

Initial and Final-Value Problems for Degenerate Parabolic Evolution Systems

R. E. SHOWALTER

1. Introduction. We begin by showing that a Cauchy problem for the linear implicit evolution equation

$$(1.1) \quad \frac{d}{dt} (Mu(t)) + Lu(t) = f(t), \quad t > 0$$

is *continuously solvable*. By this we mean that for appropriate data there exists a (not necessarily unique) solution which depends continuously on the data with respect to certain seminorms. The non-negative and Hermitian operator M may be degenerate (*i.e.*, may vanish on non-zero vectors) and L is required to satisfy a sector condition. In Section 2 we show that this Cauchy problem is resolved by an analytic semigroup. Analogous results are obtained in Section 4 for a second-order system which is equivalent to the equation

$$(1.2) \quad \frac{d}{dt} \left(\frac{d}{dt} Cu(t) + Bu(t) \right) + Au(t) = f(t), \quad t > 0$$

when C and A are non-negative and Hermitian and B satisfies a sector condition. These results on continuous solvability of Cauchy problems for (1.1) and (1.2) are extensions of certain related results in [1]. Although we do obtain new applications to initial-boundary value problems for degenerate partial differential equations, the primary objective of the above presentation is to provide the background material for our treatment below of the corresponding final-value problems.

The method of *quasi-reversibility* (QR) will be used to approximate the solutions of final-value problems for (1.1) and (1.2) in the situation described above. The equations are *parabolic*, in the sense that they are resolved by analytic semigroups, so the final-value problems are necessarily ill-posed. In a QR-method one begins with final data at time $t = T > 0$ and solves an approximating equation backward in time to obtain corresponding initial data at time $t = 0$. Then the original equation is solved forward starting from the initial data. The general QR-method was introduced by Lattes and Lions [5]. We shall implement the QR-method in Section 3 where we approximate (1.1) by

$$(1.3) \quad \frac{d}{dt} (M + \epsilon L)v_\epsilon(t) + Lv_\epsilon(t) = 0$$

and in Section 5 where we approximate (1.2) by (essentially)

$$(1.4) \quad \frac{d}{dt} \left(\frac{d}{dt} (C + \epsilon B + \epsilon^2 A)v_\epsilon(t) + (B + 2\epsilon A)v_\epsilon(t) \right) + Av_\epsilon(t) = 0$$

with $\epsilon > 0$. (We set $f(t) \equiv 0$ with no loss of generality.) Both the initial and the final value problems are continuously solvable for (1.3) and (1.4). The success of a general QR-method depends on how well the solution so obtained approximates the final data. Our method always converges to the final data and, moreover, converges on the entire interval $[0, T]$ if and only if there actually exists a solution of the final value problem!

The results on final-value problems are obtained from [7]. If M is the identity operator our Theorem 2 coincides with the results of [7] for the standard evolution equation. Similarly, if C is the identity and $B = A$, we recover the special results in [8] and in Chapter 2 of [5]. Our hypotheses permit the choice of $A = 0$ in which case (1.1) and (1.2) are equivalent, and we may obtain corresponding results by setting $C = 0$. The final-value problem for (1.1) was studied in [4] where M was replaced by $I + M_0$, M_0 being m -accretive. Such hypotheses do not allow nonuniqueness. We refer to Chapter 3 of [1] for a variety of examples of boundary value problems for partial differential equations to which our abstract results immediately apply.

We indicate some of the notation that will be used below. The algebraic dual of conjugate-linear functionals on the complex linear space V is denoted by V^* ; when V is a topological vector space we indicate by V' its topological dual of continuous conjugate-linear functionals. For the linear operator L of V into V^* we denote by $\text{Ker}(L)$ and $\text{Rg}(L)$ the kernel and range, respectively. Such an operator is called θ -sectorial if its numerical range $\{Lx(x) : x \in V\}$ is contained in the sector $S(\theta)$ of all complex numbers z whose arguments satisfy $|\arg(z)| \leq \theta$. Thus $(\pi/2)$ -sectorial means *monotone*: $\text{Re} Lx(x) \geq 0$ for all $x \in V$. (The real and imaginary parts of the complex z are denoted by $\text{Re}(z)$ and $\text{Im}(z)$, respectively.) A monotone operator L is called *strictly monotone* if $\text{Re} Lx(x) = 0$ only if $x = 0$. Suppose V is a Banach space and $0 < T$. Then for $0 < \alpha \leq 1$, $C^\alpha_T(V)$ is the space of functions $F: [0, T] \rightarrow V$ which are Hölder-continuous with exponent α :

$$\|f(t) - f(s)\|_V \leq K(t - s)^\alpha, \quad 0 \leq s \leq t \leq T,$$

for some $K > 0$, and we set $C^\alpha(V) = \cap \{C^\alpha_T(V) : T > 0\}$. The space of functions $f: [0, T] \rightarrow V$ which are continuous on $[0, T]$ and (strongly) differentiable in $(0, T)$ is denoted by $D_T(V)$, and we set $D(V) = \cap \{D_T(V) : T > 0\}$.

2. First-order equations. We shall prove that an appropriate initial-value problem for the first-order implicit evolution equation is continuously solvable.

Theorem 1. *Let the linear operators L and M map the vector space V into*

its dual V^* . Assume M is Hermitian and monotone; denote by V_m the space V with the seminorm $|x|_m \equiv Mx(x)^{1/2}$ and by V'_m the corresponding dual space. Thus, V'_m is a Hilbert space contained in V^* . Assume that for some real λ_0 and θ with $0 \leq \theta < \pi/2$, $\lambda_0 M + L$ is θ -sectorial and $\text{Rg}(\lambda M + L) \supset V'_m$ for all $\lambda > \lambda_0$. Then for each $h \in V'_m$ and $f \in C^\alpha(V'_m)$ there exists a function $u: (0, \infty) \rightarrow V$ such that $Mu \in D(V'_m)$, $Mu(0) = h$,

$$(2.1) \quad \frac{d}{dt} Mu(t) + Lu(t) = f(t), \quad t > 0.$$

Any such solution satisfies the estimate

$$(2.2) \quad |u(t)|_m \leq e^{\lambda_0 t} \|h\|_{V'_m} + \int_0^t e^{\lambda_0(t-s)} \|f(s)\|_{V'_m} ds, \quad t \geq 0.$$

If v is another solution then $u(t) - v(t) \in \text{Ker}(M) \cap \text{Ker}(L)$ for all $t > 0$.

Remarks. The continuous solvability of the Cauchy problem for (2.1) follows from the asserted existence of solutions and the *a priori* estimate (2.2). If the operator $L + \lambda M$ is strictly monotone, then uniqueness follows and the Cauchy problem is well-posed.

The proof of Theorem 1 will show that $\text{Rg}(\lambda M + L) \supset V'_m$ for all $\lambda > \lambda_0$ if it is true for some $\lambda > \lambda_0$.

Proof of Theorem 1. Since u is a solution of (2.1) if and only if the function $v(t) \equiv e^{-\lambda t} u(t)$, $\lambda > \lambda_0$, is a solution of the equation with L replaced by $\lambda M + L$, we may assume with no loss of generality that $\text{Rg}(L) \supset V'_m$ and that

$$(2.3) \quad \text{Re } Lx(x) \geq (\lambda - \lambda_0)Mx(x), \quad x \in V.$$

Let $V_m/\text{Ker}(M)$ be the indicated quotient space with the inner product inherited from V_m , and denote its completion by H . The quotient map $q: V_m \rightarrow V_m/\text{Ker}(M)$ is a strict homomorphism of V_m into H and its (continuous) dual $q^*: H' \rightarrow V'_m$ is an isomorphism. (To see that q^* is a surjection, note that each $f \in V'_m$ which vanishes on $\text{Ker}(M)$ can be factored into $f = g \circ q$ for some $g \in (V_m/\text{Ker}(M))'$. Since $V_m/\text{Ker}(M)$ is dense in H , $g \in H'$ and we have $f = q^*(g)$.) Furthermore we have

$$(q(x), q(y))_H = Mx(y), \quad x, y \in V$$

so the Riesz map $M_0: H \rightarrow H'$ satisfies

$$q^* M_0 q(x) = M(x), \quad x \in V.$$

The operator L will be factored similarly. Set $D \equiv \{x \in V: Lx \in V'_m\}$ and $D_0 \equiv q[D]$, the image in $V_m/\text{Ker}(M)$ of D under the map q . From the lemma below we obtain $\text{Ker}(M) \cap D \subset \text{Ker}(L)$; since q^* is an isomorphism of H' onto V'_m it follows that there is a function $L_0: D_0 \rightarrow H'$ for which

$$q^* L_0 q(x) = L(x), \quad x \in D.$$

We shall show that the negative of the operator $A: D_0 \rightarrow H$ defined by $A \equiv M_0^{-1}L_0$ is the generator of an analytic semigroup of contractions on H . Since M_0 is the Riesz map for H we obtain

$$(Ax, y)_H = M_0 Ax(y) = L_0 x(y), \quad x \in D_0, y \in H,$$

and setting $x = y = q(v)$ gives

$$(2.4) \quad (Ax, x)_H = Lv(v), \quad v \in D, x = q(v) \in D_0.$$

Also we have the identity

$$(2.5) \quad q^* M_0 (I + A) q(v) = (M + L)(v), \quad v \in D.$$

From (2.4), (2.5) and our hypotheses on L and M , it follows that A is m -sectorial on H , hence, $-A$ generates an analytic semigroup [3, pages 490–493]. Thus there is a unique solution $x(\cdot) \in D(H)$ of the Cauchy problem

$$(2.6) \quad \begin{aligned} x'(t) + Ax(t) &= (q^* M_0)^{-1} f(t), \quad t > 0, \\ x(0) &= (q^* M_0)^{-1} h. \end{aligned}$$

From (2.3) and (2.4) we obtain

$$\operatorname{Re}(Ax, x)_H \geq (\lambda - \lambda_0) \|x\|_H^2, \quad x \in D,$$

so the semigroup representation of $x(\cdot)$ gives

$$(2.7) \quad \|x(t)\|_H \leq e^{-(\lambda - \lambda_0)t} \|x(0)\|_H + \int_0^t e^{-(\lambda - \lambda_0)(t-s)} \|(q^* M_0)^{-1} f(s)\|_H ds.$$

The desired results will be obtained from the correspondence between solutions of the Cauchy problem for (2.1) and that of (2.6). First, if $x(\cdot)$ is the solution of (2.6) then for each $t > 0$, $x(t) \in D_0$, so there exists a $u(t) \in D$ with $q(u(t)) = x(t)$. Since $q^* M_0: H \rightarrow V'_m$ is an isomorphism, it follows that $Mu = (q^* M_0)x \in D(V'_m)$, and from (2.6) we obtain $Mu(0) = h$ and (2.1). This establishes the existence of a solution of our problem. Conversely, if u is a solution of the Cauchy problem for (2.1), then the function defined by $x(t) = q(e^{-\lambda t} u(t))$, $t > 0$, is the solution of (2.6). But $q^* M_0$ being an isomorphism and the estimate (2.7) imply the desired estimate (2.2). Note finally that if $h = 0$ and $f(t) \equiv 0$, then for any solution u of the Cauchy problem it follows from (2.2) that $Mu(t) \equiv 0$, hence, from (2.1) that $Lu(t) \equiv 0$. The uniqueness criterion now follows by linearity.

It remains only to establish the following.

Lemma. $\operatorname{Ker}(M) \cap D = \operatorname{Ker}(L)$.

Proof. From (2.3) and the Cauchy-Schwartz inequality for $Mx(y)$ we obtain $\operatorname{Ker}(L) \subset \operatorname{Ker}(M) \cap D$. For the reverse inclusion, note first that for each $x \in D$ we have

$$|Lx(y)| \leq (\text{const.}) Mx(y)^{1/2}, \quad y \in V.$$

Setting $x = y$ thus gives $Lx(x) = 0$ for all $x \in \operatorname{Ker}(M) \cap D$. But L is θ -sectorial so

we have [3, p. 311]

$$|Lx(y)| \leq (1 + \tan \theta) \operatorname{Re} Lx(x)^{1/2} \operatorname{Re} Ly(y)^{1/2}, \quad x, y \in V,$$

so it follows from $Lx(x) = 0$ that $Lx = 0$ for all $x \in \operatorname{Ker}(M) \cap D$.

Remark. The preceding lemma guarantees that a (single-valued) function L_0 is obtained when we factor L through the quotient map q . This is in contrast to the more general situations *c.f.* [1, pages 216–218] where L_0 is multi-valued.

3. Final-value problem I. Consider now the following final-value problem: given $F \in V'_m$ and $T > 0$, find $u: (0, T) \rightarrow V$ for which $Mu \in D_T(V'_m)$, $Mu(T) = F$, and

$$(3.1) \quad \frac{d}{dt} Mu(t) + Lu(t) = 0, \quad 0 < t < T.$$

All results below extend immediately to the non-homogeneous (2.1); it is necessary only to require $f \in C^q(V'_m)$ and apply the usual linearity arguments. We can give examples to show that the final-value problem for (3.1) is *ill-posed*. More generally, in the situation of Theorem 1, if the final-value problem were continuously solvable, the corresponding problem of finding $x \in D_T(H)$ with $x(T) = (q^*M_0)^{-1}F$ and

$$(3.2) \quad x'(t) + Ax(t) = 0, \quad 0 < t < T,$$

would be well-posed. Thus, the semigroup $S(t) \equiv \exp(-At)$, $t \geq 0$, would have an extension to a group $S(t)$, $-\infty < t < +\infty$. Since $S(\cdot)$ is analytic, $A \cdot S(t)$ is bounded on H for each $t > 0$, and this would imply that $A = (A \cdot S(t)) \cdot S(-t)$ is bounded on H . Hence, in the situation of Theorem 1, the final-value problem is continuously solvable if and only if L is continuous from V_m to V'_m , *i.e.*, L is dominated by M .

Our results on the final-value problem for (3.1) in the situation of Theorem 1 will be obtained from corresponding results for the final-value problem for (3.2). We shall assume for the moment that $\lambda_0 = 0$; then A is accretive and the generated semigroup consists of contractions on H . Thus, the QR-method of [7] applies to (3.2) with $x(T) = (q^*M_0)^{-1}F$ and we describe it as follows. For each $\epsilon > 0$, $A(I + \epsilon A)^{-1}$ is bounded on H so there is a unique solution y_ϵ of

$$(3.3) \quad \frac{d}{dt} (I + \epsilon A)y_\epsilon(t) + Ay_\epsilon(t) = 0, \quad 0 \leq t \leq T,$$

$$y_\epsilon(T) = (q^*M_0)^{-1}F.$$

Having obtained $y_\epsilon(0) \in H$ from (3.3), we define x_ϵ to be that solution of (3.2) for which $x_\epsilon(0) = y_\epsilon(0)$. Then one expects (at least) that $x_\epsilon(T)$ approximates $x(T)$ in some sense as $\epsilon \rightarrow 0$. The following results were proved in [7] under the additional assumption that A is $(\pi/4)$ -sectorial:

There is at most one solution $x \in D_T(H)$ of (3.2) satisfying the condition $x(T) = (q^*M_0)^{-1}F$.

The preceding QR-method is *stable* and *convergent* at the final time:

$$\|x_\epsilon(T)\|_H \leq \|(q^*M_0)^{-1}F\|_H, \quad \epsilon > 0, \text{ and}$$

$$\lim_{\epsilon \rightarrow 0} x_\epsilon(T) = (q^*M_0)^{-1}F \quad \text{in } H.$$

There exists a solution $x \in D_T(H)$ of (3.2) with $x(T) = (q^*M_0)^{-1}F$ if and only if $\lim_{\epsilon \rightarrow 0} x_\epsilon(0)$ exists in H , and then we obtain the estimates

$$\|x_\epsilon^{(n)}(t) - x^{(n)}(t)\|_H \leq o(\epsilon)/t^n, \quad \epsilon > 0, 0 < t \leq T$$

$$\|x_\epsilon^{(n)}(t) - x^{(n)}(t)\|_H \leq C_\delta \cdot \epsilon/(t - \delta)^n, \quad \epsilon > 0, 0 < \delta < t \leq T.$$

The preceding results extend easily to the situation where $\lambda_0 + A$ is $(\pi/4)$ -sectorial for some real λ_0 ; this more general situation will arise in Section 5 so it is appropriate to include it here. The point is to observe that x is a solution of (3.2) if and only if $x_0(t) \equiv e^{-\lambda_0 t}x(t)$ is a solution of the same equation with the operator $\lambda_0 + A$. Since $\lambda_0 + A$ is $(\pi/4)$ -sectorial, the reversible approximation (3.3) is replaced by

$$\frac{d}{dt}(I + \epsilon(\lambda_0 + A))y_\epsilon(t) + (\lambda_0 + A)y_\epsilon(t) = 0, \quad 0 \leq t \leq T$$

(3.4)

$$y_\epsilon(T) = e^{-\lambda_0 T}(q^*M_0)^{-1}F.$$

The approximate solution of the final-value problem for (3.2) is then obtained as that solution x_ϵ of (3.2) with initial-value $x_\epsilon(0) = y_\epsilon(0)$.

In order to extend the preceding results to the situation of Theorem 1, we recall how the equation (3.2) arose in the proof and investigate the correspondence between solutions of (3.2) and (3.4) with those of (3.1) and an appropriate approximation. The remarks following (2.7) show that u is a solution of (3.1) with $Mu(T) = F$ if and only if $x(t) \equiv q(u(t))$ is the solution of (3.2) with $x(T) = (q^*M_0)^{-1}F$. Similarly, v_ϵ is a solution of

$$\frac{d}{dt}(M + \epsilon(\lambda_0 M + L))v_\epsilon(t) + (\lambda_0 M + L)v_\epsilon(t) = 0, \quad 0 \leq t \leq T$$

(3.5)

$$Mv_\epsilon(T) = e^{-\lambda_0 T}F$$

if and only if $y_\epsilon(t) \equiv q(v_\epsilon(t))$ is the solution of (3.4). Finally, we note that q is a strict homomorphism and from (2.4) that $\lambda_0 + A$ is $(\pi/4)$ -sectorial precisely when $\lambda_0 M + L$ is $(\pi/4)$ -sectorial. These observations lead to the following result.

Theorem 2. *Let the spaces V , V_m and operators L , M be given as in Theorem 1. Let $F \in V'_m$ and assume further that $\lambda_0 M + L$ is $(\pi/4)$ -sectorial. If u_1 and*

u_2 are solutions of the final-value problem (3.1) with $Mu(T) = F$, then $u_1(t) - u_2(t) \in \text{Ker}(M) \cap \text{Ker}(L)$ for all t , $0 < t \leq T$. For each $\epsilon > 0$ let v_ϵ be a solution of (3.5) and let u_ϵ be any solution of (3.1) with $Mu_\epsilon(0) = Mv_\epsilon(0)$. Then $\|u_\epsilon(T)\|_m \leq \|F\|_{V'_m}$ and $\lim_{\epsilon \rightarrow 0} Mu_\epsilon(T) = F$ in V'_m . There exists a solution u of (3.1) with $Mu(T) = F$ if and only if $\lim_{\epsilon \rightarrow 0} Mv_\epsilon(0)$ exists in V'_m , and then we have the estimate

$$(3.6) \quad \left\| \left(\frac{d}{dt} \right)^n (Mu_\epsilon(t) - Mu(t)) \right\|_{V'_m} \leq o(\epsilon)/t^n, \quad \epsilon > 0, 0 < t \leq T,$$

$$(3.7) \quad \left\| \left(\frac{d}{dt} \right)^n (Mu_\epsilon(t) - Mu(t)) \right\|_{V'_m} \leq C_\delta \cdot \epsilon / (t - \delta)^n, \quad \epsilon > 0, 0 < \delta < t \leq T.$$

Remark. The filter $\{Mv_\epsilon(0)\}_{\epsilon > 0}$ is convergent in V'_m if and only if $\{v_\epsilon(0)\}_{\epsilon > 0}$ is Cauchy in V_m .

4. Second-order equations. We shall use Theorem 1 to show that an initial-value problem for a parabolic system is continuously solvable. This system contains the second-order equation (1.2) as well as the first-order (1.1).

Theorem 3. Let the linear operators, A , B and C map the vector space U into its dual U^* . Assume A and C are Hermitian and monotone; denote by U_a and U_c the space U with the respective seminorms $|x|_a \equiv Ax(x)^{1/2}$ and $|x|_c \equiv Cx(x)^{1/2}$. Assume that for some $\lambda_0 > 0$, $K \geq 0$, and θ with $0 \leq \theta < \pi/2$, $\lambda_0 C + B$ is θ -sectorial and $\text{Rg}(\lambda^2 C + \lambda B + A) \supset U'_a + U'_c$ for $\lambda > \lambda_0$,

$$(4.1) \quad |x|_a^2 \leq K \cdot \text{Re}(B + \lambda_0 C)x(x), \quad x \in U,$$

and that $\text{Rg}(A) = U'_a$. Then for $h_a \in U'_a$, $h_c \in U'_c$, $f_1 \in C^\alpha(U'_a)$ and $f_2 \in C^\alpha(U'_c)$ there exists a pair of functions $u_1, u_2: (0, \infty) \rightarrow U$ such that $Au_1 \in D(U'_a)$, $Cu_2 \in D(U'_c)$, $Au_1(0) = h_a$, $Cu_2(0) = h_c$, and

$$(4.2.a) \quad \frac{d}{dt} Au_1(t) - Au_2(t) = f_1(t)$$

$$(4.2.b) \quad \frac{d}{dt} Cu_2(t) + Au_1(t) + Bu_2(t) = f_2(t), \quad t > 0.$$

Any such solution pair satisfies the estimate

$$(4.3) \quad \begin{aligned} & (|u_1(t)|_a^2 + |u_2(t)|_c^2)^{1/2} \leq e^{\lambda_0 t} (\|h_a\|_{U'_a}^2 + \|h_c\|_{U'_c}^2)^{1/2} \\ & + \int_0^t e^{\lambda_0(t-s)} (\|f_1(s)\|_{U'_a}^2 + \|f_2(s)\|_{U'_c}^2)^{1/2} ds, \quad t \geq 0. \end{aligned}$$

If v_1, v_2 is another solution pair then $u_1(t) - v_1(t) \in \text{Ker}(A)$ and $u_2(t) - v_2(t) \in \text{Ker}(A) \cap \text{Ker}(B) \cap \text{Ker}(C)$ for all $t > 0$.

Proof. Define $V \equiv U \times U$ and $M: V \rightarrow V^*$ by $M[x_1, x_2] \equiv [Ax_1, Cx_2]$ for $[x_1, x_2] \in V$; it follows that $V_m = U_a \times U_c$. Set $L[x_1, x_2] \equiv [-Ax_2, Ax_1 + Bx_2]$ for

$[x_1, x_2] \in V$. We first verify the sector condition in Theorem 1. From the identities

$$\operatorname{Re}(\lambda_0 M + L)x(x) = \lambda_0 |x_1|_a^2 + \operatorname{Re}(B + \lambda_0 C)x_2(x_2),$$

$$\operatorname{Im}(\lambda_0 M + L)x(x) = 2 \operatorname{Im} Ax_1(x_2) + \operatorname{Im} Bx_2(x_2), \quad x = [x_1, x_2] \in V,$$

the estimate

$$2|\operatorname{Im} Ax_1(x_2)| \leq \frac{1}{\delta} |x_1|_a^2 + \delta |x_2|_a^2, \quad \epsilon > 0,$$

and the hypotheses above, we obtain

$$(4.4) \quad |\operatorname{Im}(\lambda_0 M + L)x(x)| \leq \max\{(\lambda_0 \delta)^{-1}, \delta K + \tan \theta\} \times \\ \operatorname{Re}(\lambda_0 M + L)x(x), \quad x \in V.$$

This shows $\lambda_0 M + L$ is θ_δ -sectorial for some θ_δ , $\theta < \theta_\delta < \pi/2$. Next we verify the range condition. Let $g_1 \in U'_a$ and $g_2 \in U'_c$. Our hypotheses show there is a $u_2 \in U$ with $(\lambda^2 C + \lambda B + A)u_2 = \lambda g_2 - g_1$ and a $u_1 \in U$ satisfying $\lambda A u_1 = A u_2 + g_1$. Thus for $\lambda > \lambda_0$ and $[g_1, g_2] \in V'_m$ there is a pair $[u_1, u_2] \in V$ for which

$$\lambda A u_1 - A u_2 = g_1, \quad A u_1 + (B + \lambda C)u_2 = g_2.$$

That is, $\operatorname{Rg}(\lambda M + L) \supset V'_m$. All hypotheses of Theorem 1 are met, so Theorem 3 follows directly.

We remark that the condition that $\operatorname{Rg}(A) = U'_a$ is equivalent to having $\operatorname{Rg}(A)$ closed, and this is equivalent to having the seminorm space U_a complete. The estimate (4.1) means that $B + \lambda_0 C$ dominates A . The range condition in Theorem 3 can be shown to follow from (4.1) in certain rather general situations. Specifically, let U_{a+c} denote U with the seminorm induced by $A + C$ and assume B is continuous from U_{a+c} into its dual U'_{a+c} . Then the estimate (4.1) shows that for λ sufficiently large, $\lambda^2 C + \lambda B + A$ is coercive over U_{a+c} . If U_{a+c} is complete, it follows that $\operatorname{Rg}(\lambda^2 C + \lambda B + A) = U'_{a+c}$, hence, contains $U'_a + U'_c \subset U'_{a+c}$. It follows as above that $\operatorname{Rg}(\lambda M + L) \supset V'_m$ for some $\lambda > \lambda_0$, hence, for all $\lambda > \lambda_0$.

The system (4.2) is equivalent to a single second-order evolution equation in the situation of Theorem 3. If we set $u(t) \equiv u_2(t)$ then we easily show that $u: (0, \infty) \rightarrow U$ satisfies

$$Cu \in D(U'_c), \quad \frac{d}{dt} Cu + Bu - f_2 \in D(U'_a),$$

$$Cu(0) = h_c, \quad \left(-\frac{d}{dt} Cu + Bu - f_2 \right)(0) = -h_a,$$

and

$$(4.5) \quad \frac{d}{dt} \left\{ \frac{d}{dt} Cu(t) + Bu(t) - f_2(t) \right\} + Au(t) = -f_1(t), \quad t > 0.$$

Conversely, if u is given as above, we can set $u_2 \equiv u$ and define u_1 by (4.2.b) to obtain a solution u_1, u_2 of the initial-value problem in Theorem 3.

Finally, we remark that Theorem 3 contains Theorem 1 as a special case. If we set $A \equiv 0$, then $U'_a = \{0\}$ so necessarily $f_1 \equiv 0$, and (4.2.a) is trivially true. But then (4.2.b) is precisely (2.1) where we identify M with C , L with A , u with u_2 , and f with f_2 . Thus, this case of Theorem 3 is just Theorem 1 with the additional assumption that $\lambda_0 > 0$. But the change-of-variable remark at the beginning of Theorem 1 shows that Theorem 1 is true for some λ_0 only if it is true for all real λ_0 , so Theorem 1 and Theorem 3 are *equivalent*.

5. Final-value problem, II. We consider here a final-value problem for the parabolic system (4.2). Since the equation (3.1) is contained as a special case of (4.2) it follows from our remarks at the beginning of Section 3 that the final-value problem for (4.2) will not be continuously solvable in general. Specifically, these remarks imply this problem is continuously solvable only if C dominates both A and B ; this follows in the situation of Theorem 3 whenever L is continuous from V_m to V'_m . By the usual linearity arguments and Theorem 3, it suffices to consider homogeneous systems. Thus we have the following problem in the situation of Theorem 3: given $T > 0$, $F_1 \in U'_a$ and $F_2 \in U'_c$, find a pair of functions $u_1, u_2: (0, T] \rightarrow U$ such that $Au_1 \in D_T(U'_a)$, $Cu_2 \in D_T(U'_c)$, $Au_1(T) = F_1$, $Cu_2(T) = F_2$, and

$$(5.1.a) \quad \frac{d}{dt} Au_1(t) - Au_2(t) = 0, \quad 0 < t < T,$$

$$(5.1.b) \quad \frac{d}{dt} Cu_2(t) + Au_1(t) + Bu_2(t) = 0.$$

Theorem 4. Let the spaces U, U_a, U_c and operators A, B, C be given as in Theorem 3. Let $F_1 \in U'_a, F_2 \in U'_c$, and assume further that the angle θ of $\lambda_0 C + B$ satisfies $0 \leq \theta < \pi/4$ and λ is sufficiently large. For each $\epsilon > 0$, let $v_1^\epsilon, v_2^\epsilon$ be a solution of the system

$$(5.2.a) \quad \frac{d}{dt} \{(1 + \epsilon\lambda)Av_1^\epsilon(t) - \epsilon Av_2^\epsilon(t)\} + \lambda Av_1^\epsilon(t) - Av_2^\epsilon(t) = 0, \quad 0 \leq t \leq T,$$

$$(5.2.b) \quad \frac{d}{dt} \{\epsilon Av_1^\epsilon(t) + (C + \epsilon(\lambda C + B))v_2^\epsilon(t)\} + Av_1^\epsilon(t) + (\lambda C + B)v_2^\epsilon(t) = 0$$

with $Av_1^\epsilon(T) = e^{-\lambda T}F_1$ and $Cv_2^\epsilon(T) = e^{-\lambda T}F_2$. Let $u_1^\epsilon, u_2^\epsilon$ be any solution of (5.1) with $Au_1^\epsilon(0) = Av_1^\epsilon(0)$ and $Cu_2^\epsilon(0) = Cv_2^\epsilon(0)$. Then

$$\|u_1^\epsilon(T)\|_a^2 + \|u_2^\epsilon(T)\|_c^2 \leq \|F_1\|_{U'_a}^2 + \|F_2\|_{U'_c}^2, \quad \epsilon > 0,$$

$\lim_{\epsilon \rightarrow 0} Au_1^\epsilon(T) = F_1$ in U'_a , and $\lim_{\epsilon \rightarrow 0} Cu_2^\epsilon(T) = F_2$ in U'_c . There exists a solution u_1, u_2 of (5.1) satisfying $Au_1(T) = F_1$ and $Cu_2(T) = F_2$ if and only if $\lim_{\epsilon \rightarrow 0} Av_1^\epsilon(0)$ exists in U'_a and $\lim_{\epsilon \rightarrow 0} Cv_2^\epsilon(0)$ exists in U'_c . If w_1, w_2 is a second solution of the final-value

problem of (5.1) with $Aw_1(T) = F_1$ and $Cw_2(T) = F_2$, then $u_1(t) - w_1(t) \in \text{Ker}(A)$ and $u_2(t) - w_2(t) \in \text{Ker}(A) \cap \text{Ker}(B) \cap \text{Ker}(C)$ for $0 < t \leq T$.

Proof. The proof of Theorem 3 shows that the desired results follow from the conclusions of Theorem 2 with λ_0 replaced by λ ; (5.2) is the reversible approximation which corresponds to (3.5). Thus, it suffices to show that the hypotheses of Theorem 2 are true for some λ sufficiently large. Specifically, we need only to demonstrate that $\lambda M + L$ is $(\pi/4)$ -sectorial for some λ sufficiently large. From (4.4) we obtain for each $\delta > 0$ and $\lambda \geq \lambda_0$

$$|\text{Im}(\lambda M + L)x(x)| \leq \max\{(\lambda\delta)^{-1}, \delta K + \tan \theta\} \text{Re}(\lambda M + L)x(x), \quad x \in V.$$

Note that K and θ are determined with λ_0 and all hypotheses hold for $\lambda \geq \lambda_0$ and the same K and θ . Since $\tan \theta < 1$, we can choose $\delta > 0$ so that $\delta K + \tan \theta \leq 1$. Then choose λ so large that $(\lambda\delta)^{-1} \leq 1$. The preceding estimate then shows $\lambda M + L$ is θ_λ -sectorial for some $\theta_\lambda \leq \pi/4$, so the proof is complete.

One can also write down estimates on the rate of convergence in the limits above; these follow directly from the analogous estimates (3.6) and (3.7). Also, we note that $\{Av_1^\epsilon(0)\}$ and $\{Cv_2^\epsilon(0)\}$ converge in U'_a and U'_c if and only if $\{v_1^\epsilon(0)\}$ and $\{v_2^\epsilon(0)\}$ are Cauchy in U_a and U_c , respectively.

We observed in Section 4 that the second component u_2 of a solution of (5.1) can be characterized as a solution of the second-order equation

$$(5.3) \quad \frac{d}{dt} \left\{ \frac{d}{dt} Cu(t) + Bu(t) \right\} + Au(t) = 0, \quad 0 < t < T.$$

One naturally inquires whether the second component of the solution of (5.2) is characterized likewise. The correspondence is inexact in the rather general case considered here. One can show that the system (5.2) is in some sense equivalent to the system

$$\begin{aligned} \frac{d}{dt} [C + \epsilon(B + 2\lambda C) + \epsilon^2(\lambda^2 C + \lambda B + A)]v_2^\epsilon + Av_1^\epsilon \\ + [(B + \lambda C) + \epsilon(\lambda^2 C + \lambda B + A)]v_2^\epsilon = 0, \end{aligned}$$

$$\frac{d}{dt} [-Av_1^\epsilon + (\lambda C + \epsilon(\lambda^2 C + \lambda B + A))v_2^\epsilon] + (\lambda^2 C + \lambda B + A)v_2^\epsilon = 0.$$

The second component is then a generalized solution of the equation

$$(5.4) \quad \frac{d}{dt} \left\{ \frac{d}{dt} [C + \epsilon(B + 2\lambda C) + \epsilon^2(\lambda^2 C + \lambda B + A)]v_2^\epsilon(t) \right. \\ \left. + [(B + 2\lambda C) + 2\epsilon(\lambda^2 C + \lambda B + A)]v_2^\epsilon(t) \right\} + (\lambda^2 C + \lambda B + A)v_2^\epsilon(t) = 0.$$

Thus, (5.4) may be viewed as the reversible approximation of (5.3). Although (5.4) is more complex than the approximation used in [5], it applies to a much

more general situation and the Theorem 4 is considerably stronger than its analogue in [5].

REFERENCES

1. R. W. CARROLL & R. E. SHOWALTER, *Singular and Degenerate Cauchy Problems*, Academic Press, New York, 1976.
2. R. E. EWING, *The approximation of certain parabolic equations backward in time by Sobolev equations*, SIAM J. Math. Anal. **6** (1975), 283–294.
3. T. KATO, *Perturbation theory for linear operators*, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, **132**, Springer-Verlag, Berlin, 1966.
4. JOHN LAGNESE, *The final value problem for Sobolev equations*, Proc. Amer. Math. Soc. **56** (1976), 247–252.
5. R. LATTES & J.-L. LIONS, *The Method of Quasi-Reversibility*, American Elsevier, New York, 1969.
6. L. E. PAYNE, *Improperly Posed Problems in Partial Differential Equations*, CBMS-NSF Regional Conference Series in Appl. Math. **22**, Soc. Ind. Appl. Math., Philadelphia, 1975.
7. R. E. SHOWALTER, *The final-value problem for evolution equations*, J. Math. Anal. Appl. **47** (1974), 563–572.
8. ———, *Quasi-reversibility of first and second order evolution equations*, in A. CARUSO & A. STONE, ed., *Improperly Posed Boundary Value Problems*, Pitman Publishing, London, 1975.

This research was supported by an NSF grant.

Received September 7, 1978

UNIVERSITY OF TEXAS, AUSTIN, TX 78712