The narrow fracture approximation by channeled flow

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Article history:
Received 14 April 2009
Available online 21 October 2009
Submitted by P. Sacks

Keywords:
Porous media
Heterogeneous
Narrow fracture
Singular Darcy system

The singular problem of non-stationary Darcy flow in a region containing a narrow channel of width $O(\epsilon)$ and high permeability $O(1/\epsilon)$ is shown to be well approximated by a problem with flow concentrated on a weighted Sobolev space over a lower-dimensional interface. The convergence of the solution as $\epsilon \to 0$ is proved for both the stationary case and the corresponding initial-boundary-value problem. The structure of the limiting problems is dependent on the rate of taper of the channel at its extremities.

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1. Introduction

Fluid flow through a fully-saturated porous medium is altered in the vicinity of a rigid wall by the sharp rise in permeability due to the inefficiency of the packing of the particles in the vicinity of the wall. Consequently, in a narrow region close to the wall the velocity is substantially higher and the flow is predominantly parallel to it; this phenomenon is known as the channeled effect [9]. Related models were used previously to describe flow through a porous medium in the vicinity of a narrow fracture which is characterized similarly as a thin interior region of high permeability. Such problems arise
e.g. from hydraulic fracturing in which narrow channels of high permeability are created in the vicinity of a well to enhance the flow rate and consequently the production. The narrow fracture approximation leads to a model like the one above for thin channel flow, and by taking advantage of the symmetry about the center surface defining the fracture, one can reduce such a problem to one of the type considered here with the high-permeability region located on the boundary [2,6]. Analogous models of heat conduction arise from regions of high conductivity, and these may also include a concentrated capacity. We include these in the discussion for comparison.

For a final example, we mention saturated gravity-driven flow of subsurface water through a hillslope bounded below by sloping bedrock. A network of narrow channels of very high permeability develops in the vicinity of the impermeable bedrock, and it is observed that most of the fluid in the system flows through this region. Such systems with high flow rate over narrow regions greatly influence the transport and flow processes and are a topic of current study [16].

We shall describe such situations with Darcy flow for which the permeability is scaled to balance the channel width and model the higher flow rates in the channel. Due to the higher permeability, the fluid flows primarily into and then tangential to the channel. The resulting model captures the tangential boundary flow coupled to the interior flow by continuity of flux and pressure. It contains two sources of singularity: a geometric one from the thinness of the channel and a material one due to the higher permeability of the channel. With the appropriate scaling, these two singularities are balanced, and a model of these and related models.

An additional challenging issue is to account for the shape of the channel, especially for any taper near the edges or boundary of the channel. Such shapes are ubiquitous in applications, but they are not commonly included in the modeling process. They are important because the rate of the tapering at the edges determines the appropriate boundary conditions (or lack thereof) that describe the resulting model [8,12].

The geometry of the model is described first. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) and denote by \( \Gamma \) a relatively-open connected portion of its boundary \( \partial \Omega \) along the top of the domain. For simplicity of representation, we assume this portion of the boundary is flat, that is, \( \Gamma \subset \mathbb{R}^{n-1} \times \{0\} \) and that \( x_0 < 0 \) for each \( x = (x_0, x_1, \ldots, x_n) \in \Omega \), where \( x \in \mathbb{R}^{n-1} \). The channel is realized as a region of the form \( \Omega^\varepsilon(u) = ((x, \omega(x_0)); (x_0, x) \in \Gamma \times (0, \varepsilon)) \). The function \( \omega(u) \) shapes the width of the channel at each \( x \in \Gamma \), and the parameter \( \varepsilon > 0 \) denotes its scale. We assume that this width function satisfies \( 0 < a \leq \omega(x) \leq 1 \) on each compact subset of \( \Gamma \), where \( a \) depends on the set, but it may approach zero near \( \partial \Gamma \) at a rate to be determined below. This assumption permits the channel to be tapered or to pinch off near its extremities.

For the single-phase flow of a slightly compressible fluid through \( \Omega^\varepsilon \equiv \Omega_1 \cup \Gamma \cup \Omega_2^\varepsilon \), Darcy’s law together with conservation of fluid mass lead to the interface problem

\[
\begin{align*}
    m_1 \frac{\partial u_1^\varepsilon}{\partial t} - \nabla \cdot k_1 \nabla u_1^\varepsilon &= m_1 f \quad \text{in } \Omega_1, \\
    u_1^\varepsilon &= 0 \quad \text{on } \partial \Omega_1 - \Gamma, \\
    u_1^\varepsilon &= u_2^\varepsilon, \quad k_1 \frac{\partial u_1^\varepsilon}{\partial n} - \frac{k_2}{\varepsilon} \frac{\partial u_2^\varepsilon}{\partial x_n} &= g \quad \text{on } \Gamma, \\
    m_2 \frac{\partial u_2^\varepsilon}{\partial t} - \nabla \cdot \frac{k_2}{\varepsilon} \nabla u_2^\varepsilon &= m_2 f \quad \text{in } \Omega_2^\varepsilon, \\
    \frac{k_2}{\varepsilon} (\nabla u_2^\varepsilon \cdot \mathbf{n}) &= 0 \quad \text{on } \partial \Omega_2^\varepsilon - \Gamma,
\end{align*}
\]

at each \( t > 0 \) for the fluid density \( u_1^\varepsilon(\cdot, t) \) in \( \Omega_1 \) and \( u_2^\varepsilon(\cdot, t) \) in \( \Omega_2^\varepsilon \), and these satisfy the initial conditions

\[
    u_1(\cdot, 0) = u_0^1(\cdot) \quad \text{on } \Omega_1, \quad u_2(\cdot, 0) = u_0^2(\cdot) \quad \text{on } \Omega_2^\varepsilon.
\]

Thus, the region is drained along \( \partial \Omega_1 - \Gamma \) and there is no flow across \( \partial \Omega_2^\varepsilon - \Gamma \), where the outward normal is indicated by \( \mathbf{n} \). This latter condition would follow if the region were symmetric about \( \Gamma \times \{\varepsilon\} \). The given initial density distributions \( u_0^1(\cdot) \) complete the initial-boundary-value problem. Corresponding non-homogeneous problems with known pressure on \( \partial \Omega_1 - \Gamma \) and flow-rate along \( \partial \Omega_2^\varepsilon - \Gamma \) can be reduced to this case. The permeability in \( \Omega_2^\varepsilon \) has been scaled by \( \varepsilon \) to indicate the high flow rate, and this will be shown to balance the width \( \varepsilon \) of the channel, so the flow in \( \Omega_2^\varepsilon \) is closely approximated by surface flow along \( \Gamma \). It will be seen below that \( k_2 \) is the effective tangential permeability and \( \frac{k_2}{\varepsilon} \) is the effective normal permeability for channel flow; see [6] for substantial discussion and further perspective. The coefficients \( m_1, m_2 \) are obtained from the porosity and from the compressibility of either the fluid or the medium. We include for comparison the concentrated capacity model in which also \( m_2 \) is scaled by \( \frac{1}{\varepsilon} \), but this has nothing to do with porous media.

2. Preliminaries

We use standard notation and results on function spaces. \( L^2(\Omega) \) is the Hilbert space of (equivalence classes of) Lebesgue square summable functions on \( \Omega \), and \( H^m(\Omega) \), \( m \geq 1 \), with the norm \( \| \cdot \|_{m,\Omega} \) is the Sobolev space of functions in \( L^2(\Omega) \) for which each weak derivative up to order \( m \) belongs to \( L^2(\Omega) \). The space \( H^1_0(\Omega) \) is the closure in \( H^1(\Omega) \) of those infinitely
differentiable functions which have compact support in \( \Omega \). The trace \( \gamma(v) \) of a \( v \in H^1(\Omega) \) is its boundary value in \( H^{1/2}(\partial \Omega) \). The spaces with fractional exponents are defined by interpolation. Corresponding spaces of vector-valued functions are denoted by bold-face symbols, \( L^2(\Omega) \), \( H^0(\Omega) \). The space of those functions of \( L^2(\Omega) \) whose divergence belongs to \( L^2(\Omega) \) is denoted by \( L^2_{div}(\Omega) \). These have a normal trace on the boundary. See [1,13–15].

Assume the interface \( \Gamma \) is an open bounded connected subset of \( \mathbb{R}^{n-1} \) and that it lies locally on one side of its boundary, \( \partial \Gamma \), a \( C^1 \) manifold. Let \( \delta(\tilde{x}) \) be the distance from \( \tilde{x} \in \Gamma \) to \( \partial \Gamma \) and \( 0 \leq \alpha < 1 \). Define \( W(\alpha) \) to be the space obtained by completing \( H^{1}(\Gamma) \) in the weaker norm

\[
\| v \|_{W(\alpha)} = \left\{ \int_{\Gamma} (v(\tilde{x})^2 + \delta(\tilde{x})^\alpha \| \tilde{\nabla} v(\tilde{x}) \|^2) \, d\tilde{x} \right\}^{1/2}.
\]

For details, let

\[
0 \leq \alpha < 1 \quad \text{and} \quad \| v \|_{W(\alpha)} = \left\{ \int_{\Gamma} (v(\tilde{x})^2 + \delta(\tilde{x})^\alpha \| \tilde{\nabla} v(\tilde{x}) \|^2) \, d\tilde{x} \right\}^{1/2}.
\]

Here and in the following, \( \tilde{\nabla} \) denotes the \( \mathbb{R}^{n-1} \)-gradient in directions tangent to \( \Gamma \). It is known that the embedding \( W(\alpha) \rightarrow L^2(\Gamma) \) is compact and the trace operator \( \gamma : W(\alpha) \rightarrow L^2(\partial \Gamma) \) is continuous [3,7]. Here we assume the width function satisfies

\[
\omega(\tilde{x}) \geq c\delta^\alpha(\tilde{x}) \quad \text{a.e. } \tilde{x} \in \Gamma
\]  

for some \( c > 0 \), and we say \( \Gamma \) is weakly tapered. Then define \( H^1_{\text{loc}}(\Gamma) \) to be the completion of \( H^1(\Gamma) \) with the norm

\[
\| v \|_{H^1_{\text{loc}}(\Gamma)} = \left\{ \int_{\Gamma} (v(\tilde{x})^2 + \omega(\tilde{x}) \| \tilde{\nabla} v(\tilde{x}) \|^2) \, d\tilde{x} \right\}^{1/2}.
\]

As above, the embedding \( H^1_{\text{loc}}(\Gamma) \rightarrow L^2(\Gamma) \) is compact and the trace operator \( \gamma : H^1_{\text{loc}}(\Gamma) \rightarrow L^2(\partial \Gamma) \) is continuous. More generally, we have the following [12].

**Theorem 2.1.** Let the bounded domain \( \Gamma \) be given as above and let \( 0 \leq \alpha < 1 \). Suppose there is a function \( \alpha(\cdot) \) on \( \partial \Gamma \) for which \( 0 \leq \alpha(\tilde{x}) \leq \alpha \) for each \( \tilde{x} \in \partial \Gamma \). Assume the function \( \omega(\cdot) \) satisfies (2.2) and that at each point of \( \partial \Gamma \) there is a neighborhood \( N \) in \( \mathbb{R}^{n-1} \) and constants \( 0 < c(N) < C(N) \) such that

1. for each \( \tilde{x} \in N \cap \partial \Gamma \) there is an \( \tilde{x}_0 \in \partial \Gamma \) such that \( \| \tilde{x}_0 - \tilde{x} \| = \delta(\tilde{x}) \), and
2. for each \( \tilde{x} \in N \cap \partial \Gamma \), \( c(N) \leq \frac{\omega(\tilde{x})}{\delta(\tilde{x})^{1-\alpha(\tilde{x})}} \leq C(N) \).

Then the trace map is continuous from \( H^1_{\text{loc}}(\Gamma) \) into \( L^2(\partial \Gamma) \), its kernel is the closure of \( C_0^\infty(\Gamma) \) in \( H^1_{\text{loc}}(\Gamma) \), and the range is dense in \( L^2(\partial \Gamma) \).

In the contrary case we call \( \Gamma \) strongly tapered if

\[
\omega(\tilde{x}) \leq C\delta^\alpha(\tilde{x}) \quad \text{a.e. } \tilde{x} \in \Gamma
\]

for some \( C > 0 \), and then \( C_0^\infty(\Gamma) \) is dense in \( H^1_{\text{loc}}(\Gamma) \), so \( H^1_{\text{loc}}(\Gamma) \)' is a space of distributions on \( \Gamma \) and \( L^2(\Gamma) \subset H^1_{\text{loc}}(\Gamma) \).

We recall some classical results for unbounded operators and the Cauchy problem; see [5,13] or the first chapter of [14] for details. Let \( V \) be a Hilbert space, and denote its dual space of continuous linear functionals by \( V' \). A bilinear form \( a(\cdot,\cdot) : V \times V \rightarrow \mathbb{R} \) is \( V \)-elliptic if there is \( c_0 > 0 \) for which

\[
a(u,u) \geq c_0 \| u \|_V^2, \quad u \in V.
\]

The Lax–Milgram theorem shows this is a convenient sufficient condition for the associated problem to be well-posed.

**Theorem 2.2.** If \( a(\cdot,\cdot) \) is bilinear, continuous and \( V \)-elliptic, then for each \( f \in V' \) there is a unique

\[
u \in V : \quad a(u,v) = f(v), \quad v \in V.
\]

An unbounded linear operator \( A : D \rightarrow H \) with domain \( D \) in the Hilbert space \( H \) is accretive if

\[
(Ax,x)_H \geq 0, \quad x \in D,
\]

and it is \( m \)-accretive if, in addition, \( A + I \) maps \( D \) onto \( H \). Sufficient conditions for the initial-value problem to be well-posed are provided by the Hille–Yoshida theorem.
**Theorem 2.3.** Let the operator \( A : \mathcal{D} \to H \) be \( m \)-accretive on the Hilbert space \( H \). Then for every \( u^0 \in \mathcal{D}(A) \) and \( f \in C^1([0, \infty), H) \) there is a unique solution \( u \in C^1([0, \infty), H) \) of the initial-value problem

\[
\frac{du}{dt}(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = u^0.
\]

(2.4)

If additionally \( A \) is self-adjoint, then for each \( u^0 \in H \) and Hölder continuous \( f \in C^\beta([0, \infty), H), 0 < \beta < 1 \), there is a unique solution \( u \in C([0, \infty), H) \cap C^1((0, \infty), H) \) of (2.4).

Finally, the standard finite-difference approximation of (2.4) leads to the stationary problem with \( \lambda > 0 \),

\[
u \in \mathcal{D}(A): \quad \lambda u + A(u) = \lambda F \quad \text{in} \ H,
\]

for the resolvent of the operator \( A \). It is precisely the \( m \)-accretive operators for which this problem is always solvable with \( \|u\|_H \leq \|F\|_H \).

3. The stationary problem

With the family of domains \( \Omega^\epsilon = \Omega_1 \cup \Gamma \cup \Omega_2^\epsilon \) given above for each value of the parameter with \( 0 < \epsilon < 1 \), the stationary problem corresponding to the initial-value problem (1.1) takes the weak form

\[
u^\epsilon \in V^\epsilon \cap \mathcal{H}^1(\Omega^\epsilon): \left\{ \begin{array}{l}
\int_{\Omega_1} \lambda m_1 u^\epsilon \cdot \nabla v dx + \int_{\Omega_2} k_1 \nabla u^\epsilon \cdot \nabla v dx + \int_{\Omega_2^\epsilon} \lambda m_2 u^\epsilon \cdot \nabla v dx + \int_{\Omega_2^\epsilon} \frac{k_2}{\epsilon} \nabla u^\epsilon \cdot \nabla v dx = 0,
\end{array} \right.
\]

(3.5)

in the space \( V^\epsilon = \{v \in H^1(\Omega^\epsilon): v = 0 \text{ on } \partial \Omega_1 - \Gamma \} \). This is the exact or \( \epsilon \)-problem to be solved, and it depends on the thin domain \( \Omega_2^\epsilon \) and the high permeability \( \frac{k_2}{\epsilon} \) through the scale parameter \( \epsilon > 0 \). We expect the last term on the left side to be approximated for small values of \( \epsilon \) by averaging across the narrow channel,

\[
\frac{1}{\epsilon} \int_{\Omega_2^\epsilon} k_2 \nabla u \cdot \nabla v dx \approx \int_{\Gamma} \frac{1}{\epsilon} \int_{\lambda} k_2 \tilde{\nabla} u \cdot \tilde{\nabla} v \omega(\tilde{x}) d\tilde{x},
\]

(3.6)

where \( \tilde{\nabla} \) denotes the gradient in the variable \( \tilde{x} \) in \( \Gamma \), and this will be established in our work below.

3.1. The scaled problem

Since our primary interest is the dependence of the solution on \( \epsilon \), we shall reformulate the problem in a space that is independent of this parameter. In order to eliminate this dependence on the domain, we scale \( \Omega_2^\epsilon \) in the direction normal to \( \Gamma \) by \( x_N = \epsilon z \) to get an equivalent problem on the domain \( \Omega = \Omega_1 \cup \Gamma \cup \Omega_2 \) with \( \Omega_2 \equiv \Omega_2^1 = \{(\tilde{x}, \omega(\tilde{x}) z) \in \mathbb{R}^d: (\tilde{x}, z) \in \Gamma \times (0, 1)\} \). The corresponding bilinear form is

\[
a^\epsilon(u, v) = \int_{\Omega_1} k_1 \nabla u \cdot \nabla v dx + \int_{\Omega_2^\epsilon} k_2 \tilde{\nabla} u \cdot \tilde{\nabla} v \omega(\tilde{x}) d\tilde{x} + \int_{\Omega_2^\epsilon} \frac{k_2}{\epsilon} \tilde{\nabla} u \cdot \tilde{\nabla} v d\tilde{x} dz.
\]

(3.7)

This form is continuous on \( V = \{v \in H^1(\Omega): v = 0 \text{ on } \partial \Omega_1 - \Gamma\} \), and the scaled problem is

\[
u^\epsilon \in V \cap \mathcal{H}^1(\Omega): \left\{ \begin{array}{l}
\int_{\Omega_1} \lambda m_1 u^\epsilon \cdot \nabla v dx + \epsilon \int_{\Omega_2^\epsilon} \lambda m_2 u^\epsilon \cdot \nabla v d\tilde{x} dz + a^\epsilon(u^\epsilon, v) = 0, \\
\int_{\Omega_1} \lambda m_1 F v dx + \epsilon \int_{\Omega_2^\epsilon} \lambda m_2 F v d\tilde{x} dz + \int_{\Gamma} g y(v) d\tilde{x}, \quad \forall v \in V.
\end{array} \right.
\]

(3.8)

For each \( \epsilon > 0 \) the bilinear form (3.7) is clearly \( V \)-elliptic, so the problem (3.8) is well-posed. Moreover, the solution \( u^\epsilon \) satisfies
\[ \lambda m_1 u_1^\epsilon - \nabla \cdot k_1 \nabla u_1^\epsilon = \lambda m_1 F \quad \text{in } \Omega_1, \]
\[ u_1^\epsilon = 0 \quad \text{on } \partial \Omega_1 - \Gamma, \]
\[ u_1^\epsilon = u_2^\epsilon, \quad k_1 \partial_{\nu} u_1^\epsilon - \frac{k_2}{\epsilon^2} \partial_{\nu} u_2^\epsilon = g \quad \text{on } \Gamma, \]
\[ \epsilon \lambda m_2 u_2^\epsilon - \tilde{\nabla} \cdot k_2 \tilde{\nabla} u_2^\epsilon - \frac{k_2}{\epsilon^2} \partial_{\nu} u_2^\epsilon = \epsilon \lambda m_2 F \quad \text{in } \Omega_2, \]
\[ \left( \tilde{\nabla} u_2^\epsilon, \frac{1}{\epsilon^2} \partial_{\nu} u_2^\epsilon \right) \cdot \hat{n} = 0 \quad \text{on } \partial \Omega_2 - \Gamma. \] (3.9)

This is the stationary form of the interface problem (1.1) after the rescaling. Here we see the role of the effective tangential permeability \( k_2 \) and the effective normal permeability \( \frac{k_2}{\epsilon^2} \).

**The estimates.** Denote by \( \chi_j \) the characteristic function of \( \Omega_j \), \( j = 1, 2 \), and set \( u^\epsilon = u_1^\epsilon \chi_1 + u_2^\epsilon \chi_2 \). Due to the boundary conditions of the space \( V \), the gradient controls the entire \( H^1(\Omega) \) norm on \( V \). Testing (3.8) with \( v = u^\epsilon \), we obtain
\[ C_1 \left( \| u_1^\epsilon \|_{0, \Omega_1}^2 + \| u_2^\epsilon \|_{0, \Omega_2}^2 \right) \leq C_2 \left( \| u_1^\epsilon \|_{0, \Omega_1}^2 + \| \nabla u_1^\epsilon \|_{0, \Omega_1}^2 + \epsilon \| u_2^\epsilon \|_{0, \Omega_2}^2 + \| \tilde{\nabla} u_2^\epsilon \|_{0, \Omega_2}^2 + \| \frac{1}{\epsilon} \partial_{\nu} u_2^\epsilon \|_{0, \Omega_2}^2 \right) \]
\[ \leq \| F \|_{0, \Omega_1} \| u_1^\epsilon \|_{0, \Omega_1} + \| g \|_{0, \Omega_1} \| u_2^\epsilon \|_{1, \Omega_1} \leq \tilde{C} \| u_1^\epsilon \|_{1, \Omega_1} \] where \( C_1, C_2, \tilde{C} \) are positive constants. It follows that
\[ \| u_1^\epsilon \|_{0, \Omega_1}^2 + \| \nabla u_1^\epsilon \|_{0, \Omega_1}^2 + \epsilon \| u_2^\epsilon \|_{0, \Omega_2}^2 + \| \tilde{\nabla} u_2^\epsilon \|_{0, \Omega_2}^2 + \| \frac{1}{\epsilon} \partial_{\nu} u_2^\epsilon \|_{0, \Omega_2}^2 \leq C \] (3.10)
for some generic positive constant \( C \).

**The limit.** The estimate (3.10) implies that there is a subsequence, which we denote again by \( \{ u^\epsilon \} \), and a \( u^* = u_1^* \chi_1 + u_2^* \chi_2 \in V \) such that \( u^\epsilon \rightharpoonup u^* \) in \( H^1(\Omega) \) and strongly in \( L^2(\Omega) \). For any \( v \in V \), as \( \epsilon \to 0 \) we have
\[ \int_{\Omega_1} k_1 \nabla u_1^\epsilon \cdot \nabla v \, dx \to \int_{\Omega_1} k_1 \nabla u_1^* \cdot \nabla v \, dx, \quad \text{and} \]
\[ \int_{\Omega_2} k_2 \tilde{\nabla} u_2^\epsilon \cdot \tilde{\nabla} v \, d\tilde{x} \to \int_{\Omega_2} k_2 \tilde{\nabla} u_2^* \cdot \tilde{\nabla} v \, d\tilde{x}. \]

Since the right side of (3.8) is bounded for \( v \in V \) fixed, we conclude the existence of the limit
\[ \ell(v) = \lim_{\epsilon \to 0} \int_{\Omega_2} \frac{k_2}{\epsilon^2} \partial_{\nu} u_2^\epsilon \partial_{\nu} v \, d\tilde{x} \, d\tilde{z}, \]
and due to the a priori estimates we conclude \( \ell \in V' \). In addition, there must exist \( \zeta \in L^2(\Omega_2) \) such that \( \epsilon^{-1} \partial_{\nu} u_2^\epsilon \rightharpoonup \zeta \) in \( L^2(\Omega_2) \). Also \( \| \partial_{\nu} u_2^\epsilon \|_{0, \Omega_2} \leq \epsilon C \), so \( \| \partial_{\nu} u_2^\epsilon \|_{0, \Omega_2} \to 0 \), and we know \( \partial_{\nu} u_2^\epsilon \rightharpoonup \partial_{\nu} u_2 \) in \( L^2(\Omega_2) \), so \( \partial_{\nu} u_2 \equiv 0 \) and \( u_2 \) is independent of \( z \) in \( \Omega_2 \). Taking the limit in (3.8), we find that \( u^* = u_1^* \chi_1 + u_2^* \chi_2 \) satisfies
\[ u^* \in V: \quad \partial_{\nu} u_2 = 0 \quad \text{in } \Omega_2, \quad \text{and} \]
\[ \int_{\Omega_1} \lambda m_1 u_1^* v \, dx + \int_{\Omega_1} k_1 \nabla u_1^* \cdot \nabla v \, dx + \int_{\Omega_2} k_2 \tilde{\nabla} u_2^* \cdot \tilde{\nabla} v \, d\tilde{x} + \ell(v) = \int_{\Omega_1} \lambda m_1 F v \, dx + \int_{\Gamma} g(v) \, d\tilde{x}, \quad \forall v \in V. \] (3.11)

Define now the subspace \( W = \{ v \in V: \partial_{\nu} v = 0 \quad \text{on } \Omega_2 \} \). We have shown that for some subsequence we obtain a weak limit, \( u^* \rightharpoonup u^* \) in \( V \) with \( u^* \in W \), and since the linear functional \( \ell(\cdot) \) vanishes on \( W \), this limit satisfies
\[ u^* \in W: \quad \int_{\Omega_1} \lambda m_1 u^* v \, dx + a(\lambda^*, v) = \int_{\Omega_1} \lambda m_1 F v \, dx + \int_{\Gamma} g(v) \, d\tilde{x} \quad \text{for all } v \in W, \] (3.12)
where the limit bilinear form on $W$ is defined by

$$ a^0(u, v) = \int_{\Omega_1} k_1 \nabla u \cdot \nabla v \, dx + \int_{\Omega_2} k_2 \tilde{\nabla} u \cdot \tilde{\nabla} v \, d\tilde{z}. $$

(3.13)

This continuous bilinear form is $W$-elliptic, so we see that $u^*$ is the only solution and the original sequence $\{u^\epsilon\}$ converges weakly to $u^*$. In summary, the problem (3.12) characterizes the limit $u^*$ of the stationary problems (3.8).

### 3.2. Strong convergence

On the space $V$ we take the scalar product

$$ \langle v, w \rangle = \int_{\Omega_1} k_1 \nabla v \cdot \nabla w \, dx + \int_{\Omega_2} k_2 \tilde{\nabla} v \cdot \tilde{\nabla} w \, dx. $$

(3.14)

This scalar product $\langle \cdot, \cdot \rangle$ is equivalent to the usual $H^1(\Omega)$ scalar product, that is, the $V$-norm $\|v\|_V = \langle v, v \rangle^{1/2}$ is equivalent to the $H^1(\Omega)$ norm, so from the weak convergence $u^\epsilon \rightharpoonup u^*$ in $H^1(\Omega)$ we know

$$ \|u^*\|_V \leq \liminf_{\epsilon \downarrow 0} \|u^\epsilon\|_V. $$

Now, for $0 < \epsilon \leq 1$, the solution $u^\epsilon$ of (3.8) satisfies

$$ \|u^\epsilon\|_V^2 \leq \epsilon \int_{\Omega_2} \lambda m_2(u^\epsilon)^2 \, d\tilde{z} \, dz + \epsilon \langle u^\epsilon, u^\epsilon \rangle = -\int_{\Omega_1} \lambda m_1(u^\epsilon)^2 \, dx + \int_{\Omega_2} \lambda m_1 Fu^\epsilon \, dx + \int_{\Gamma} \epsilon m_2 Fu^\epsilon \, d\tilde{z} + \int_{\Gamma} g\gamma u^\epsilon \, d\tilde{x}, $$

so from weak lower-semicontinuity of the first term we obtain

$$ \limsup_{\epsilon \downarrow 0} \|u^\epsilon\|_V^2 \leq -\int_{\Omega_1} \lambda m_1 (u^*)^2 \, dx + \int_{\Omega_2} \lambda m_1 Fu^* \, dx + \int_{\Gamma} g\gamma (u^*) \, d\tilde{x}. $$

But with (3.12) this gives

$$ \limsup_{\epsilon \downarrow 0} \|u^\epsilon\|_V^2 \leq a^0(u^*, u^*) = \|u^*\|_V^2, $$

so $\lim_{\epsilon \downarrow 0} \|u^\epsilon\|_V = \|u^*\|_V$. Together with the weak convergence of the sequence, this implies $\|u^\epsilon - u^*\|_V \to 0$, and so we have strong convergence $u^\epsilon \to u^*$ in $H^1(\Omega)$.

### An alternative system

The solution of the limiting problem can be characterized by a boundary-value problem on $\Omega_1$ and $\Gamma$. First we rewrite (3.11). Since $C_0^\infty(\Omega_1) \subseteq V$, for any $\varphi \in C_0^\infty(\Omega_1)$ we obtain

$$ \int_{\Omega_1} \lambda m_1 u_1 \varphi \, dx + \int_{\Omega_1} k_1 \nabla u_1 \cdot \nabla \varphi \, dx = \int_{\Omega_1} \lambda m_1 F \varphi \, dx, $$

i.e., $\lambda m_1 u_1 - \nabla \cdot k_1 \nabla u_1 = \lambda m_1 F \in L^2(\Omega_1)$, so $k_1 \nabla u_1 \in L^2_{div}(\Omega_1)$ and the normal trace $k_1 \nabla u_1 \cdot \hat{n} \in H^{-1/2}(\partial \Omega_1)$ is well defined. Moreover, we know that for any $v \in V$ the Stokes’ formula [15]

$$ \langle k_1 \nabla u_1 \cdot \hat{n}, \gamma \nu \rangle_{H^{-1/2}(\partial \Omega_1), H^{1/2}(\partial \Omega_1)} = \int_{\Omega_1} k_1 \nabla u_1 \cdot \nabla \nu \, dx + \int_{\Gamma} \nabla \cdot (k_1 \nabla u_1) \nu \, d\tilde{x} $$

must hold. Substituting these into (3.11), we conclude

$$ \langle k_1 \nabla u_1 \cdot \hat{n}, \gamma \nu \rangle + \int_{\Omega_2} k_2 \tilde{\nabla} u_2 \cdot \tilde{\nabla} \nu \, dx + \int_{\Gamma} g\gamma \nu \, d\tilde{x} \quad \text{for all } v \in V. $$

(3.15)

Since the functions in $W$ are independent of $z$ for $(\hat{x}, z) \in \Omega_2$, we have for each pair $u, v \in W$

$$(u, v)_{H^1(\Omega_2)} = \int_{\Omega_2} \int_{\partial \Omega_2} (u(\hat{x})v(\hat{x}) + \nabla u(\hat{x}) \cdot \nabla v(\hat{x})) \, d\tilde{x} \, dz = \int_{\Gamma} (u(\hat{x})v(\hat{x}) + \tilde{\nabla} u(\hat{x}) \cdot \tilde{\nabla} v(\hat{x})) \omega(\hat{x}) \, d\tilde{x}. $$
This is equivalent to the scalar product

\[(u, v)_{H^1_0(\Gamma)} = \int_{\Gamma} (u(\vec{x})v(\vec{x}) + \omega(\vec{x})\tilde{\nabla}u(\vec{x}) \cdot \tilde{\nabla}v(\vec{x})) d\vec{x}\]

of the weighted Sobolev space

\[H^1_0(\Gamma) = \{ u \in L^2(\Gamma) : \omega^{1/2} \tilde{\nabla}u \in L^2(\Gamma) \} \]

Furthermore, we see \( W \) is equivalent to the space

\[V_\Gamma = \{ v \in H^1(\Omega_1) : v|_{\Gamma} \in H^1_0(\Gamma), v|_{\partial \Omega_1 - \Gamma} = 0 \} \]

in the sense of boundary trace. Thus, the solution of problem (3.12) is characterized by

\[u^* \in V_\Gamma : \int_{\Omega_1} \lambda_1 u^* v \, dx + \int_{\Gamma} k_1 \tilde{\nabla}u^* \cdot \tilde{\nabla}v \, d\vec{x} + \int_{\Gamma} k_2 \omega \tilde{\nabla}u^* \cdot \tilde{\nabla}v \, d\vec{x} = \int_{\Omega_1} \lambda_1 F v \, dx + \int_{\Gamma} g \gamma(v) \, d\vec{x} \quad \text{for all} \; v \in V_\Gamma, \]

(3.16)

and this means it determines a pair \( u_1 = X_1 u^* \in H^1(\Omega_1), u_2 = \gamma(u^*) \in H^1_0(\Gamma) \) which satisfies the system

\[
\begin{align*}
\lambda_1 u_1 - \nabla \cdot k_1 \nabla u_1 &= \lambda_1 F, & \text{in} \; \Omega_1, \\
u_1 &= 0 & \text{on} \; \partial \Omega_1 - \Gamma, \\
u_1 &= u_2 & \text{on} \; \Gamma, \quad \text{and} \\
\int_{\Gamma} k_2 \omega \nabla u_2 \cdot \nabla \gamma v \, d\vec{x} + (k_1 \nabla u_1 \cdot \vec{n}, \gamma v)_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} &= \int_{\Gamma} g \gamma v \, d\vec{x} & \text{for all} \; v \in V.
\end{align*}
\]

(3.17a) - (3.17c)

In the situation of Theorem 2.1, the variational identity (3.17d) is equivalent to

\[
\begin{align*}
-\tilde{\nabla} \cdot k_2 \omega \tilde{\nabla} u_2 + k_1 \partial_2 u_1 &= g & \text{in} \; \Gamma, \\
u_2 &= 0 & \text{on} \; \partial \Gamma.
\end{align*}
\]

(3.17e) - (3.17f)

However, in the strongly tapered case of (2.3), the last condition (3.17f) is deleted, since the trace is meaningless and the variational equation is equivalent to Eq. (3.17e) in \( H^1_0(\Gamma) \). See [12] for such examples. Thus, the limiting form of the singular problem (3.8) is the elliptic boundary-value problem on \( \Omega_1 \) with the (non-local and possibly degenerate) elliptic boundary constraint. We summarize the above as follows.

**Theorem 3.1.** Let the regions \( \Omega^\epsilon \) and the rescaled \( \Omega \), the constants \( k_1, k_2, m_1, m_2 > 0, \lambda \geq 0 \), and functions \( F \in L^2(\Omega), g \in L^2(\Gamma) \) be given. Define the bilinear form (3.7) for each \( 0 < \epsilon \leq 1 \) on the space \( V \). Then each scaled problem (3.8) has a unique solution, \( u^\epsilon \), these satisfy the estimates (3.10) and converge strongly \( u^\epsilon \to u^* \) in \( V \), where \( u^* \) satisfies (3.11). Finally, the limit \( u^* \) is characterized as the solution of the well-posed limit problem (3.12) or its equivalent form (3.16).

### 3.2.1. Remarks on minimization and penalty

Set \( f^\epsilon(v) = \int_{\Omega_1} \lambda_1 m_1 F v \, dx + \epsilon \int_{\Omega_2} \lambda_2 m_2 F v \, dx + \int_{\Gamma} g \gamma(v) \, d\vec{x} \). Eq. (3.8) shows that \( u^\epsilon \) is characterized by the minimization of

\[
\phi^\epsilon(v) = \frac{1}{2} \left( \int_{\Omega_1} \lambda_1 v^2 \, dx + \int_{\Omega_2} \epsilon \lambda_2 m_2 v^2 \, dx + a^\epsilon(v, v) \right) - f^\epsilon(v), \quad v \in V.
\]

According to (3.11), the limit \( u^* \) satisfies

\[
u^* \in W : \int_{\Omega_1} \lambda_1 m_1 u^* v \, dx + (u^*, v)_V + \ell(v) = f^0(v) \quad \text{for all} \; v \in V
\]

and is characterized by (3.12), that is,

\[
u^* \in W : \int_{\Omega_1} \lambda_1 m_1 u^* v \, dx + (u^*, v)_V = f^0(v) \quad \text{for all} \; v \in W.
\]
This shows that \( u^* \) is obtained by the minimization of
\[
\varphi(v) = \frac{1}{2} \left( \frac{\lambda m_1 v^2}{\Omega_1} + \langle v, v \rangle_{\mathcal{V}} - f^0(v) \right), \quad v \in \mathcal{V},
\]
over the subspace \( \mathcal{W} \). This is the same as minimizing \( \varphi(v) + I_{\mathcal{W}}(v) \) over all of \( \mathcal{V} \), where
\[
I_{\mathcal{W}}(v) = \begin{cases} 
0 & \text{if } v \in \mathcal{W}, \\
+\infty & \text{if } v \notin \mathcal{W},
\end{cases}
\]
is the indicator function of \( \mathcal{W} \).

Furthermore, if \( \partial I_{\mathcal{W}}(v) \) denotes the subgradient of the convex \( I_{\mathcal{W}}(v) \), then \( \ell \in \partial I_{\mathcal{W}}(u^*) \) is the Lagrange multiplier that realizes the constraint \( u^* \in \mathcal{W} \). The last term in (3.7) is the penalty term and (3.8) is a penalty method to approximate (3.12).

### 3.3. The concentrated capacity model

Suppose that in the interface problem (1.1), we assume that not only the permeability \( k_2 \) but also \( m_2 \) is scaled by \( \frac{1}{\epsilon} \) in \( \Omega_2 \). Such an assumption is meaningless for porous media, since the porosity is bounded by 1, but it is appropriate in analogous heat conduction problems with a concentrated capacity along the highly-conducting interface or boundary.

However, the problem (3.8) with the factor \( \epsilon \) deleted from the two terms can be used as a fracture model with highly anisotropic permeability. We include this case to show what assumptions are required to arrive at the narrow fracture model described in [2].

**Theorem 3.2.** Let the region \( \Omega \), the constants \( k_1, k_2, \lambda m_1 > 0 \), and functions \( F \in L^2(\Omega) \), \( g \in L^2(\Gamma) \) be given. For each \( 0 < \epsilon \leq 1 \), consider the problem

\[
\begin{align*}
u^\epsilon & \in \mathcal{V} : \\
& \int_{\Omega_1} \lambda m_1 u^\epsilon v \, dx + \int_{\Omega_2} \lambda m_2 u^\epsilon v \, dx + a^\epsilon(u^\epsilon, v) \\
& = \int_{\Omega_1} \lambda m_1 F v \, dx + \int_{\Omega_2} \lambda m_2 F v \, dx + \int_{\Gamma} g \gamma(v) \, d\tilde{x}, \quad \forall \mathcal{V} \in \mathcal{V}.
\end{align*}
\]

This problem has a unique solution, \( u^\epsilon \), these satisfy the estimates (3.10) and converge strongly \( u^\epsilon \to u^* \) in \( \mathcal{V} \), where the limit \( u^* \) satisfies

\[
\begin{align*}
u^* & \in \mathcal{W} : \\
& \int_{\Omega_1} \lambda m_1 u^* v \, dx + \int_{\Gamma} \lambda m_2 \omega u^* v \, d\tilde{x} + \int_{\Gamma} k_1 \nabla u^* \cdot \nabla v \, dx + \int_{\Gamma} k_2 \omega \nabla u^* \cdot \nabla v \, d\tilde{x} \\
& = \int_{\Omega_1} \lambda m_1 F v \, dx + \int_{\Gamma} \lambda m_2 \hat{F} v \, d\tilde{x} + \int_{\Gamma} g \gamma(v) \, d\tilde{x} \quad \text{for all } v \in \mathcal{W},
\end{align*}
\]

and the channel average of \( F \) in \( \Omega_2 \) is given by
\[
\hat{F}(\tilde{x}) = \frac{1}{\omega(\tilde{x})} \int_0^{\omega(\tilde{x})} F(\tilde{x}, z) \, dz, \quad \tilde{x} \in \Gamma.
\]

Note as before that the limit \( u^* \in \mathcal{V}_\Gamma \) determines a pair \( u_1 \in H^1(\Omega_1), u_2 \in H^1_\omega(\Gamma) \) which satisfies

\[
\begin{align*}
\lambda m_1 u_1 - \nabla \cdot k_1 \nabla u_1 &= \lambda m_1 F \quad \text{in } \Omega_1, \\
u_1 &= 0 \quad \text{on } \partial \Omega_1 - \Gamma, \quad (3.21a) \\
u_1 &= u_2 \quad \text{on } \Gamma, \quad \text{and} \\
\int_{\Gamma} \lambda m_2 \omega u_2 v \, d\tilde{x} + \int_{\Gamma} k_2 \omega \nabla u_2 \cdot \nabla v \, d\tilde{x} + \langle k_1 \nabla u_1 \cdot \hat{n}, \gamma v \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} \\
&= \int_{\Gamma} \lambda m_2 \hat{F} v \, d\tilde{x} + \int_{\Gamma} g v \, d\tilde{x} \quad \text{for all } v \in \mathcal{V}_\Gamma.
\end{align*}
\]
In the weakly tapered situation of Theorem 2.1, the variational identity is equivalent to

$$\begin{align*}
\lambda m_2 \omega u_2 - \nabla \cdot k_2 \omega \nabla u_2 + k_1 \partial_t u_1 &= \lambda m_2 \omega F + g \quad \text{in } \Gamma, \\
u_2 &= 0 \quad \text{on } \partial \Gamma.,
\end{align*}$$

(3.21d)

(3.21e)

and in the strongly tapered case of (2.3), the last condition (3.21e) is deleted.

4. The evolution problems

We apply Theorem 3.1 to show the dynamics of the initial-boundary-value problem (1.1) is governed by an analytic semigroup on the Hilbert space $H = L^2(\Omega)$, and the limiting form corresponds similarly to an analytic semigroup on the Hilbert space $H_0 = L^2(\Omega_1)$. Then we establish the convergence as $\epsilon \to 0$ of solutions of the corresponding evolution problems.

4.1. Well-posed problems

Let $H_{\epsilon}$ denote $H$ with the norm $\|u\|_{H_{\epsilon}} = \|m_1^{1/2} \chi_1 u + (\epsilon m_2)^{1/2} \chi_2 u\|_{L^2(\Omega)}$, so its Riesz map is the multiplication function $m_{\epsilon} = m_1 \chi_1 + \epsilon m_2 \chi_2$ from $H_{\epsilon}$ to $H_0$. Similarly, $m_0 = m_1$ is the Riesz map from $H_0$ to $H_0$, where $\|u\|_{H_0} = \|m_1^{1/2} u\|_{L^2(\Omega)}$. Note that $V \subset H_{\epsilon}$ and $W \subset H_0$ are dense and continuous inclusions.

Define the operators $A^\epsilon : D^\epsilon \to H_{\epsilon}$ with domains $D^\epsilon \subset V$ by $u^\epsilon \in D^\epsilon$ and $A^\epsilon(u^\epsilon) = F \in H_0'$ if $u^\epsilon \in V$: $A^\epsilon(u^\epsilon, v) = F(v)$ for all $v \in V$. Similarly the operator $A_0^0 : D_0 \to H_0$ with domain $D_0 \subset W$ is determined by $u^0 \in D_0$ and $A_0^0(u^0) = F \in H_0'$ if $u^0 \in W$: $A_0^0(u^0, v) = F(v)$ for all $v \in W$. If we set $\epsilon = 0$, then the scaled problem (3.8) is equivalent to $A^\epsilon(u^\epsilon) = \lambda m_{\epsilon}(F - u^\epsilon)$ for $F \in H_0$, and the limit problem (3.12) is equivalent to $A_0^0(u^*) = \lambda m_0(F - u^*)$ for $F \in H_0$.

Each of the operators $m_{\epsilon}^{-1}A^\epsilon$ is $m$-accretive on $H_{\epsilon}$, that is, $\|((I + \epsilon m_{\epsilon}^{-1}A^\epsilon)^{-1}F\|_{H_{\epsilon}} \leq \|F\|_{H_{\epsilon}}$ for each $\epsilon > 0$ and $F \in H_{\epsilon}$. Likewise $(I + \epsilon m_0^{-1}A_0)^{-1}$ is a contraction on $H_0$ for each $\epsilon > 0$. These operators are also self-adjoint, since the corresponding bilinear forms are symmetric, so $m_{\epsilon}^{-1}A^\epsilon$ and $m_0^{-1}A_0$ generate analytic semigroups on $H_{\epsilon}$ and $H_0$, respectively.

The Hille–Yoshida Theorem 2.3 shows that the corresponding initial-value problems are well-posed. Applying it to the operator $m_{\epsilon}^{-1}A^\epsilon$ in $H_{\epsilon}$, we obtain the scaled problem.

**Theorem 4.1.** For every $u_0 \in L^2(\Omega)$ and $F \in C^0([0, \infty), L^2(\Omega))$, there is a unique $u^\epsilon \in C([0, \infty), L^2(\Omega)) \cap C^1((0, \infty), L^2(\Omega))$ with $u^\epsilon(t) \in D^\epsilon$ for each $t > 0$ such that $u^\epsilon(t) = \chi_1 u^\epsilon_1(t) + \chi_2 u^\epsilon_2(t)$ satisfies the scaled problem

$$\begin{align*}
\frac{\partial u^\epsilon_1}{\partial t} &- \nabla \cdot k_1 \nabla u^\epsilon_1 = m_1 F \quad \text{in } \Omega_1, \\
u^\epsilon_1 &= 0 \quad \text{on } \partial \Omega_1 - \Gamma, \\
u^\epsilon_1 &= u^\epsilon_2, \\
\frac{\partial u^\epsilon_2}{\partial t} &- \nabla \cdot k_2 \nabla u^\epsilon_2 = \frac{k_2}{\epsilon^2} \partial_\Gamma \partial_\epsilon \partial_\gamma u^\epsilon_2 = \epsilon m_2 F \quad \text{in } \Omega_2, \\
\left(k_2 \nabla u^\epsilon_2, \frac{k_2}{\epsilon^2} \partial_\Gamma \partial_\epsilon \partial_\gamma u^\epsilon_2\right) \cdot \hat{n} &= 0 \quad \text{on } \partial \Omega_2 - \Gamma, \\
\left.u^\epsilon_1(-, 0)\right|_{\Omega_1} &= u_0(\cdot) \quad \text{on } \Omega_1, \\
\left.u^\epsilon_2(-, 0)\right|_{\Omega_2} &= u_0(\cdot) \quad \text{on } \Omega_2.
\end{align*}$$

(4.22a)

at each $t > 0$, and these satisfy the initial conditions

$$\begin{align*}
u^\epsilon_1(-, 0) &= u_0(\cdot) \quad \text{on } \Omega_1, \\
u^\epsilon_2(-, 0) &= u_0(\cdot) \quad \text{on } \Omega_2.
\end{align*}$$

(4.22b)

Note that this is a rather strong solution, since $\nabla \cdot k_j \nabla u^\epsilon_j(t) \in L^2(\Omega_j)$ for each $t > 0, j = 1, 2$.

Similarly from the operator $m_0^{-1}A_0$ in $H_0$ we obtain the limiting problem. When the fracture is weakly tapered, this takes the following form.

**Theorem 4.2.** For every $u_0 \in L^2(\Omega_1)$ and $F \in C^0([0, \infty), L^2(\Omega_1))$, there is a unique $u^* \in C([0, \infty), L^2(\Omega_1)) \cap C^1((0, \infty), L^2(\Omega_1))$ with $u^*(t) \in D_0$ for each $t > 0$ such that the functions $u_1(t) = u^*(t)|_{\Omega_1} \in H^1(\Omega_1)$, $u_2(t) = \gamma(u^*(t)) \in H^1_0(\Gamma)$ satisfy

$$\begin{align*}
\frac{\partial u_1}{\partial t} &- \nabla \cdot k_1 \nabla u_1 = m_1 F \quad \text{in } \Omega_1, \\
u_1 &= 0 \quad \text{on } \partial \Omega_1 - \Gamma, \\
u_1 &= u_2 \quad \text{on } \Gamma, \quad \text{and} \\
u_1 &+ k_1 \partial_\gamma u_1 = 0 \quad \text{in } \Gamma, \\
u_2 &= 0 \quad \text{on } \partial \Gamma.
\end{align*}$$

(4.23a)

(4.23b)

(4.23c)

(4.23d)

(4.23e)
at each \( t > 0 \) and the initial condition

\[
 u_1(\cdot, 0) = u_0(\cdot) \quad \text{on } \Omega_1.
\] (4.23f)

In particular, each term of Eq. (4.23a) belongs to \( L^2(\Omega_1) \), so the solution is rather strong. As before, in the strongly tapered case, the last condition (4.23e) is deleted.

### 4.2. Strong convergence

For the stationary problems, we have shown that \((m_\epsilon + A^\epsilon)^{-1}m_\epsilon F \to (m_0 + A^0)^{-1}m_0 F\) in the \( V \)-norm, hence, in \( H^1(\Omega) \) so also in \( H \). However, for the corresponding dynamic problems, with \( \epsilon > 0 \) we have an evolution in \( H_\epsilon = L^2(\Omega) \) whereas the limit is an evolution in \( H_0 = L^2(\Omega_1) \), and these are not immediately comparable, so we shall work directly in the corresponding evolution spaces, \( \mathcal{V} \equiv L^2(0, T; V) \) and \( \mathcal{W} \equiv L^2(0, T; W) \). The Cauchy problem leads to the Hilbert space

\[
 \mathcal{W}^{1,2}(0, T) \equiv \left\{ u \in \mathcal{V} : \frac{du}{dt} \in \mathcal{V}' \right\}
\]

with the norm \( \|u\|_{\mathcal{W}^{1,2}(0, T)} = (\|u\|^2_{\mathcal{V}} + \|\frac{du}{dt}\|^2_{\mathcal{V}'})^{1/2} \), and this space is contained in \( C([0, T], H) \) with continuous imbedding, that is,

\[
 \|u\|_{C([0, T], H)} \leq C \|u\|_{\mathcal{W}^{1,2}(0, T)}, \quad u \in \mathcal{W}^{1,2}(0, T).
\]

See any one of \([1,14,15]\).

The solution of (4.22) satisfies

\[
 u^\epsilon \in \mathcal{V} : \forall v \in \mathcal{V} \cap \mathcal{W}^{1,2}(0, T; H) \text{ with } v(T) = 0,
\]

\[
 -\int_0^T \left( m_\epsilon u^\epsilon(t), \frac{dv}{dt}(t) \right)_{L^2(\Omega)} dt + \int_0^T a^\epsilon(u^\epsilon(t), v(t)) dt = \int_0^T (m_\epsilon F(t), v(t))_{L^2(\Omega)} dt + (m_\epsilon u_0, v(0))_{L^2(\Omega)}.
\]

This is the weak formulation of the Cauchy problem

\[
 u^\epsilon \in \mathcal{V} : \forall v \in \mathcal{V} \cap \mathcal{W}^{1,2}(0, T; H) \text{ with } v(T) = 0,
\]

and the solution \( u^\epsilon \) satisfies the identity

\[
 \frac{1}{2}(m_\epsilon u^\epsilon(T), u^\epsilon(T))_{L^2(\Omega)} + \int_0^T a^\epsilon(u^\epsilon(t), u^\epsilon(t)) dt = \int_0^T (m_\epsilon F(t), u^\epsilon(t))_{L^2(\Omega)} dt + \frac{1}{2}(m_\epsilon u_0, u_0)_{L^2(\Omega)}.
\] (4.24)

This implies that \( \|u^\epsilon\|_{\mathcal{V}}, \|\frac{du^\epsilon}{dt}\|_{L^2(0,T;H_0)} \) are bounded, so there is a weakly convergent subsequence, \( u^\epsilon \rightharpoonup u^* \) in \( \mathcal{V} \) with limit \( u^* \in \mathcal{W} \). Then the evolution equation shows that \( \frac{du^*}{dt} \rightharpoonup \frac{du^*}{dt} \) in \( \mathcal{V}' \), so we obtain

\[
 u^* \in \mathcal{W} : \forall v \in \mathcal{V} \cap \mathcal{W}^{1,2}(0, T; H_0) \text{ with } v(T) = 0,
\]

\[
 -\int_0^T \left( m_0 u^*(t), \frac{dv}{dt}(t) \right)_{L^2(\Omega_1)} dt + \int_0^T a^0(u^*(t), v(t)) dt = \int_0^T (m_0 F(t), v(t))_{L^2(\Omega_1)} dt + (m_0 u_0, v(0))_{L^2(\Omega_1)}.
\]

As before, this characterizes the solution of

\[
 u^* \in \mathcal{W} : m_0 u^*(T) + A^0(u^*(T)) = m_0 F(\cdot) \quad \text{in } \mathcal{V}', \quad u^*(0) = \chi_1 u_0,
\]

which has only one solution \([11]\), so the original sequence converges weakly to \( u^* \) and this is also the solution of (4.23). Moreover, we have

\[
 \frac{1}{2}(m_0 u^*(T), u^*(T))_{L^2(\Omega_1)} + \int_0^T a^0(u^*(t), u^*(t)) dt = \int_0^T (m_0 F(t), u^*(t))_{L^2(\Omega_1)} dt + \frac{1}{2}(m_0 u_0, u_0)_{L^2(\Omega_1)}.
\] (4.25)
and this will be used to show strong convergence $u^\epsilon \to u^*$ in $V$. From the weak convergence, we have

$$\int_0^T [u^\epsilon(t), u^*(t)] dt \leq \liminf_{\epsilon \downarrow 0} \int_0^T [u^\epsilon, u^*] dt.$$  

This follows since the $V$-norm from the scalar product (3.14) is equivalent to the $H^1(\Omega)$-norm. Also from (4.24) we have

$$\int_0^T [u^\epsilon, u^*] dt \leq \int_0^T a^0(u^\epsilon(t), u^*(t)) dt = -\frac{1}{2} \left( m_0 u^\epsilon(T), u^*(T) \right)_{L^2(\Omega)} + \int_0^T [m F(t), u^*(t)]_{L^2(\Omega)} dt + \frac{1}{2} (m_0 u_0, u_0)_{L^2(\Omega)}.$$ 

Then using the (weak) continuity of the linear map $u \to u(T)$ from $\{u \in W: m_0^{1/2} \frac{du}{dt} \in \mathcal{W}'\}$ to $H_0$, we take the lim sup above to get

$$\limsup_{\epsilon \downarrow 0} \int_0^T [u^\epsilon, u^*] dt \leq \int_0^T a^0(u^\epsilon(t), u^*(t)) dt = \int_0^T [u^*(t), u^*(t)] dt,$$

so we have established $\lim_{\epsilon \downarrow 0} \int_0^T [u^\epsilon(t), u^*(t)] dt = \int_0^T (u^\epsilon(t), u^*(t)) dt$ and, hence, strong convergence in $V$. Recalling that from the evolution equation we have the strong convergence $m_0 \frac{du^\epsilon}{dt} \to m_0 \frac{du^*}{dt}$ in $\mathcal{W}'$, we have

**Theorem 4.3.** In the situation of Theorems 4.1 and 4.2, the sequence converges strongly $u^\epsilon \to u^*$ in $V = L^2(0, T; V)$ and in $C([0, T], H_0)$.

### 4.3. The concentrated capacity model

We obtain the analogous results for the evolution problem corresponding to Theorem 3.2. The approximation evolves in $H = L^2(\Omega)$ with the norm $\|u\|_H = \|(m_1^{1/2} \chi_1 + m_2^{1/2} \chi_2) u\|_{L^2(\Omega)}$; its Riesz map is the multiplication function $m_1 \chi_1 + m_2 \chi_2$ from $H$ to $H'$. Similarly, $H_0$ is defined to be the closure of $W$ in $H$, and as above we find it is equivalent to the weighted $L^2$ space with the scalar product

$$(u, v)_{L^2(\Omega)} = \int_{\Omega} m_1 u(x) v(x) dx + \int_{\Gamma} m_2 u(\tilde{\chi}) v(\tilde{\chi}) d\tilde{x}.$$ 

Note that $V \subset H$ and $W \subset H_0$ are dense and continuous inclusions.

By the same arguments given previously, we obtain the following.

**Theorem 4.4.** For every $u_0 \in L^2(\Omega)$ and $F \in C^0([0, \infty), L^2(\Omega))$, there is a unique $u^\epsilon \in C([0, \infty), L^2(\Omega)) \cap C^1((0, \infty), L^2(\Omega))$ with $u^\epsilon(t) \in D^\epsilon$ for each $t > 0$ and $u^\epsilon(t) = \chi_1 u_1^\epsilon(t) + \chi_2 u_2^\epsilon(t)$ satisfies the scaled problem

$$m_1 \frac{\partial u_1^\epsilon}{\partial t} - \nabla \cdot k_1 \nabla u_1^\epsilon = m_1 F \quad \text{in } \Omega_1,$$

$$u_1^\epsilon = 0 \quad \text{on } \partial \Omega_1 - \Gamma,$$

$$u_1^\epsilon = u_2^\epsilon, \quad k_1 \partial_1 u_1^\epsilon - \frac{k_2}{\epsilon} \partial_2 u_2^\epsilon = 0 \quad \text{on } \Gamma,$$

$$m_2 \frac{\partial u_2^\epsilon}{\partial t} - \tilde{\nabla} \cdot k_2 \tilde{\nabla} u_2^\epsilon - \frac{k_2}{\epsilon^2} \partial_2 \partial_2 u_2^\epsilon = m_2 F \quad \text{in } \Omega_2,$$

$$(k_2 \tilde{\nabla} u_2^\epsilon, \frac{k_2}{\epsilon^2} \partial_2 u_2^\epsilon) \cdot \tilde{n} = 0 \quad \text{on } \partial \Omega_2 - \Gamma,$$  

(4.26a)

at each $t > 0$, and these satisfy the initial conditions

$$u_1^\epsilon(\cdot, 0) = u_0(\cdot) \quad \text{on } \Omega_1, \quad u_2^\epsilon(\cdot, 0) = u_0(\cdot) \quad \text{on } \Omega_2.$$  

(4.26b)
Also, there is a unique $u^* \in C([0, \infty), L^2(\Omega)) \cap C^1([0, \infty), L^2(\Omega))$ with $u^*(t) \in D^0$ for each $t > 0$, such that the functions $u_1(t) = u^*(t)|\Omega_1 \in H^1(\Omega_1)$, $u_2(t) = \gamma(u^*(t)) \in H^1_\omega(\Gamma)$ satisfy

\begin{align}
\frac{\partial u_1}{\partial t} - \nabla \cdot k_1 \nabla u_1 &= m_1 F \quad \text{in } \Omega_1, \\
u_1 &= 0 \quad \text{on } \partial \Omega_1 - \Gamma, \\
u_1 &= u_2 \quad \text{on } \Gamma,
\end{align}

and

\begin{align}
\frac{\partial u_2}{\partial t} - \tilde{\nabla} \cdot k_2 \tilde{\nabla} u_2 + k_1 \partial_2 u_1 &= m_2 \omega \tilde{F} \quad \text{in } \Gamma, \\
u_2 &= 0 \quad \text{on } \partial \Gamma,
\end{align}

at each $t > 0$ and the initial condition

\begin{align}
u_1(\cdot, 0) &= u_0(\cdot) \quad \text{on } \Omega_1, \\
u_2(\cdot, 0) &= \tilde{u}_0(\cdot) \quad \text{on } \Gamma.
\end{align}

Finally, we have strong convergence $u^* \to u^*$ in $V = L^2(0, T; V)$ and in $C([0, T], H_0)$.

References