Semilinear Parabolic Equations With Preisach Hysteresis

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Abstract

A coupled system consisting of a semilinear parabolic partial differential equation and a family of ordinary differential equations which is capable of modeling a very general class of hysteresis effects will be realized as an abstract Cauchy problem. Accretiveness estimates and maximality conditions are established in a product of L^1 spaces for the closure of the operator associated with this problem. Thus, the Cauchy problem corresponding to the closed operator admits a unique integral solution by way of the Crandall-Liggett theory. Special cases of the system include a one-dimensional derivation from Maxwell's equations for a ferromagnetic body under slowly varying field conditions, the Super-Stefan problem, and other partial differential equations with hysteresis terms appearing in the literature.

Key Words: porous medium equation, Super-Stefan problem, phase change, free boundary problem, hysteresis, memory, Preisach model.

1 Introduction

We shall consider here the well-posedness of the initial-boundary-value problem for a semilinear (possibly) degenerate parabolic partial differential equation with a hysteresis nonlinearity in the energy. This will include evolution equations of the form of a generalized porous medium equation

$$\text{(1)} \qquad \frac{\partial}{\partial t}\left(a(u) + \mathcal{H}(u)\right) - \Delta u = f$$

in which $a(\cdot)$ is a continuous monotone function and $\mathcal{H}$ is a hysteresis functional, that is, the output $\mathcal{H}(u)$ depends not only on the current value of the input u, but also on the history of the input.

As an elementary but generic example of hysteresis, we mention a functional that arises in the description of the Super-Stefan problem [14]. This functional provides an example of a simple but basic form of hysteresis. The example depends on three parameters, α, β, and ϵ, with $0 < \epsilon$, $\alpha < \beta$.

Denote by $[x]_+$ and $[x]_-$, respectively, the positive and negative parts of the real number x. The output $w(t) = \mathcal{H}(u(t))$ varies for $t > 0$ according to the following:

$$\text{if } u > \beta + \epsilon, \text{ then } w = 1;$$
$$\text{if } u < \alpha - \epsilon, \text{ then } w = -1;$$
$$\text{if } \alpha - \epsilon < u < \beta + \epsilon, \text{ then } |w| \leq 1 \text{ and}$$

$$w'(t) = \begin{cases} \left[\frac{u'(t)}{\epsilon}\right]_+ & \text{if } w = \frac{u-\beta}{\epsilon}, \\ 0 & \text{if } \frac{u-\beta}{\epsilon} < w < \frac{u-\alpha}{\epsilon}, \\ \left[\frac{u'(t)}{\epsilon}\right]_- & \text{if } w = \frac{u-\alpha}{\epsilon}. \end{cases}$$

Thus, for example, suppose that $u(0) = 0$ and $w(0) = 0$. As long as $u(t)$ remains between α and β, $w(t) = 0$. If $u(t)$ increases to β and then beyond $\beta + \epsilon$, then $w(t)$ increases to $+1$ where it remains until $u(t)$ gets down to $\alpha + \epsilon$. If $u(t)$ decreases below $\alpha - \epsilon$, then $w(t)$ will drop to -1 and remain there until $u(t)$ again reaches $\beta - \epsilon$, and so on. Such a function arises naturally in the description of the Super-Stefan problem [14] in which $\frac{w(t)+1}{2}$ represents the fraction of melt (water) in the ice/water in terms of the temperature $u(t)$; α is the freezing temperature and β is the melting temperature. The limiting case obtained from $\epsilon \to 0$ is the 'relay' hysteresis functional that is basic to the Preisach representation of a very general class of hysteresis functionals. This class will be included in our theory below. When $\alpha = \beta$ this reduces to the classical Stefan free boundary problem whose weak formulation is of the form (1) but with $\mathcal{H}$ replaced by the Heaviside function (or graph) $H(x) = \frac{1}{2}(1 + \text{sgn}(x))$, $\text{sgn}(x) \equiv 1$ if $x > 0$, $\text{sgn}(x) \equiv -1$ if $x < 0$, and $\text{sgn}(0) = [-1, 1]$. Also, our system can be used to produce a Krasnosel'skii-Pokrovskii hysteron.

Hysteresis is a very important and general concept, and one should consult the recent survey [15] for a concise description of recent results on the development and application of mathematical models of hysteresis. For an excellent well-motivated introduction to this topic, see the monograph [16]. Due to the complex description of the operators traditionally used to represent hysteresis [12], their addition to systems of differential equations leads to substantial technical problems for the development of a good theory.

We shall show here that it is possible to include an extensive class of hysteresis functionals in a fashion that is strikingly compatible with standard methods for differential equations. The idea is to obtain the hysteresis output as the weighted average of the solutions of a system of ordinary differential equations subject to constraints. The use of ordinary differential equations with constraints to represent hysteresis is certainly not a new idea; specifically, the special form appearing below in (3) is based on the construction in the earlier work [11] in which hysteresis

occurs on the boundary. By choosing each of the component equations of the system to simulate a simple relay (such as the example above) from a family parameterized by their switching values, α and β, we are able to include the very general Preisach hysteresis models. The system consisting of the semilinear parabolic equation coupled to this hysteresis model is then resolved as an application of the theory of evolution equations in general Banach space. A related construction was recently announced in [24] with corresponding results to appear; the distinction there is that the coupling function is specified as a function of the two variables u, v instead of the function of the difference of the variables.

The model problem (1) of a parabolic diffusion equation coupled to hysteresis has been studied extensively by A. Visintin. In the case of a linear equation he has proved existence by backward-difference discretization and compactness methods applied to monotone steps that describe the discretization. In [21] is covered the (regularized) case without jumps in the defining functions of the hysteresis model (such as the example above with $\epsilon > 0$), and this is extended in [22] to the case with jumps ($\epsilon = 0$ above). The uniformly parabolic (nondegenerate) semilinear case with a general Preisach hysteresis is covered in [23]. The uniqueness of a solution was established in [9].

We shall use freely the methods of convex analysis; see [1], [2], [6], or [19], for example. Specifically sgn^{-1} is the maximal monotone graph obtained as the subgradient (see Definition 1 in Section 2) of the convex indicator function (see below) of the interval $[-1, 1]$ in R.

Let us describe the system to be considered here. Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$, and let ν denote the unit outward normal. Let $j, k, q : R \to (-\infty, +\infty]$ be convex lower-semi-continuous functions for which j and q are quadratically upper-bounded, k is quadratically lower-bounded, and whose subgradients satisfy $0 \in \partial k(0)$, $0 \in \partial j(0)$, and ∂k, ∂q are single-valued. We shall denote these continuous monotone functions by $a(\cdot) = \partial k(\cdot)$ and $b(\cdot) = \partial q(\cdot)$. Let $m > 0$, $S_m = \{s = (\alpha, \beta) \in R^2 : -m \leq \alpha < \beta \leq m\}$, and $(S_m, \mathcal{B}, \mu)$ a finite measure space that contains the Borel measurable subsets of S_m. Let $\text{sgn}_s^{-1} \equiv \partial\zeta_s$, $s = (\alpha, \beta) \in S_m$, where $\zeta_s : R \to \{0, +\infty\}$ is the indicator function given by $+\infty$ on $R \setminus [\alpha, \beta]$ and 0 on $[\alpha, \beta]$. We consider the degenerate parabolic system of coupled equations with Neumann type boundary conditions

$$(2) \qquad \frac{\partial}{\partial t} a(u(x,t)) + \frac{\partial}{\partial t} \int_{S_m} b(v(x,s,t)) \, d\mu(s) - \Delta u(x,t) \;=\; f(x,t) \quad x \in \Omega, \; t \in (0,T],$$

$$(3) \qquad \frac{\partial}{\partial t} b(v(x,s,t)) + \text{sgn}_s^{-1}(v(x,s,t) - u(x,t)) \;\ni\; 0 \quad s \in S_m,$$

$$(4) \qquad -\frac{\partial}{\partial \nu} u(\tau,t) \;\in\; \partial j(u(\tau,t)) \quad \tau \in \partial\Omega,$$

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for which the initial conditions $a(u(x, 0))$ and $b(v(x, s, 0))$ are specified.

In order to see how the constrained ordinary differential equation (3) produces the desired hysteresis, we construct a hysteresis model as follows. Let a maximal monotone graph $b(\cdot)$ be given; our hysteresis model will be of the type *generalized play* described by horizontal translates of $w \in b(u)$. The simple functional described above is given by the choice $b = \sigma_\epsilon$, where

$$\sigma_\epsilon(r) = \begin{cases} 1 & \text{if } r \geq \epsilon \\ \frac{r}{\epsilon} & \text{if } -\epsilon < r < \epsilon \\ -1 & \text{if } r \leq -\epsilon. \end{cases}$$

Thus, we introduce a new variable, v, in order to represent the phase constraints:

$$w \in b(v), \ u - 1 \leq v \leq u + 1.$$

Finally, we use the sgn^{-1} graph to realize these constraints. Let $u(t)$ be a time-dependent input to this generalized play model, and let $w(t)$ be the corresponding output or response. There is at each time a corresponding phase variable $v(t)$ which is related to $w(t)$ and $u(t)$ as above, and so it is required that $w(t)$ be non-decreasing when $v(t) = u(t) - 1$, non-increasing when $v(t) = u(t) + 1$, and stationary $(w'(t) = 0)$ in the interior region, $u - 1 < v < u + 1$. This is equivalent to requiring that $w(t), v(t)$ satisfy

$$w(t) \in b(v(t)), \quad w' + \text{sgn}^{-1}(v(t) - u(t)) \ni 0.$$

Thus, we are led to ordinary differential equations of the form

$$w(t) \in b(v(t)), \quad w'(t) + c(v(t) - u(t)) \ni 0$$

with maximal monotone graphs $b(\cdot)$ and $c(\cdot)$ as models of hysteresis in which the output is the solution $w(t)$ with input $u(t)$.

Although we are restricted here to (single-valued) functions $b(\cdot)$, the Heaviside graph $b = H$ as well as the other general examples are obtained through the Preisach representation implicit in the integral in (2). The use of the hysteresis loops produced by σ_ϵ as a substitute in the construction of the Preisach model is discussed in [16, p.31]. In the case $a(r) = r$ and $b = \sigma_\epsilon$, the partial differential equation (2) corresponds to a one-dimensional derivation from Maxwell's equations for a ferromagnetic body under slowly varying field conditions [13,23]. Also in this case, if we allow μ to be a Dirac mass, then (2) corresponds to the Super-Stefan problem [14,22]. If we choose $b(r) = r$, then (3) produces a Krasnosel'skii-Pokrovskii hysteron for each $s \in S_m$ [12,15].

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Systems of the general form of (2)-(4) appear in many other contexts in which (3) is frequently a local storage or capacity in immobile (nondiffusive) sites. A similar quasilinear system in which all three of $a(\cdot)$, $b(\cdot)$, and $c(\cdot)$ are monotone functions with power growth rates was developed in [20]. The device used there to prove regularity of solutions, i.e., to show the difference scheme also converges in a dual Sobolev space, could be used to get properties of solutions here. Also the technique of approximating the generalized solutions by smooth solutions of a corresponding problem with regularized functions is applicable here. We also remark that a maximum principle is immediate from the estimates in Section 4 for the system (2)-(4). The maximum principle together with an obvious choice of k gives $\frac{\partial}{\partial t} a(u(x,t)) \equiv 0$, and in this case (2) takes the form of (P1) in [21]. See [18] for additional related systems. For control of Stefan problems by hysteresis functionals, see [8,10].

Our objective is to show that the dynamics of problem (2)-(4) is determined by a nonlinear semigroup of contractions on the Banach space $L^1(\Omega) \times L^1(\Omega \times S_m)$. The negative of the generator of this contraction semigroup is the closure $\mathbb{C}$ in $\left[L^1(\Omega) \times L^1(\Omega \times S_m)\right]^2$ of an operator $C \subset \left[L^2(\Omega) \times L^2(\Omega \times S_m)\right]^2$ for which the resolvent equation $(I + \eta C)(a(u), b(v)) \ni (f, 0)$, with $\eta > 0$, $u \in H^2(\Omega)$, $f, a(u) \in L^2(\Omega)$, and $v, b(v) \in L^2(\Omega \times S_m)$, takes the form

$$
\begin{aligned}
a(u) + \int_{S_m} b(v) - \eta \Delta u &\ni f \quad \text{in } \Omega, \\
b(v) + \eta \operatorname{sgn}_s^{-1}(v - u) &\ni 0 \quad \text{in } \Omega \times S_m, \\
-\frac{\partial u}{\partial \nu} &\in \partial j(u) \quad \text{on } \partial \Omega.
\end{aligned}
$$

This will follow from our construction in Section 4 of the operator $\mathbb{C}$ and the verification that it is an *m-accretive* operator on this Banach space. A (possibly multi-valued) operator or relation $\mathbb{C}$ in a Banach space X is a collection of related pairs $[x, y] \in X \times X$ denoted by $y \in \mathbb{C}(x)$; the domain $D(\mathbb{C})$ is the set of all such x and the range $Rg(\mathbb{C})$ consists of all such y. The operator $\mathbb{C}$ is called *accretive* if for all $y_1 \in \mathbb{C}(x_1)$, $y_2 \in \mathbb{C}(x_2)$ and $\varepsilon > 0$

$$
\|x_1 - x_2\| \leq \|x_1 - x_2 + \varepsilon(y_1 - y_2)\| .
$$

This is equivalent to requiring that $(I + \varepsilon\mathbb{C})^{-1}$ is a contraction on $Rg(I + \varepsilon\mathbb{C})$ for every $\varepsilon > 0$. If, in addition, $Rg(I + \varepsilon\mathbb{C}) = X$ for some (equivalently, for all) $\varepsilon > 0$, then $\mathbb{C}$ is called *m-accretive*. In Section 5 we shall recall the nonlinear semigroup theory and describe its application to (2)-(4).

2   The Resolvent Equation

Let λ denote Lebesgue measure on R^n. Let $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ be the trace map [17]. Any pointwise statement made in regard to an element of a function space is assumed to be made in terms of a finite-valued representative of the equivalence class associated with that element.

Let $j : R \to (-\infty, +\infty]$ be proper ($j \not\equiv +\infty$), convex, and lower-semi-continuous. Define $\Phi_1 : L^2(\Omega) \times L^2(\Omega \times S_m) \to (-\infty, +\infty]$ by

$$\Phi_1(u, v) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} j(\gamma u) & \text{if } u \in H^1(\Omega) \text{ and } j(\gamma u) \in L^1(\partial\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Note that Φ_1 does not depend on the second component v, however, defining Φ_1 as above will allow for simpler notation in some of the results to follow.

Remark 1 *The function Φ_1 is proper, convex, and lower-semi-continuous* [1].

The essential elements of the proof of the following lemma are contained in [5].

Lemma 1 *Let $r : R \to R$ be Lebesgue measurable and bounded, $\sigma(x) = \int_0^x r(t)\, dt$, and $u \in H^1(\Omega)$. Then*

$$\frac{d}{dx_i}\sigma(u) = r(u)\frac{du}{dx_i} \ \ a.e. \ \ for \ i = 1, 2, ..., n\,, \ and$$
$$\sigma(u) \in H^1(\Omega).$$

We prove an additional elementary lemma.

Lemma 2 *Let $\sigma : R \to R$ be a Lipschitz function. For every $u \in H^1(\Omega)$, we have $\gamma\sigma(u) = \sigma(\gamma u)$.*

Proof: Fix $u \in H^1(\Omega)$. Choose a sequence $\{u_n\}$ in $C^\infty(\overline{\Omega})$ such that $u_n \to u$ in $H^1(\Omega)$. We clearly have $\sup_n \|\sigma(u_n)\|_{L^2(\Omega)} < +\infty$. Note that Lemma (1) implies $\sup_n \||\nabla\sigma(u_n)|\|_{L^2(\Omega)} < +\infty$. Hence, there exists a $w \in H^1(\Omega)$ and a subsequence $\{\sigma(u_n)\}$ (after a change of notation) such that $u_n \rightharpoonup w$ in $H^1(\Omega)$. Hence, $\sigma(u_n) \rightharpoonup w$ in $L^2(\Omega)$. We also have $\sigma(u_n) \rightharpoonup \sigma(u)$ in $L^2(\Omega)$ and so $w = \sigma(u)$. The compactness of the trace operator implies $\gamma\sigma(u_n) \to \gamma\sigma(u)$ in $L^2(\partial\Omega)$. After a change of notation we have $\gamma\sigma(u_n)(\tau) \to \gamma\sigma(u)(\tau)$ at almost every $\tau \in \partial\Omega$. We also have (after a change of notation) that $\gamma u_n(\tau) \to \gamma u(\tau)$ at almost every $\tau \in \partial\Omega$, and therefore $\sigma(\gamma u_n(\tau)) \to \sigma(\gamma u(\tau))$ at almost every $\tau \in \partial\Omega$. After noting that $\gamma\sigma(u_n)(\tau) = \sigma(\gamma u_n(\tau))$ for all n and all $\tau \in \partial\Omega$ [17], we have $\gamma\sigma(u)(\tau) = \sigma(\gamma u(\tau))$ at almost every $\tau \in \partial\Omega$. $\square$

Definition 1 *If H is a real Hilbert space and $\Psi : H \to (-\infty, +\infty]$ is proper, convex, and lower-semi-continuous, then the symbol ∂ applied to Ψ produces a relation in $H \times H$ defined as follows: For $u \in \text{Dom}(\Psi) \equiv \{u \in H : \Psi(u) < +\infty\}$, define $\partial\Psi(u) \equiv \{w \in H : \langle w, v - u \rangle_H \leq \Psi(v) - \Psi(u) \text{ for all } v \in H\}$. If $w \in \partial\Psi(u)$, we say w is a subgradient of Ψ at u.*

Remark 2 *If Ψ is as in Definition (1), then $\partial\Psi$ is an m-accretive operator in H [1,2].*

Let $k : R \to (-\infty, +\infty]$ be proper, convex, and lower-semi-continuous. Define $\Phi_2 : L^2(\Omega) \times L^2(\Omega \times S_m) \to (-\infty, +\infty]$ by

$$\Phi_2(u,v) = \begin{cases} \displaystyle\int_\Omega k(u) & \text{if } k(u) \in L^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Remark 3 *The function Φ_2 is proper, convex, and lower-semi-continuous [1].*

Proposition 1 *Assume $0 \in \partial j(0)$ and $0 \in \partial k(0)$. Then $\partial\Phi_1 + \partial\Phi_2$ is m-accretive.*

Proof: A sufficient condition for $\partial\Phi_1 + \partial\Phi_2$ to be m-accretive is that for all $(u,v) \in \text{Dom}\Phi_1$ and all $\eta > 0$ we have

$$\Phi_1\left((I + \eta\partial\Phi_2)^{-1}(u,v)\right) \leq \Phi_1(u,v) \quad [2].$$

Fix $(u,v) \in \text{Dom}\Phi_1$ and $\eta > 0$. Note that $(f_1, f_2) = (I + \eta\partial\Phi_2)^{-1}(u,v)$ implies $f_1(x) = (I + \eta\partial k)^{-1}(u(x))$ at almost every $x \in \Omega$ [1,19]. Using $f_1 = (I + \eta\partial k)^{-1}(u)$, the fact that $(I + \eta\partial k)^{-1} : R \to R$ is a (monotone) contraction [2], Lemma (1), and Lemma (2), we obtain

$$
\begin{aligned}
& \Phi_1\left((I + \eta\partial\Phi_2)^{-1}(u,v)\right) \\
(5) \qquad & = \tfrac{1}{2}\int_\Omega |\nabla(I + \eta\partial k)^{-1}(u)|^2 + \int_{\partial\Omega} j\left((I + \eta\partial k)^{-1}(\gamma u)\right).
\end{aligned}
$$

Note that $j\left((I + \eta\partial k)^{-1}(t)\right) \leq j(t)$, $t \in R$, and hence for every $w \in H^1(\Omega)$ we have

$$(6) \qquad j\left((I + \eta\partial k)^{-1}(\gamma w(\tau))\right) \leq j(\gamma w(\tau)), \quad \tau \in \partial\Omega.$$

Since $(I + \eta\partial k)^{-1} : R \to R$ is a contraction, Lemma (1) implies

$$(7) \qquad \int_\Omega |\nabla(I + \eta\partial k)^{-1}(w)|^2 \leq \int_\Omega |\nabla w|^2, \quad w \in H^1(\Omega).$$

Using (5), (6), and (7) we obtain

$$\Phi_1\left((I + \eta\partial\Phi_2)^{-1}(u,v)\right) \leq \Phi_1(u,v).$$

$\square$

Remark 4 *The function $\Phi_1 + \Phi_2$ is convex and lower-semi-continuous. The function $\Phi_1 + \Phi_2$ is proper provided $\text{Dom}(j) \cap \text{Dom}(k) \neq \emptyset$. Hence, under the hypothesis of Proposition (1) we have $\partial(\Phi_1 + \Phi_2) = \partial\Phi_1 + \partial\Phi_2$ [2].*

For each $s = (\alpha, \beta) \in S_m$, define $\zeta_s : R \to \{0, +\infty\}$ by

$$\zeta_s(t) = \begin{cases} 0 & \text{if } \alpha \leq t \leq \beta \\ +\infty & \text{otherwise .} \end{cases}$$

Let $\text{sgn}_s^{-1} \equiv \partial\zeta_s$.

After noting that $(x, s) \mapsto \zeta_s(v(x, s) - u(x))$ is $\lambda \times \mu$-measurable, $(u, v) \in L^2(\Omega) \times L^2(\Omega \times S_m)$, we define $\Phi_3 : L^2(\Omega) \times L^2(\Omega \times S_m) \to \{0, +\infty\}$ by

$$\Phi_3(u, v) = \int_{\Omega \times S_m} \zeta_s(v(x, s) - u(x)) \, d(\lambda \times \mu)(x, s).$$

Remark 5 *Note that $\text{Dom}(\Phi_3) = \{(u, v) \in L^2(\Omega) \times L^2(\Omega \times S_m) : \Phi_3(u, v) = 0\}$.*

Lemma 3 *The function Φ_3 is proper, convex, and lower-semi-continuous.*

Proof: The convexity of Φ_3 is clear. To see that Φ_3 is proper, let $v(x, s) = \alpha$ and $u = 0$ to get $\Phi_3(u, v) = 0$. We need to show $A = \{(u, v) \in L^2(\Omega) \times L^2(\Omega \times S_m) : \Phi_3(u, v) = 0\}$ is closed. Assume $\{(u_n, v_n)\} \subset A$ and $(u_n, v_n) \to (u, v)$ in $L^2(\Omega) \times L^2(\Omega \times S_m)$. After a change of notation we may assume $v_n(x, s) - u_n(x) \to v(x, s) - u(x)$ at almost every $(x, s) \in \Omega \times S_m$. Therefore,

$$\begin{aligned} 0 &= \liminf \int_{\Omega \times S_m} \zeta_s(v_n(x, s) - u_n(x)) \, d(\lambda \times \mu)(x, s) \\ &\geq \int_{\Omega \times S_m} \liminf \zeta_s(v_n(x, s) - u_n(x)) \, d(\lambda \times \mu)(x, s) \\ &\geq \int_{\Omega \times S_m} \zeta_s(v(x, s) - u(x)) \, d(\lambda \times \mu)(x, s) \\ &\geq 0. \end{aligned}$$

Hence, $\Phi_3(u, v) = 0$. □

Lemma 4 *We have $(f_1, f_2) \in \partial\Phi_3(u, v)$ iff $u \in L^2(\Omega), v \in L^2(\Omega \times S_m), f_2 \in L^2(\Omega \times S_m), f_1 = -\int_{S_m} f_2 \, d\mu$, and $f_2(x, s) \in \partial\zeta_s(v(x, s) - u(x))$ at almost every $(x, s) \in \Omega \times S_m$.*

Proof: Fix $(f_1, f_2) \in \partial \Phi_3(u, v)$. For every $(g_1, g_2) \in L^2(\Omega) \times L^2(\Omega \times S_m)$ we have

(8)
$$\int_\Omega f_1(g_1 - u) \, d\lambda + \int_{\Omega \times S_m} f_2(g_2 - v) \, d(\lambda \times \mu) \leq \Phi_3(g_1, g_2).$$

Let $g_1 = u + h$ and $g_2 = v + h$, with $h \in L^2(\Omega)$, in (8) to get $\int_\Omega h f_1 \, d\lambda + \int_{\Omega \times S_m} h f_2 \, d(\lambda \times \mu) \leq 0$, i.e. $\int_\Omega h \left(f_1 + \int_{S_m} f_2 \, d\mu \right) d\lambda \leq 0$. Hence, $f_1 = -\int_{S_m} f_2 \, d\mu$. Let $f_1 = -\int_{S_m} f_2 \, d\mu$ in (8) so that for every $(g_1, g_2) \in L^2(\Omega) \times L^2(\Omega \times S_m)$ we have

(9)
$$\int_{\Omega \times S_m} f_2(u - g_1 + g_2 - v) \, d(\lambda \times \mu) \leq \Phi_3(g_1, g_2).$$

Let $g_1 = 0$ in (9) so that for every $g_2 \in L^2(\Omega \times S_m)$ we have

(10)
$$\int_{\Omega \times S_m} f_2(x, s) \left(g_2(x, s) - (v(x, s) - u(x)) \right) d(\lambda \times \mu)(x, s)$$
$$\leq \int_{\Omega \times S_m} \zeta_s(g_2(x, s)) \, d(\lambda \times \mu)(x, s).$$

Let $\{t_j\}$ be an enumeration of the rational numbers. Define

$$N_{t_j} = \{(x, s) \in \Omega \times S_m : \alpha \leq t_j \leq \beta \text{ and } f_2(x, s)(t_j - (v(x, s) - u(x))) > 0\}.$$

If $(\lambda \times \mu)(N_{t_j}) > 0$, then letting

$$g_2(x, s) = \begin{cases} t_j & \text{if } (x, s) \in N_{t_j} \\ v(x, s) - u(x) & \text{otherwise} \end{cases}$$

in (10) gives

$$0 < \int_{N_{t_j}} f_2(x, s)(t_j - (v(x, s) - u(x))) \, d(\lambda \times \mu)(x, s) \leq 0.$$

Hence, $(\lambda \times \mu)(N_{t_j}) = 0$ for each j. Let $N_1 = \bigcup_{j=1}^{\infty} N_{t_j}$, $N_2 = \{(x, s) \in \Omega \times S_m : \zeta_s(v(x, s) - u(x)) = +\infty\}$, and $N = N_1 \cup N_2$. Note that $(\lambda \times \mu)(N) = 0$. Fix $(x, s) \in \Omega \times S_m \setminus N$. If $t \in R \setminus [\alpha, \beta]$, then

$$f_2(x, s)(t - (v(x, s) - u(x))) < +\infty = \zeta_s(t) = \zeta_s(t) - \zeta_s(v(x, s) - u(x)).$$

If $t \in [\alpha, \beta]$, then choose $\{t_{j_k}\} \subset [\alpha, \beta]$ so that $t_{j_k} \to t$. For each k we have $f_2(x, s)(t_{j_k} - (v(x, s) - u(x))) \leq 0$. Hence, $f_2(x, s)(t - (v(x, s) - u(x))) \leq 0$. Hence,

$$f_2(x, s)(t - (v(x, s) - u(x))) \leq 0 = \zeta_s(t) - \zeta_s(v(x, s) - u(x)).$$

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We have shown

(11) $f_2(x, s) \in \partial \zeta_s(v(x, s) - u(x))$ at almost every $(x, s) \in \Omega \times S_m$.

For the converse we have (11) $\Rightarrow$ (10) $\Rightarrow$ (9) $\Rightarrow$ (8), and therefore $(f_1, f_2) \in \partial \Phi_3(u, v)$. $\square$

Proposition 2 *Assume $0 \in \partial j(0)$ and $0 \in \partial k(0)$. Then $\partial \Phi_1 + \partial \Phi_2 + \partial \Phi_3$ is m-accretive.*

Proof: Note that if the measure $\mu(\cdot)$ is replaced by the measure $\frac{1}{\mu(S_m)}\mu(\cdot)$ in the definition of Φ_3, then Φ_3 remains unchanged, and therefore $\partial \Phi_3$ remains unchanged. For the remainder of this proof we assume, without loss of generality, that $\mu(S_m) = 1$. Fix $(f_1, f_2) \in L^2(\Omega) \times L^2(\Omega \times S_m)$. Let $u \in L^2(\Omega)$. We can use Jensen's inequality, the fact that $(I + \partial \zeta_s)^{-1}(\alpha) = \alpha$, and the fact that $(I + \partial \zeta_s)^{-1} : R \to R$ is a contraction, to obtain

$$\int_\Omega \left(\int_{S_m} \left(f_2(x, s) - (I + \partial \zeta_s)^{-1}(f_2(x, s) - u(x)) \right) d\mu(s) \right)^2 d\lambda(x)$$

$$\leq \int_\Omega \int_{S_m} \left(f_2(x, s) - (I + \partial \zeta_s)^{-1}(f_2(x, s) - u(x)) \right)^2 d\mu(s) \, d\lambda(x)$$

$$= \int_\Omega \int_{S_m} \left(f_2(x, s) + (I + \partial \zeta_s)^{-1}(\alpha) - (I + \partial \zeta_s)^{-1}(f_2(x, s) - u(x)) - \alpha \right)^2 d\mu(s) \, d\lambda(x)$$

$$\leq \int_\Omega \int_{S_m} (2|f_2(x, s)| + |u(x)| + 2|\alpha|)^2 \, d\mu(s) \, d\lambda(x) < +\infty.$$

Therefore, $\int_{S_m} \left(f_2 - (I + \partial \zeta_s)^{-1}(f_2 - u) \right) d\mu \in L^2(\Omega)$. Define $T : L^2(\Omega) \times L^2(\Omega \times S_m) \to L^2(\Omega) \times L^2(\Omega \times S_m)$ by

$$T(u, v) = \left(I + \tfrac{1}{2}\partial \Phi_1 + \tfrac{1}{2}\partial \Phi_2 \right)^{-1} \left(\tfrac{1}{2}\left(f_1 + \int_{S_m} (f_2 - (I + \partial \zeta_s)^{-1}(f_2 - u)) \, d\mu \right), \tfrac{1}{2}v \right).$$

We can use the fact that $\left(I + \tfrac{1}{2}\partial \Phi_1 + \tfrac{1}{2}\partial \Phi_2 \right)^{-1} : L^2(\Omega) \times L^2(\Omega \times S_m) \to L^2(\Omega) \times L^2(\Omega \times S_m)$ is a contraction, Jensen's inequality, and the fact that $(I + \partial \zeta_s)^{-1} : R \to R$ is a contraction, to obtain

$$\|T(u_1, v_1) - T(u_2, v_2)\|_{L^2(\Omega) \times L^2(\Omega \times S_m)}$$

$$\leq \tfrac{1}{2} \left\| \int_{S_m} (I + \partial \zeta_s)^{-1}(f_2 - u_1) \, d\mu - \int_{S_m} (I + \partial \zeta_s)^{-1}(f_2 - u_2) \, d\mu \right\|_{L^2(\Omega)} + \tfrac{1}{2}\|v_1 - v_2\|_{L^2(\Omega \times S_m)}$$

$$= \tfrac{1}{2} \left(\int_\Omega \left(\int_{S_m} ((I + \partial \zeta_s)^{-1}(f_2 - u_1) - (I + \partial \zeta_s)^{-1}(f_2 - u_2)) \, d\mu \right)^2 d\lambda \right)^{\frac{1}{2}} + \tfrac{1}{2}\|v_1 - v_2\|_{L^2(\Omega \times S_m)}$$

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$$\leq \tfrac{1}{2} \left(\int_\Omega \int_{S_m} |(I + \partial\zeta_s)^{-1}(f_2 - u_1) - (I + \partial\zeta_s)^{-1}(f_2 - u_2)|^2 \, d\mu \, d\lambda \right)^{\frac{1}{2}} + \tfrac{1}{2}\|v_1 - v_2\|_{L^2(\Omega \times S_m)}$$

$$\leq \tfrac{1}{2} \left(\int_\Omega \int_{S_m} |u_1 - u_2|^2 \, d\mu \, d\lambda \right)^{\frac{1}{2}} + \tfrac{1}{2}\|v_1 - v_2\|_{L^2(\Omega \times S_m)}$$

$$= \tfrac{1}{2} \left(\int_\Omega |u_1 - u_2|^2 \, d\lambda \right)^{\frac{1}{2}} + \tfrac{1}{2}\|v_1 - v_2\|_{L^2(\Omega \times S_m)}$$

$$= \tfrac{1}{2}\|(u_1, v_1) - (u_2, v_2)\|_{L^2(\Omega) \times L^2(\Omega \times S_m)}.$$

Hence, T is a strict contraction. Let (u_0, v_0) be the fixed point of T. Note that $T(u_0, v_0) = (u_0, v_0)$ implies

$$(12) \quad \left(f_1 + \int_{S_m} \left(f_2 - (I + \partial\zeta_s)^{-1}(f_2 - u_0) \right) d\mu - 2u_0, -v_0 \right) \in \partial\Phi_1(u_0, v_0) + \partial\Phi_2(u_0, v_0).$$

Remark (4) and statement (12) imply that for every $(g_1, g_2) \in L^2(\Omega) \times L^2(\Omega \times S_m)$ we have

$$\int_\Omega \left((g_1 - u_0) \left(f_1 + \int_{S_m} (f_2 - (I + \partial\zeta_s)^{-1}(f_2 - u_0)) \, d\mu - 2u_0 \right) \right) d\lambda$$

$$- \int_{\Omega \times S_m} (g_2 - v_0) v_0 \, d(\lambda \times \mu)$$

$$(13) \qquad \leq \Phi_1(g_1, g_2) + \Phi_2(g_1, g_2) - \Phi_1(u_0, v_0) - \Phi_2(u_0, v_0).$$

Letting $g_1 = u_0$ and $g_2 = v_0 - \mathrm{sgn}_0(v_0)$ in (13) gives $\int_{\Omega \times S_m} |v_0| \leq 0$. Hence,

$$(14) \quad \left(f_1 + \int_{S_m} \left(f_2 - (I + \partial\zeta_s)^{-1}(f_2 - u_0) \right) d\mu - 2u_0, 0 \right) \in \partial\Phi_1(u_0, 0) + \partial\Phi_2(u_0, 0).$$

If we let $w_0(x, s) = (I + \partial\zeta_s)^{-1}(f_2(x, s) - u_0(x)) + u_0(x)$, then $f_2(x, s) - w_0(x, s) \in \partial\zeta_s(w_0(x, s) - u_0(x))$. Therefore, Lemma (4) implies

$$(15) \qquad \left(\int_{S_m} (w_0 - f_2) \, d\mu, f_2 - w_0 \right) \in \partial\Phi_3(u_0, w_0).$$

At this point we need to modify statement (14). Since $\Phi_1(u_0, 0) = \Phi_1(u_0, w_0)$ and $\Phi_2(u_0, 0) = \Phi_2(u_0, w_0)$, we have

$$(16) \quad \left(f_1 + \int_{S_m} \left(f_2 - (I + \partial\zeta_s)^{-1}(f_2 - u_0) \right) d\mu - 2u_0, 0 \right) \in \partial\Phi_1(u_0, w_0) + \partial\Phi_2(u_0, w_0).$$

Adding (15) and (16) gives

$$(17) \qquad \left(f_1 + \int_{S_m} \left(w_0 - (I + \partial\zeta_s)^{-1}(f_2 - u_0) \right) d\mu - 2u_0, f_2 - w_0 \right)$$

$$(18) \qquad \in \partial\Phi_1(u_0, w_0) + \partial\Phi_2(u_0, w_0) + \partial\Phi_3(u_0, w_0).$$

After simplifying (17) we have $(f_1, f_2) \in (I + \partial\Phi_1 + \partial\Phi_2 + \partial\Phi_3)(u_0, w_0)$. In other words, $I + \partial\Phi_1 + \partial\Phi_2 + \partial\Phi_3$ is onto $L^2(\Omega) \times L^2(\Omega \times S_m)$. Hence, $\partial\Phi_1 + \partial\Phi_2 + \partial\Phi_3$ is m-accretive. □

Remark 6 *The function $\Phi_1 + \Phi_2 + \Phi_3$ is convex and lower-semi-continuous. The function $\Phi_1 + \Phi_2 + \Phi_3$ is proper provided $\mathrm{Dom}(j) \cap \mathrm{Dom}(k) \neq \emptyset$. Hence, under the hypothesis of Proposition (2) we have $\partial(\Phi_1 + \Phi_2 + \Phi_3) = \partial\Phi_1 + \partial\Phi_2 + \partial\Phi_3$ [2].*

Let $q : R \to (-\infty, +\infty]$ be proper, convex, and lower-semi-continuous. Define $\Phi_4 : L^2(\Omega) \times L^2(\Omega \times S_m) \to (-\infty, +\infty]$ by

$$\Phi_4(u, v) = \begin{cases} \displaystyle\int_{\Omega \times S_m} q(v) & \text{if } q(v) \in L^1(\Omega \times S_m) \\ +\infty & \text{otherwise.} \end{cases}$$

Remark 7 *The function Φ_4 is proper, convex, and lower-semi-continuous [1].*

Proposition 3 *Assume $0 \in \partial j(0)$, $0 \in \partial k(0)$, and $|q(t)| \leq c(t^2 + 1)$ for all $t \in R$. Then $\partial\Phi_1 + \partial\Phi_2 + \partial\Phi_3 + \partial\Phi_4$ is m-accretive.*

Proof: The operator $\partial\Phi_1 + \partial\Phi_2 + \partial\Phi_3$ is m-accretive by Proposition (2), and therefore $\partial\Phi_1 + \partial\Phi_2 + \partial\Phi_3 + \partial\Phi_4$ will be m-accretive if $\mathrm{int}(\mathrm{Dom}(\partial\Phi_4)) \cap \mathrm{Dom}(\partial\Phi_1 + \partial\Phi_2 + \partial\Phi_3) \neq \emptyset$ [2]. We have $\mathrm{Dom}(\partial\Phi_1 + \partial\Phi_2 + \partial\Phi_3) \neq \emptyset$ since $\partial(\Phi_1 + \Phi_2 + \Phi_3) = \partial\Phi_1 + \partial\Phi_2 + \partial\Phi_3$. We also have $\mathrm{int}(\mathrm{Dom}(\partial\Phi_4)) = L^2(\Omega) \times L^2(\Omega \times S_m)$ since Φ_4 is continuous [6,19]. Hence, $\mathrm{int}(\mathrm{Dom}(\partial\Phi_4)) \cap \mathrm{Dom}(\partial\Phi_1 + \partial\Phi_2 + \partial\Phi_3) \neq \emptyset$. □

Remark 8 *The function $\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4$ is convex and lower-semi-continuous. The function $\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4$ is proper provided $\mathrm{Dom}(j) \cap \mathrm{Dom}(k) \neq \emptyset$ and $|q(t)| \leq c(t^2 + 1)$ for all $t \in R$. Hence, under the hypothesis of Proposition (3) we have $\partial(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4) = \partial\Phi_1 + \partial\Phi_2 + \partial\Phi_3 + \partial\Phi_4$ [2].*

3 Coercivity

Proposition 4 *Assume $0 \in \partial j(0)$, $0 \in \partial k(0)$, $|q(t)| \leq c_1(t^2 + 1)$ for all $t \in R$, and $k(t) \geq c_2 t^2 - c_3$ for all $t \in R$, with $c_2, c_3 > 0$. Then $\mathrm{Rg}(\partial\Phi_1 + \partial\Phi_2 + \partial\Phi_3 + \partial\Phi_4) = L^2(\Omega) \times L^2(\Omega \times S_m)$.*

Proof: We need to verify that $\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4$ is coercive on $L^2(\Omega) \times L^2(\Omega \times S_m)$, i.e.

$$\frac{\Phi_1(u,v)+\Phi_2(u,v)+\Phi_3(u,v)+\Phi_4(u,v)}{\|u\|_{L^2(\Omega)}+\|v\|_{L^2(\Omega\times S_m)}} \longrightarrow +\infty,$$

$$\text{as } \|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega\times S_m)} \to +\infty \quad [2].$$

If the coercivity condition did not hold we could find a sequence $\{(u_n, v_n)\}$ in $L^2(\Omega) \times L^2(\Omega \times S_m)$ and a constant K_1 such that

$$\Phi_1(u_n, v_n) + \Phi_2(u_n, v_n) + \Phi_3(u_n, v_n) + \Phi_4(u_n, v_n) \leq K_1(\|u_n\|_{L^2(\Omega)} + \|v_n\|_{L^2(\Omega\times S_m)}),$$

$$\text{with } \|u_n\|_{L^2(\Omega)} + \|v_n\|_{L^2(\Omega\times S_m)} \to +\infty.$$

Note that $\zeta_s(t) \geq t^2 - m^2$ for all $s \in S_m$ and all $t \in R$. Using this lower bound on ζ_s and the lower bound on k from the hypothesis, we obtain

$$\frac{1}{2}\|\|\nabla u_n\|\|^2_{L^2(\Omega)} + \int_{\partial\Omega} j(\gamma u_n) + c_2\|u_n\|^2_{L^2(\Omega)} + \|v_n - u_n\|^2_{L^2(\Omega\times S_m)} + \int_{\Omega\times S_m} q(v_n)$$

(19)
$$\leq K_2(\|u_n\|_{L^2(\Omega)} + \|v_n\|_{L^2(\Omega\times S_m)} + 1).$$

There exists constants a_1 and a_2 such that $j(t) \geq a_1 t + a_2$ for all $t \in R$ [2,19]. Hence,

$$\int_{\partial\Omega} j(\gamma u_n) \geq a_3 + a_1 \int_{\partial\Omega} \gamma u_n \text{ , where } a_3 = a_2|\partial\Omega|.$$

Similarly, there are constants b_1 and b_2 such that $q(t) \geq b_1 t + b_2$ for all $t \in R$. Hence,

$$\int_{\Omega\times S_m} q(v_n) \geq b_3 + b_1 \int_{\Omega\times S_m} v_n \text{ , where } b_3 = b_2|\Omega \times S_m|.$$

Inequality (19) can now be used to obtain

$$\frac{1}{2}\|\|\nabla u_n\|\|^2_{L^2(\Omega)} + a_3 + a_1 \int_{\partial\Omega} \gamma u_n + c_2\|u_n\|^2_{L^2(\Omega)} + \|v_n - u_n\|^2_{L^2(\Omega\times S_m)} + b_3 + b_1 \int_{\Omega\times S_m} v_n$$

(20)
$$\leq K_2\left(\|u_n\|_{L^2(\Omega)} + \|v_n\|_{L^2(\Omega\times S_m)} + 1\right).$$

Upon dividing inequality (20) by $\frac{1}{2}\|\|\nabla u_n\|\|^2_{L^2(\Omega)} + c_2\|u_n\|^2_{L^2(\Omega)} + \|v_n - u_n\|^2_{L^2(\Omega\times S_m)}$ and using $\|v_n\|_{L^2(\Omega\times S_m)} \leq \|v_n - u_n\|_{L^2(\Omega\times S_m)} + (\mu(S_m))^{\frac{1}{2}} \|u_n\|_{L^2(\Omega)}$, we obtain

(21)
$$\frac{a_3+b_3+a_1\int_{\partial\Omega}\gamma u_n +b_1\int_{\Omega\times S_m} v_n}{\frac{1}{2}\|\|\nabla u_n\|\|^2_{L^2(\Omega)}+c_2\|u_n\|^2_{L^2(\Omega)}+\|v_n-u_n\|^2_{L^2(\Omega\times S_m)}} + 1$$
$$\leq \frac{K_3\left(\|u_n\|_{L^2(\Omega)}+\|v_n-u_n\|_{L^2(\Omega\times S_m)}+1\right)}{\frac{1}{2}\|\|\nabla u_n\|\|^2_{L^2(\Omega)}+c_2\|u_n\|^2_{L^2(\Omega)}+\|v_n-u_n\|^2_{L^2(\Omega\times S_m)}}.$$

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To see that the right side of (21) tends to zero as $n \to +\infty$, consider the following three cases:

1. $\|u_n\|_{L^2(\Omega)}$ is unbounded and $\|v_n - u_n\|_{L^2(\Omega \times S_m)}$ is bounded.
2. $\|u_n\|_{L^2(\Omega)}$ and $\|v_n - u_n\|_{L^2(\Omega \times S_m)}$ are both unbounded.
3. $\|u_n\|_{L^2(\Omega)}$ is bounded (and hence $\|v_n - u_n\|_{L^2(\Omega \times S_m)}$ is unbounded).

We will therefore obtain a contradiction if it can be shown that

$$\frac{\|\gamma u_n\|_{L^2(\partial\Omega)} + \|v_n\|_{L^2(\Omega \times S_m)}}{\frac{1}{2}\|\|\nabla u_n\|\|^2_{L^2(\Omega)} + c_2\|u_n\|^2_{L^2(\Omega)} + \|v_n - u_n\|^2_{L^2(\Omega \times S_m)}} \to 0, \text{ as } n \to +\infty.$$

Let $M = \left(\min\left(\frac{1}{2}, c_2\right)\right)^{-1}$. Using $\frac{1}{M}\|u_n\|^2_{H^1(\Omega)} \le \frac{1}{2}\|\|\nabla u_n\|\|^2_{L^2(\Omega)} + c_2\|u_n\|^2_{L^2(\Omega)}$ and $\|v_n\|_{L^2(\Omega \times S_m)} \le (\mu(S_m))^{\frac{1}{2}}\|u_n\|_{L^2(\Omega)} + \|v_n - u_n\|_{L^2(\Omega \times S_m)}$, we obtain

(22)
$$\frac{\|\gamma u_n\|_{L^2(\partial\Omega)} + \|v_n\|_{L^2(\Omega \times S_m)}}{\frac{1}{2}\|\|\nabla u_n\|\|^2_{L^2(\Omega)} + c_2\|u_n\|^2_{L^2(\Omega)} + \|v_n - u_n\|^2_{L^2(\Omega \times S_m)}}$$
$$\le \frac{M\left(\|\gamma u_n\|_{L^2(\partial\Omega)} + (\mu(S_m))^{\frac{1}{2}}\|u_n\|_{L^2(\Omega)} + \|v_n - u_n\|_{L^2(\Omega \times S_m)}\right)}{\|u_n\|^2_{H^1(\Omega)} + M\|v_n - u_n\|^2_{L^2(\Omega \times S_m)}}.$$

If $\|u_n\|_{H^1(\Omega)}$ is bounded, then $\|\gamma u_n\|_{L^2(\partial\Omega)}$ is bounded and $\|v_n - u_n\|_{L^2(\Omega \times S_m)}$ is unbounded. Hence, if $\|u_n\|_{H^1(\Omega)}$ is bounded, then the right side of (22) tends to zero as $n \to +\infty$. On the other hand, if $\|u_n\|_{H^1(\Omega)}$ is unbounded, then the right side of (22) also tends to zero as $n \to +\infty$. Therefore, the coercivity condition holds. Thus, $\mathrm{Rg}\left(\partial(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4)\right) = L^2(\Omega) \times L^2(\Omega \times S_m)$. However, under the assumptions $0 \in \partial j(0)$, $0 \in \partial k(0)$, and $|q(t)| \le c(t^2 + 1)$ for all $t \in R$, we have $\partial(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4) = \partial\Phi_1 + \partial\Phi_2 + \partial\Phi_3 + \partial\Phi_4$. Hence, $\mathrm{Rg}(\partial\Phi_1 + \partial\Phi_2 + \partial\Phi_3 + \partial\Phi_4) = L^2(\Omega) \times L^2(\Omega \times S_m)$. $\square$

4 The M-Accretive Operator

Let $Z = \Phi_1 + \Phi_3$ and note that Z is proper, convex, and lower-semi-continuous. Define $\Lambda : H^1(\Omega) \to (H^1(\Omega))^*$ by

$$(\Lambda u)(\varphi) = \int_\Omega \nabla u \cdot \nabla\varphi \quad \text{for } u, \varphi \in H^1(\Omega).$$

Lemma 5 *Assume $|j(t)| \le c(t^2 + 1)$ for all $t \in R$. Then $(f, g) \in \partial Z(u, v)$ iff $u \in H^1(\Omega)$, $f \in L^2(\Omega)$, $g \in L^2(\Omega \times S_m)$, $v \in L^2(\Omega \times S_m)$, $g(x, s) \in \partial\zeta_s(v(x, s) - u(x))$ at almost every $(x, s) \in \Omega \times S_m$, and there exists a $w \in L^2(\partial\Omega)$, with $w(\tau) \in \partial j(\gamma u(\tau))$ at almost every $\tau \in \partial\Omega$, such that $\int_\Omega fh + \int_{\Omega \times S_m} gh = (\Lambda u)(h) + \int_{\partial\Omega} w\gamma h$ for all $h \in H^1(\Omega)$.*

Proof: If $(f, g) \in \partial Z(u, v)$, then for all $(\psi_1, \psi_2) \in H^1(\Omega) \times L^2(\Omega \times S_m)$ we have

$$\int_\Omega f(\psi_1 - u) + \int_{\Omega \times S_m} g(\psi_2 - v)$$

(23)
$$\leq \tfrac{1}{2} \int_\Omega |\nabla \psi_1|^2 + \int_{\partial\Omega} j(\gamma\psi_1) + \int_{\Omega \times S_m} \zeta_s(\psi_2 - \psi_1) - \tfrac{1}{2} \int_\Omega |\nabla u|^2 - \int_{\partial\Omega} j(\gamma u).$$

Let $\psi_1 = u + h$ and $\psi_2 = v + h$, with $h \in H^1(\Omega)$, in inequality (23) to get

(24)
$$\int_\Omega fh + \int_{\Omega \times S_m} gh \leq \tfrac{1}{2} \int_\Omega |\nabla(u+h)|^2 - \tfrac{1}{2} \int_\Omega |\nabla u|^2$$
$$+ \int_{\partial\Omega} j(\gamma(u+h)) - \int_{\partial\Omega} j(\gamma u) \quad \text{for all } h \in H^1(\Omega).$$

Now let $h = \psi_1 - u$, with $\psi_1 \in H^1(\Omega)$, in inequality (24) to get

$$\int_\Omega f(\psi_1 - u) + \int_{\Omega \times S_m} g(\psi_1 - u) \leq \tfrac{1}{2} \int_\Omega |\nabla \psi_1|^2 - \tfrac{1}{2} \int_\Omega |\nabla u|^2$$

(25)
$$+ \int_{\partial\Omega} j(\gamma\psi_1) - \int_{\partial\Omega} j(\gamma u) \quad \text{for all } \psi_1 \in H^1(\Omega).$$

Define $\Gamma_1 : H^1(\Omega) \to R$ by $\Gamma_1(\psi) = \tfrac{1}{2}(\Lambda\psi)(\psi)$ and $\Gamma_2 : H^1(\Omega) \to R$ by $\Gamma_2(\psi) = \int_{\partial\Omega} j(\gamma\psi)$. Note that Γ_1 and Γ_2 are convex. Note that Γ_1 is continuous. Since Γ_2 is locally upper-bounded, Γ_2 is continuous [6,19]. At this point we would like to consider ∂ acting on Γ_1 to produce a relation in $H^1(\Omega) \times (H^1(\Omega))^*$, i.e. for $u \in \text{Dom}(\Gamma_1) \equiv \{u \in H^1(\Omega) : \Gamma_1(u) < +\infty\}$, define $\partial\Gamma_1(u) \equiv \{w^* \in (H^1(\Omega))^* : w^*(v - u) \leq \Gamma_1(v) - \Gamma_1(u) \text{ for all } v \in H^1(\Omega)\}$. If $w^* \in \partial\Gamma_1(u)$, we say w^* is a subgradient of Γ_1 at u [6,19]. We will also take this meaning for ∂ acting on Γ_2 and $\Gamma_1 + \Gamma_2$. We clearly have $\partial\Gamma_1 + \partial\Gamma_2 \subset \partial(\Gamma_1 + \Gamma_2)$. A sufficient condition for $\partial\Gamma_1 + \partial\Gamma_2 = \partial(\Gamma_1 + \Gamma_2)$ is that Γ_1 is continuous at some point in $\text{Dom}(\Gamma_1) \cap \text{Dom}(\Gamma_2)$ [6,19]. This condition is clearly satisfied since Γ_1 is continuous and $\text{Dom}(\Gamma_1) \cap \text{Dom}(\Gamma_2) = H^1(\Omega)$. Note that if $T_{f,g} \in (H^1(\Omega))^*$ is given by $T_{f,g}(\psi) = \int_\Omega f\psi + \int_{\Omega \times S_m} g\psi$, then (25) implies $T_{f,g}$ is a subgradient of $\Gamma_1 + \Gamma_2$ at u. Hence, $T_{f,g} = d_1 + d_2$ for some subgradient d_1 of Γ_1 at u and some subgradient d_2 of Γ_2 at u. The subgradients of Γ_1 and Γ_2 are readily characterized [6,19]. These characterizations imply there exists a $w \in L^2(\partial\Omega)$, with $w(\tau) \in \partial j(\gamma u(\tau))$ at almost every $\tau \in \partial\Omega$, such that

$$\int_\Omega fh + \int_{\Omega \times S_m} gh = (\Lambda u)(h) + \int_{\partial\Omega} w\gamma h \quad \text{for all } h \in H^1(\Omega).$$

To see that $g(x, s) \in \partial\zeta_s(v(x, s) - u(x))$ at almost every $(x, s) \in \Omega \times S_m$, we let $\psi_1 = u$ and $\psi_2 = u + \phi$, with $\phi \in L^2(\Omega \times S_m)$, in (23) to get

(26)
$$\int_{\Omega \times S_m} g(\phi - (v - u)) \leq \int_{\Omega \times S_m} \zeta_s(\phi) \quad \text{for all } \phi \in L^2(\Omega \times S_m).$$

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As was seen in Lemma (4) , inequality (26) holds iff $g(x, s) \in \partial \zeta_s(v(x, s) - u(x))$ at almost every $(x, s) \in \Omega \times S_m$. For the converse, we first note that $T_{f,g} \in \partial \Gamma_1(u) + \partial \Gamma_2(u)$, which in turn implies (25). We can let $\phi = \psi_2 - \psi_1$, with $\psi_2 \in L^2(\Omega \times S_m)$ and $\psi_1 \in H^1(\Omega)$, in (26) to get

$$\int_{\Omega \times S_m} g(\psi_2 - \psi_1 - (v - u))$$

(27)
$$\leq \int_{\Omega \times S_m} \zeta_s(\psi_2 - \psi_1) \quad \text{for all } (\psi_1, \psi_2) \in H^1(\Omega) \times L^2(\Omega \times S_m).$$

Adding (25) and (27) gives (23). Hence, $(f, g) \in \partial Z(u, v)$. $\qquad \square$

Lemma 6 *Let $\sigma : R \to R$ be a monotone Lipschitz function such that $\sigma(0) = 0$. Assume $|j(t)| \leq c(t^2 + 1)$ for all $t \in R$. If $(f_i, g_i) \in \partial Z(u_i, v_i)$ for $i = 1, 2$, then*

$$\langle f_1 - f_2, \sigma(u_1 - u_2) \rangle_{L^2(\Omega)} + \langle g_1 - g_2, \sigma(v_1 - v_2) \rangle_{L^2(\Omega \times S_m)} \geq 0.$$

Proof: Assume $(f_i, g_i) \in \partial Z(u_i, v_i)$ for $i = 1, 2$. Note that $u_1, u_2 \in H^1(\Omega)$, and hence Lemma (1) implies $\sigma(u_1 - u_2) \in H^1(\Omega)$. Using Lemma (5) we have

$$\langle f_1 - f_2, \sigma(u_1 - u_2) \rangle_{L^2(\Omega)} + \langle g_1 - g_2, \sigma(v_1 - v_2) \rangle_{L^2(\Omega \times S_m)}$$
$$= -\int_{\Omega \times S_m} g_1 \sigma(u_1 - u_2) + (\Lambda u_1)(\sigma(u_1 - u_2)) + \int_{\partial \Omega} w_1 \gamma \sigma(u_1 - u_2)$$
$$+ \int_{\Omega \times S_m} g_2 \sigma(u_1 - u_2) - (\Lambda u_2)(\sigma(u_1 - u_2)) - \int_{\partial \Omega} w_2 \gamma \sigma(u_1 - u_2)$$

(28)
$$+ \int_{\Omega \times S_m} (g_1 - g_2) \sigma(v_1 - v_2).$$

Note that Lemma (2) gives $\gamma \sigma(u_1 - u_2) = \sigma(\gamma u_1 - \gamma u_2)$. After simplifying (28) and using $\gamma \sigma(u_1 - u_2) = \sigma(\gamma u_1 - \gamma u_2)$, we obtain

$$\langle f_1 - f_2, \sigma(u_1 - u_2) \rangle_{L^2(\Omega)} + \langle g_1 - g_2, \sigma(v_1 - v_2) \rangle_{L^2(\Omega \times S_m)}$$
$$= (\Lambda(u_1 - u_2))(\sigma(u_1 - u_2)) + \int_{\partial \Omega} (w_1 - w_2) \sigma(\gamma u_1 - \gamma u_2)$$

(29)
$$+ \int_{\Omega \times S_m} (g_1 - g_2)(\sigma(v_1 - v_2) - \sigma(u_1 - u_2)).$$

Let

$$r(t) = \begin{cases} \sigma'(t) & \text{if } \sigma'(t) \text{ exists} \\ 0 & \text{otherwise.} \end{cases}$$

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Note that Lemma (1) gives

$$(\Lambda(u_1 - u_2))(\sigma(u_1 - u_2)) = \int_\Omega |\nabla(u_1 - u_2)|^2 r(u_1 - u_2) \geq 0.$$

The second term on the right side of (29) is nonnegative since ∂j is a monotone graph, σ is a monotone function, and $\sigma(0) = 0$. The last term on the right side of (29) is nonnegative since σ is a monotone function and each $\partial \zeta_s$ is a monotone graph. $\qquad\square$

Definition 2 *The operator $C \subset L^2(\Omega) \times L^2(\Omega \times S_m)$ is defined as follows: $(f, g) \in C(a, b)$ if there exists $(u, v) \in L^2(\Omega) \times L^2(\Omega \times S_m)$ such that $(f, g) \in \partial Z(u, v)$, with $a(x) \in \partial k(u(x))$ at almost every $x \in \Omega$ and $b(x, s) \in \partial q(v(x, s))$ at almost every $(x, s) \in \Omega \times S_m$.*

Proposition 5 *Assume ∂k and ∂q are functions, and $|j(t)| \leq c(t^2 + 1)$ for all $t \in R$. Then C is accretive in $L^1(\Omega) \times L^1(\Omega \times S_m)$.*

Proof: Fix $\eta > 0$ and assume $(f_i, g_i) \in (I + \eta C)(a_i, b_i)$ for $i = 1, 2$. Hence, for $i = 1, 2$ we have

$$(f_i - a_i, g_i - b_i) \in \eta \partial Z(u_i, v_i),$$

with $a_i(x) = \partial k(u_i(x))$ at almost every $x \in \Omega$ and $b_i(x, s) = \partial q(v_i(x, s))$ at almost every $(x, s) \in \Omega \times S_m$. Let σ_ϵ be the Yosida approximation to the maximal monotone signum graph, i.e.

$$\sigma_\epsilon(t) = \begin{cases} 1 & \text{if } t \geq \epsilon \\ \frac{t}{\epsilon} & \text{if } -\epsilon < t < \epsilon \\ -1 & \text{if } t \leq -\epsilon. \end{cases}$$

Using Lemma (6) we have

(30) $\quad \langle f_1 - a_1 - (f_2 - a_2), \sigma_\epsilon(u_1 - u_2) \rangle_{L^2(\Omega)} + \langle g_1 - b_1 - (g_2 - b_2), \sigma_\epsilon(v_1 - v_2) \rangle_{L^2(\Omega \times S_m)} \geq 0.$

Inequality (30) implies

$$\|f_1 - f_2\|_{L^1(\Omega)} + \|g_1 - g_2\|_{L^1(\Omega \times S_m)}$$
$$\geq \int_{|u_1 - u_2| \geq \epsilon} |a_1 - a_2|\, d\lambda + \tfrac{1}{\epsilon} \int_{0 < |u_1 - u_2| < \epsilon} (a_1 - a_2)(u_1 - u_2)\, d\lambda$$
(31) $$+ \int_{|v_1 - v_2| \geq \epsilon} |b_1 - b_2|\, d(\lambda \times \mu) + \tfrac{1}{\epsilon} \int_{0 < |v_1 - v_2| < \epsilon} (b_1 - b_2)(v_1 - v_2)\, d(\lambda \times \mu).$$

Note that

$$\lim_{\epsilon \downarrow o} \left| \frac{1}{\epsilon} \int_{0<|u_1-u_2|<\epsilon} (a_1 - a_2)(u_1 - u_2)\, d\lambda \right| \leq \lim_{\epsilon \downarrow 0} \int_{0<|u_1-u_2|<\epsilon} |a_1 - a_2|\, d\lambda = 0$$

Similarly,

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_{0<|v_1-v_2|<\epsilon} (b_1 - b_2)(v_1 - v_2)\, d(\lambda \times \mu) = 0.$$

Note that the hypothesis implies

$$\|a_1 - a_2\|_{L^1(\Omega)} = \int_{|u_1-u_2|>0} |a_1 - a_2|\, d\lambda \text{ and } \|b_1 - b_2\|_{L^1(\Omega \times S_m)} = \int_{|v_1-v_2|>0} |b_1 - b_2|\, d(\lambda \times \mu).$$

Therefore, taking limits in (31) gives

$$\|f_1 - f_2\|_{L^1(\Omega)} + \|g_1 - g_2\|_{L^1(\Omega \times S_m)} \geq \|a_1 - a_2\|_{L^1(\Omega)} + \|b_1 - b_2\|_{L^1(\Omega \times S_m)}.$$

In other words, for each $\eta > 0$ the map $(I + \eta C)^{-1} : \text{Rg}(I + \eta C) \to L^2(\Omega) \times L^2(\Omega \times S_m)$ is a contraction in the norm $\|\cdot\|_{L^1(\Omega) \times L^1(\Omega \times S_m)}$. □

Proposition 6 *Under the hypothesis of Proposition (4), we have* $\text{Rg}(I+C) = L^2(\Omega) \times L^2(\Omega \times S_m)$.

Proof: If $(f,g) \in L^2(\Omega) \times L^2(\Omega \times S_m)$, then Proposition (4) gives $(u,v) \in L^2(\Omega) \times L^2(\Omega \times S_m)$ such that

(32) $$(f,g) \in \partial\Phi_1(u,v) + \partial\Phi_2(u,v) + \partial\Phi_3(u,v) + \partial\Phi_4(u,v).$$

It is easy to show $(a,b) \in \partial\Phi_2(u,v)$ iff $a, u \in L^2(\Omega)$, $v \in L^2(\Omega \times S_m)$, $b = 0$, and $a(x) \in \partial k(u(x))$ at almost every $x \in \Omega$ [1,19]. Similarly, $(a,b) \in \partial\Phi_4(u,v)$ iff $b, v \in L^2(\Omega \times S_m)$, $u \in L^2(\Omega)$, $a = 0$, and $b(x,s) \in \partial q(v(x,s))$ at almost every $(x,s) \in \Omega \times S_m$. Therefore, (32) implies

(33) $$(f - a, g - b) \in \partial\Phi_1(u,v) + \partial\Phi_3(u,v),$$

for some $a \in L^2(\Omega)$, with $a(x) \in \partial k(u(x))$ at almost every $x \in \Omega$, and some $b \in L^2(\Omega \times S_m)$, with $b(x,s) \in \partial q(v(x,s))$ at almost every $(x,s) \in \Omega \times S_m$. Since $\partial\Phi_1 + \partial\Phi_3 \subset \partial Z$, we have $(f - a, g - b) \in \partial Z(u,v)$. Using definition (2) we get $(f - a, g - b) \in C(a,b)$, i.e. $(f,g) \in (I+C)(a,b)$. □

We define the closure of C, to be denoted by $\mathbb{C}$, to be the closure of $\{((a,b),(f,g)) : (f,g) \in C(a,b)\}$ in $[L^1(\Omega) \times L^1(\Omega \times S_m)]^2$

Proposition 7 *Under the hypotheses of Propositions (4) and (5), $\mathbb{C}$ is m-accretive in $L^1(\Omega) \times L^1(\Omega \times S_m)$.*

Proof: We will first show $\mathbb{C}$ is maximal, i.e. $\text{Rg}(I + \mathbb{C}) = L^1(\Omega) \times L^1(\Omega \times S_m)$. Fix $(f, g) \in L^1(\Omega) \times L^1(\Omega \times S_m)$. Choose $(f_n, g_n) \in L^2(\Omega) \times L^2(\Omega \times S_m)$ such that $(f_n, g_n) \to (f, g)$ in $L^1(\Omega) \times L^1(\Omega \times S_m)$. Proposition (6) allows for $(a_n, b_n) \in L^2(\Omega) \times L^2(\Omega \times S_m)$ such that $(I + C)(a_n, b_n) \ni (f_n, g_n)$ for each n. Note that $\{(a_n, b_n)\}$ is Cauchy in $L^1(\Omega) \times L^1(\Omega \times S_m)$ since $(a_n, b_n) = (I + C)^{-1}(f_n, g_n)$, $\{(f_n, g_n)\}$ is Cauchy in $L^1(\Omega) \times L^1(\Omega \times S_m)$, and $(I + C)^{-1} : \text{Rg}(I + C) \to L^2(\Omega) \times L^2(\Omega \times S_m)$ is a contraction in the norm $\|\cdot\|_{L^1(\Omega) \times L^1(\Omega \times S_m)}$ by Proposition (5). Hence, there exists $(a, b) \in L^1(\Omega) \times L^1(\Omega \times S_m)$ such that $(a_n, b_n) \to (a, b)$ in $L^1(\Omega) \times L^1(\Omega \times S_m)$. Therefore, we have $((a_n, b_n), (f_n - a_n, g_n - b_n)) \to ((a, b), (f - a, g - b))$ in $(L^1(\Omega) \times L^1(\Omega \times S_m)) \times (L^1(\Omega) \times L^1(\Omega \times S_m))$, with each $((a_n, b_n), (f_n - a_n, g_n - b_n))$ in the graph of C. Hence, $\mathbb{C}(a, b) \ni (f - a, g - b)$, i.e. $(I + \mathbb{C})(a, b) \ni (f, g)$. We will now show $\mathbb{C}$ is accretive in $L^1(\Omega) \times L^1(\Omega \times S_m)$. Fix $\eta > 0$ and assume $(I + \eta\mathbb{C})(a_i, b_i) \ni (f_i, g_i)$ for $i = 1, 2$. Then $\mathbb{C}(a_i, b_i) \ni (\frac{f_i - a_i}{\eta}, \frac{g_i - b_i}{\eta})$ for $i = 1, 2$. We can choose sequences $\{(a_{1,n}, b_{1,n})\}$, $\{(a_{2,n}, b_{2,n})\}$, $\{(v_{1,n}, w_{1,n})\}$, and $\{(v_{2,n}, w_{2,n})\}$ in $L^2(\Omega) \times L^2(\Omega \times S_m)$ such that $C(a_{i,n}, b_{i,n}) \ni (v_{i,n}, w_{i,n})$, $\|(a_{i,n}, b_{i,n}) - (a_i, b_i)\|_{L^1(\Omega) \times L^1(\Omega \times S_m)} \to 0$, and $\left\|(v_{i,n}, w_{i,n}) - \left(\frac{f_i - a_i}{\eta}, \frac{g_i - b_i}{\eta}\right)\right\|_{L^1(\Omega) \times L^1(\Omega \times S_m)} \to 0$ for $i = 1, 2$. Note that

$$\|(a_{1,n}, b_{1,n}) - (a_{2,n}, b_{2,n})\|_{L^1(\Omega) \times L^1(\Omega \times S_m)}$$

$$= \|(I + \eta C)^{-1}(\eta v_{1,n} + a_{1,n}, \eta w_{1,n} + b_{1,n}) - (I + \eta C)^{-1}(\eta v_{2,n} + a_{2,n}, \eta w_{2,n} + b_{2,n})\|_{L^1(\Omega) \times L^1(\Omega \times S_m)}$$

$$\leq \|(\eta v_{1,n} + a_{1,n}, \eta w_{1,n} + b_{1,n}) - (\eta v_{2,n} + a_{2,n}, \eta w_{2,n} + b_{2,n})\|_{L^1(\Omega) \times L^1(\Omega \times S_m)}.$$

Taking limits in the above inequality gives

$$\|(a_1, b_1) - (a_2, b_2)\|_{L^1(\Omega) \times L^1(\Omega \times S_m)} \leq \|(f_1, g_1) - (f_2, g_2)\|_{L^1(\Omega) \times L^1(\Omega \times S_m)}.$$

In other words, for all $\eta > 0$ the map $(I + \eta\mathbb{C})^{-1} : L^1(\Omega) \times L^1(\Omega \times S_m) \to L^1(\Omega) \times L^1(\Omega \times S_m)$ is a contraction. $\square$

5 The Evolution Equation

Under the hypotheses of Propositions (4) and (5), the nonlinear semigroup theory implies that the Cauchy problem

$$w'(t) + \mathbb{C}(w(t)) \ni f(t), \quad 0 \leq t \leq T,$$

$$w(0) = w_0 ,$$

has a unique *integral solution* $w \in C([0,T]; L^1(\Omega) \times L^1(\Omega \times S_m))$, provided $f \in L^1([0,T] : L^1(\Omega) \times L^1(\Omega \times S_m))$ and $w_0 \in \overline{\mathrm{Dom}\,(\mathbb{C})}$. This follows because $\mathbb{C}$ is m-accretive in the Banach space $X = L^1(\Omega) \times L^1(\Omega \times S_m)$. For such an operator, one can approximate the derivative in the evolution equation by a backward-difference quotient of step size $h > 0$ and the function $f(t)$ by the step function $f^h(t)$ $(= f_k^h$ for $kh \le t < (k+1)h)$ and get a unique solution $\{w_k^h : 1 \le k\}$ of

$$\frac{w_k^h - w_{k-1}^h}{h} + \mathbb{C}(w_k^h) \ni f_k^h , \qquad k = 1, 2, \dots ,$$

with $w_0^h = w_0$. Since $\mathbb{C}$ is m-accretive, this scheme is uniquely solved recursively to obtain w_k^h and, hence, the piecewise-constant approximate solution $w^h(t)$ $(= w_k^h$ for $kh \le t < (k+1)h)$ of the Cauchy problem. The fundamental result is the following.

Theorem (Crandall-Liggett). Assume $\mathbb{C}$ is m-accretive, $w_0 \in \overline{D(\mathbb{C})}$, $f \in L^1([0,T], X)$ and that $f^h \to f$ in $L^1([0,T], X)$. Then $w^h \to w(\cdot)$ uniformly as $h \to 0$ and $w(\cdot) \in C([0,T], X)$.

Thus $w(\cdot)$ is an obvious candidate for a solution of the Cauchy problem. It can be uniquely characterized as an *integral solution*. This rather technical characterization does not require any differentiability of the solution. However, if f is Lipschitz continuous and $w_0 \in D(\mathbb{C})$, it is known that w is also Lipschitz continuous. Moreover, if $f_1, f_2 \in L^1([0,T], X)$ and w_1, w_2 are integral solutions of

$$w_j' + \mathbb{C}(w_j) \ni f_j , \quad 0 \le t \le T , \quad j = 1, 2 ,$$

then

$$\|w_1(t) - w_2(t)\| \le \|w_1(0) - w_2(0)\| + \int_0^t \|f_1(s) - f_2(s)\|\, ds , \quad 0 \le t \le T .$$

For an introduction to the abstract Cauchy problem in Banach space and its applications to initial-boundary-value problems for partial differential equations, see [3]. For further details, refinements and perspective, see [1,4,7].

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