L^2(S \times I)$, we have $F = \beta D D_* \beta_*(\sigma)$, and hence, $\partial\Phi_2 \subset \beta D D_* \beta_*$. But $\partial\Phi_2$ is maximal and $\beta D D_* \beta_*$ is accretive, so they are necessarily equal.

Next we check that we have

$$(A_1 \sigma, A_2 \sigma) \geq 0, \qquad \sigma \in \operatorname{Dom}(A_1), \quad (4.4.a)$$

$$(A_1 \sigma, \partial\varphi_\varepsilon(\sigma)) \geq 0, \qquad \sigma \in \operatorname{Dom}(A_1), \quad (4.4.b)$$

$$A_2 + \partial\varphi \text{ is m-accretive}. \quad (4.4.c)$$

The first follows from (4.1) and (4.2); the second follows from the Chain rule, since each component of $\partial\varphi_\varepsilon$ is Lipschitz and monotone. To get the third, we need only verify that $A_2 + \partial\varphi + I$ is onto. That is, we need to solve

$$\sigma \in W_S : \beta DD_* \beta_* \sigma + \partial\varphi(\sigma) + \sigma = g,$$

with all terms in $L^2(S \times I)$. This is equivalent to solving the system

$$\begin{aligned} v + D_* \beta_* \sigma &= 0, \\ \sigma + \partial\varphi(\sigma) &= \beta Dv + g, \end{aligned}$$

for which we know from Section 3 (see (3.7)) that we have a strong solution, $v \in V$, $\sigma \in W_S$ as desired. From Proposition 2.17 and Theorem 4.4 of [3] we see that $A_1 + A_2 + \partial\varphi$ is m-accretive in $L^2(S \times I)$, so Eq. (4.3) has a *regular* solution, i.e., a solution with all terms in $L^2(S \times I)$.

We would like to solve (4.3) with $\varepsilon = 0$ in order to obtain a regular solution of the resolvent equation for $\mathbb{C}$, i.e., (3.2). So assume now that $f \in V$ and $g \in L^2(S; W)$. For each $\varepsilon > 0$, let $\sigma_\varepsilon \in L^2(S; W)$ be the (regular) solution of (4.3) and define $v_\varepsilon \in V$ by the equation

$$v_\varepsilon + D_* \beta_* \sigma_\varepsilon = f,$$

so that we have

$$Dv_\varepsilon + DD_* \beta_* \sigma_\varepsilon = Df.$$

Taking the scalar product with Dv_ε gives

$$\|Dv_\varepsilon\|^2 + (DD_* \beta_* \sigma_\varepsilon, Dv_\varepsilon) = (Df, Dv_\varepsilon).$$

Since $\sigma_\varepsilon \in L^2(S; W)$ and $\mathbb{D}_* \sigma_\varepsilon \in L^2(S; V)$, we obtain from (4.1) and (4.2) that $DD_* \beta_* \sigma_\varepsilon = D\beta^* \mathbb{D}_{*\sigma\varepsilon} = \beta^* \mathbb{D}\mathbb{D}_* \sigma_\varepsilon$, so this gives

$$\|Dv_\varepsilon\|^2 + (\mathbb{D}\mathbb{D}_* \sigma_\varepsilon, \beta Dv_\varepsilon) = (Df, Dv_\varepsilon).$$

Now by substituting $\varepsilon \mathbb{D}\mathbb{D}_* \sigma_\varepsilon + \sigma_\varepsilon + \partial\varphi(\sigma_\varepsilon) - g = \beta Dv_\varepsilon$ we obtain

$$\begin{aligned} &\|Dv_\varepsilon\|^2 + \varepsilon\|\mathbb{D}\mathbb{D}_* \sigma_\varepsilon\|^2 + \|\mathbb{D}_* \sigma_\varepsilon\|^2 + (\mathbb{D}\mathbb{D}_* \sigma_\varepsilon, \partial\varphi(\sigma_\varepsilon)) - (\mathbb{D}_* \sigma_\varepsilon, \mathbb{D}_* g) \\ &\qquad = (Df, Dv_\varepsilon). \end{aligned}$$

The fourth term on the left side is nonnegative by (4.4.b) and Theorem 4.4 of [3], so we obtain the estimate

$$\|Dv_\varepsilon\|^2 + \varepsilon\|\mathbb{D}\mathbb{D}_* \sigma_\varepsilon\|^2 + \|\mathbb{D}_* \sigma_\varepsilon\|^2 \le (\mathbb{D}_* \sigma_\varepsilon, \mathbb{D}_* g) + (Df, Dv_\varepsilon).$$

From the preceding a priori estimates, we obtain the existence of a subsequence for which

$$v_\varepsilon \rightharpoonup v \in V$$

$$Dv_\varepsilon \rightharpoonup Dv \in L^2(I)$$

$$\sigma_\varepsilon \rightharpoonup \in W_S, \qquad L^2(S \times I), L^2(S; W)$$

$$\mathbb{D}_* \sigma_\varepsilon \rightharpoonup \mathbb{D}_* \sigma \in L^2(S \times I)$$

$$\varepsilon \mathbb{D}\mathbb{D}_* \sigma_\varepsilon \to 0 \in L^2(S \times I).$$

From the definition of the subgradient and (4.3) we obtain

$$\begin{aligned}(g + \beta Df)(\rho - \sigma_\varepsilon) &- (\varepsilon \mathbb{D}_* \sigma_\varepsilon, \mathbb{D}_*(\rho - \sigma_\varepsilon)) \\ &-(D_* \beta_* \sigma_\varepsilon, D_* \beta_*(\rho - \sigma_\varepsilon)) \\ &-(\sigma_\varepsilon, \rho - \sigma_\varepsilon) + \varphi(\sigma_\varepsilon) \le \varphi(\rho), \qquad \rho \in L^2(S; W).\end{aligned}$$

By taking the limit infimum, we get

$$\begin{aligned}(g + \beta Df)(\rho - \sigma) &- (D_* \beta_* \sigma, D_* \beta_*(\rho - \sigma)) - (\sigma, \rho - \sigma) \\ &+ \varphi(\sigma) \le \varphi(\rho), \qquad \rho \in L^2(S; W).\end{aligned}$$

That is, $\sigma \in L^2(S; W)$ is the solution of

$$\beta DD_* \beta_* \sigma + \partial\varphi(\sigma) + \sigma \ni g + \beta Df, \tag{4.5}$$

and with v defined by

$$v \in V : v + D_* \beta_* \sigma = f,$$

the pair $[v, \sigma]$ satisfies

$$(I + \mathbb{C})([v, \sigma]) \ni [f, g]$$

together with the estimate

$$\|Dv\|^2 + \|\mathbb{D}_* \sigma\|^2 \le (D_* \sigma, D_* g) + (Df, Dv).$$

Thus, we have shown that the corresponding solutions $\{\sigma_\varepsilon\}$ of the regularized equation (4.3) converge weakly to the strong solution of (4.5), and that this solution is regular. By the uniqueness of solutions of (4.5) and of weak limits, this holds not only for the subsequence chosen above but for the original sequence.

In addition, the preceding shows that the resolvent of the operator $\mathbb{C}$ is *stable* under the norm of $V \times L^2(S; W)$. That is, the lower semicontinuous norm

$$\Phi([v, \sigma]) \equiv \left(\|Dv\|^2 + \|\mathbb{D}_* \sigma\|^2\right)^{1/2}$$

is a *Liapunov functional* for the Cauchy problem (3.1). In particular, we have shown that

$$\Phi\left((I + \mathbb{C})^{-1}[f, g]\right) \leq \Phi([f, g]), \qquad [f, g] \in V \times L^2(S; W).$$

For any $\delta > 0$, we can replace $\mathbb{C}$ by $\delta\mathbb{C}$ in the above without loss of generality, since this amounts to replacing β by $\delta\beta$ and φ by $\delta\varphi$ Neither of these substitutions alters the hypotheses. Thus, we have

$$\Phi\left((I + \delta\mathbb{C})^{-1}[f, g] \leq \Phi([f, g]), \qquad [f, g] \in V \times L^2(S; W)\right), \delta > 0.$$

When additionally $f(t) = 0$ and $g(t) = 0$, this implies that each closed ball in $H = L^2(0, 1) \times L^2(S \times (0, 1))$ of the form

$$B_R \equiv \left\{[v, \sigma] \in V \times L^2(S; W) : \Phi([v, \sigma]) \leq R\right\}$$

is *invariant* under the evolution equation in (3.1). For the nonhomogeneous case we shall show that each solution remains in such a ball, B_R, and thereby is a regular solution.

THEOREM R. *In the situation of Theorem S, assume in addition that the function $\beta(\cdot,\cdot)$ is independent of x, that is, $\beta(\cdot) \in L^2(S)$, and assume $[f(\cdot), g(\cdot)] \in L^1(0, T; V \times L^2(S; W))$ and $[v_0, \sigma_0] \in V \times L^2(S; W)$. Then the strong solution $[v(t), \sigma(t)]$ from Theorem S satisfies*

$$\left(\|Dv(t)\|^2 + \|\mathbb{D}_* \sigma(t)\|^2\right)^{1/2} \leq \left(\|Dv_0\|^2 + \|\mathbb{D}_* \sigma_0\|^2\right)^{1/2}$$
$$+ \int_0^t \left(\|Df(s)\|^2 + \|\mathbb{D}_* g(s)\|^2\right)^{1/2} ds, \qquad 0 \leq t \leq T, \tag{4.6}$$

hence, $\sigma \in L^\infty(0, T; L^2(S, W))$.

Proof. In considerably more general situations than the above, one can approximate the abstract Cauchy Problem (1.5) by a backward difference equation. Thus, let $h_n > 0$ be the size for the nth step in the approximation of the solution $\mathbf{x}(t)$ of (1.5) by a step function $\mathbf{x}^n(t)$ which has the value $\mathbf{x}_k^n$ on the corresponding interval, $kh_n < t \leq (k + 1)h_n$. If the nonhomogeneous term $\mathbf{f}(t)$ is replaced by a step function $\mathbf{f}^n$ which likewise takes on the value $\mathbf{f}_k^n$ on each interval $(kh_n, (k + 1)h_n]$, then the approxi-

mate solution of (1.5) is obtained by solving the backward difference scheme

$$\frac{\mathbf{x}_k^n - \mathbf{x}_{k-1}^n}{h_n} + \mathbb{C}(\mathbf{x}_k^n) \ni \mathbf{f}_k^n,$$

$$\mathbf{x}_0^n = \mathbf{x}_0.$$

Thus, the approximate solution is given recursively by

$$\mathbf{x}_k^n = (I + h_n\mathbb{C})^{-1}(\mathbf{x}_{k-1}^n + h_n\mathbf{f}_k^n), \qquad k = 1, 2, \dots .$$

It is known that if $\|\mathbf{f} - \mathbf{f}^n\|_{L^1(0,T;H)} \to 0$, then $\|\mathbf{x}(t)\text{-}\mathbf{x}^n(t)\|_H \to 0$, uniformly for $t \in [0, 1]$. See [4] for this and related results.

In our situation, since the norm Φ is not increased by the resolvent $(I + h_n\mathbb{C})^{-1}$, we have

$$\Phi(\mathbf{x}_k^n) \le \Phi(\mathbf{x}_{k-1}^n + h_n\mathbf{f}_k^n), \qquad k = 1, 2, \dots,$$

from which we obtain

$$\Phi(\mathbf{x}_k^n) \le \Phi(\mathbf{x}_0) + h_n(\Phi(\mathbf{f}_1^n) + \Phi(\mathbf{f}_2^n) + \dots \Phi(\mathbf{f}_k^n)), \qquad k = 1, 2, \dots .$$

By using the lower semicontinuity of Φ on the left side to take the limit, we get

$$\Phi(\mathbf{x}(t)) \le \Phi(\mathbf{x}_0) + \int_0^t (\Phi(\mathbf{f}(s))\, ds, \qquad 0 \le t \le T,$$

and this is the desired estimate (4.6). ∎

For the plasticity problems, the estimate (4.6) is a substantial regularity result for solutions. In particular, whereas a strong solution is one for which the *average* stress $\beta_*\sigma$ is regular in the sense that $\beta_*\sigma \in W$, that is, it is differentiable, the regular solution is one for which *each component* of the stress is differentiable, i.e., $\sigma(s, \cdot) \in W$ for a.e. $s \in S$. The proof given above for Theorem R depends on the assumptions (4.4). We note that (4.4.a) follows from the lack of dependence of $\beta(s)$ on x, and (4.4.c) also follows rather generally. However, although the verification of (4.4.b) appears to be easy in one dimension, it is difficult to find examples in $\mathbb{R}^n$ which satisfy this condition.

If we have $\partial\varphi_s = 0$ or, more generally, $\partial\varphi_s : W \to W$ is bounded uniformly in s, for $s \in S_0$, then from the restriction to S_0 of the identity $\sigma + \partial\varphi(\sigma) = \beta Dv + g$ we obtain a regularity result for the velocity in the stationary resolvent equation. That is, we get $Dv \in W$ and consequently

$D_* Dv \in L^2(I)$. This occurs, for example, when $\partial\varphi_s$ arises from kinematic hardening or from a viscosity regularization, respectively. A corresponding regularity result for the displacement of a regular solution of the Cauchy problem is the following.

COROLLARY. *Assume additionally that* $\partial\varphi_s = 0$ *for* $s \in S_0$ *and that* $u_0 \in V$ *with* $Du_0 \in W$. *Let* $[v,(t), \sigma(t)]$ *be the regular solution from Theorem R, and denote the displacement by* $u(t) = u_o + \int_0^t v(\tau)\, d\tau$. *Then* $u \in W^{1,\infty}(0, T; V)$ *and* $Du \in L^\infty(0, T; W)$, *i.e.,* $D_* Du \in L^\infty(0, T; L^2(I))$.

Proof. For $s \in S_0$ we have

$$\frac{\partial}{\partial t}(\sigma(s,t) - \beta(s) Du(t)) = g(s,t)$$

in $W^{1,1}(0, T; L^2(S_0 \times I)) \cap L^1(0, T; L^2(S_0, W))$. Integrate this to obtain

$$\sigma(s,t) - \beta(s) Du(t) = \int_0^t g(s,\tau)\, d\tau + \sigma_0(s) - \beta(s) Du(0)$$

in $W^{2,1}(0, T; L^2(S_0 \times I)) \cap W^{1,1}(0, T; L^2(S_0, W))$. The first term and, hence, also the second term belongs to $L^\infty(0, T; L^2(S_0, W))$; after multiplying by $\beta(s)$ and integrating over S_0 we find with the aid of (3.6) that $Du \in L^\infty(0, T; W)$. ∎

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