PDE-Models with Hysteresis on the Boundary

Ulrich HORNUNG
Fak. f. Informatik, UniBwM
D-85577 Neubiberg, Germany
ulrich@informatik.unibw-muenchen.de

Ralph E. SHOWALTER
Department of Mathematics
The University of Texas
Austin, TX 78712 U.S.A.
show@math.utexas.edu

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Abstract
A general parabolic equation of the form of the porous media equation is considered with nonlinear boundary conditions that model hysteresis phenomena. Conditions of this type describe certain adsorption processes in porous media. Several examples are given and numerical results are shown.

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1 The Problem
Let a bounded domain Ω in $\mathbb{R}^n$ with smooth boundary Γ be given. We assume that $\Gamma = \Gamma_D \cup \Gamma_H$ and denote the outer normal vector on $\Gamma_H$ by $\nu$. Then we study the
following initial boundary value problem for functions $u$ on $\Omega$ and $v$, $w$ on $\Gamma_H$.

\[
\begin{aligned}
\partial_t a(u) - \Delta u &\geq f & t > 0, x \in \Omega & \quad (a) \\
\partial_t w + \partial_y u & = g, \ w \in b(v) & t > 0, x \in \Gamma_H & \quad (b) \\
\partial_y u & \in c(v - u) & t > 0, x \in \Gamma_H & \quad (c) \\
u & = u_D & t > 0, x \in \Gamma_D & \quad (d) \\
a(u) & = a_I & t = 0, x \in \Omega \\
w & = w_I & t = 0, x \in \Gamma_H
\end{aligned}
\]

Here, the following functions are given: $f$ on $[0,T] \times \Omega$, $g$ on $[0,T] \times \Gamma_H$, $u_D$ on $[0,T] \times \Gamma_D$, $a_I$ on $\Omega$, and $w_I$ on $\Gamma_H$.

Each of $a(.)$, $b(.)$, and $c(.)$ is a maximal monotone graph in $\mathbb{R}^2$, see [5]. The special interest in problem (1) is motivated by the fact that the two conditions (1b,c) on $\Gamma_H$ may model hysteresis phenomena on the boundary. Specifically, consider the maximal monotone graph in $\mathbb{R}^2$ given by

\[
\text{sgn} (z) = \left\{
\begin{array}{ll}
\{-1\}, & z < 0 \\
\[-1,1\], & z = 0 \\
\{1\}, & z > 0
\end{array}
\right.
\]

and its inverse

\[
\text{sgn}^{-1}(z) = \left\{
\begin{array}{ll}
(-\infty,0], & z = -1 \\
\{0\}, & -1 < z < 1 \\
[0,\infty), & z = 1
\end{array}
\right.
\]

If we choose $c(z) = \text{sgn}^{-1}(\frac{z}{\delta})$ (figure 1 shows $c$ with $\delta = 1$), then the condition (1c) becomes a constraint for $v$, namely

\[
u - \delta \leq v \leq u + \delta,
\]

Figure 1: Graph of $c$
and also for $\partial_v u$, namely

$$
\partial_v u \begin{cases} 
\leq 0, & v = u - \delta \\
= 0, & u - \delta < v < u + \delta \\
\geq 0, & v = u + \delta
\end{cases}
$$

If $b$ and $b^{-1}$ are functions, one can rewrite this as

$$
\partial_v u \begin{cases} 
\leq 0, & w = b(u - \delta) \\
= 0, & b(u - \delta) < w < b(u + \delta) \\
\geq 0, & w = b(u + \delta)
\end{cases}
$$

The condition (1b) is an ordinary differential equation for $w$; for the choice $g \equiv 0$ the variable $w$ remains constant as long as the constraint (2) is strictly satisfied. If the constraint is active, then $\partial_v u$ - the control - and $w$ are selected such that via condition (1b) the corresponding equality in (2) is maintained. Therefore, one gets

$$
b(u - \delta) \leq w \leq b(u + \delta). \tag{3}
$$

Thus the relationship between $u$ and $w \in b(v)$ is an example of a generalized play, see [10] [11] (figure 4). In this case, the function $b$ prescribes the general shape of a loop in the $u - w$-plane, and the number $\delta$ its width the $u$-direction. On the right hand boundary of the loop the path is always directed upwards, and on the left hand boundary the path is always directed downwards. As soon as the variation of $u$ leads into the interior of the loop, the value of $w$ remains constant until the boundary is reached again.

Furthermore, if $b$ is a multiple of the signum function

$$
b = \gamma \text{ sgn },
$$

then conditions (1b,c) model a perfect relay, see [10] [11] (figure 6). One may consider this case as a degenerate play. As in the previous case, in the interior of the loop all lines are horizontal, whereas on the boundary of the loop one may distinguish the following situations: on the upper and lower part $w$ is constant and the normal flux is zero, i.e., $u$ satisfies a unilateral boundary condition; on the right and left part $u$ is constant, i.e., it satisfies a Dirichlet condition, and $w$ is governed by an ordinary differential equation.

System (1) consists of a general porous media equation (1a) - which becomes degenerate whenever $a(u)$ vanishes - in the interior of $\Omega$ subject to a nonlinear dynamic Neumann condition (1b,c) on the boundary $\Gamma_H$ and a Dirichlet condition (1d) on $\Gamma_D$. The variable $w$ is the internal state of the hysteron, $v - u$ is the order parameter, and $u$ is the external input.

A rather remarkable variety of boundary conditions can be obtained from (1b,c). For example, if $b \equiv 0$ we have an explicit Neumann boundary condition, and if $c \equiv 0$ it
is homogeneous. (Clearly any general solvability results cannot allow simultaneously $c = b = 0$, for this forces $g = 0$.) If $b(0) = R$ (i.e., $b^{-1} = 0$), then $v \equiv 0$ and we have a nonlinear Neumann constraint, and if in addition

$$c(z) = \begin{cases} 
  0, & z < 0 \\
  [0, \infty), & z = 0
\end{cases},$$

it is the Signorini condition

$$u \geq 0, \quad \partial_n u \geq 0, \quad u \partial_n u = 0.$$ 

If $c(0) = R$ we get $v = u$ on $\Gamma$ and this satisfies a nonlinear dynamic boundary condition

$$\partial_t b(u) + \partial_n u \geq g$$

of Neumann type. If $b(0) = c(0) = R$ we have the homogeneous Dirichlet boundary condition. If both $b^{-1}$ and $c$ are functions, one gets a nonlinear adsorption condition of the form

$$\partial_t w + \partial_n u = g, \quad \partial_n u = c(b^{-1}(w) - u).$$

For previous work on some of these various classes, we refer to [2] [4] [6].

Theoretical results for the problem (1) are given in the paper [8]. There well-posedness for the corresponding stationary and the evolution problem are proved using methods that had previously been developed in [12]. In this paper we concentrate on special examples and their numerical solution. Papers that deal with problems closely related to those of the present paper are [1] [9] [13] [14] [15] where parabolic problems with a hysteresis source term are studied.

Adsorption in porous media may be governed by conditions on the surfaces of the solid material that are of hysteresis type (see [3] pp 357 ff. where experimental evidence for hysteresis in adsorption processes is described). In that case $u$ is the concentration of a chemical species that is dissolved in the fluid occupying the pores, and $w$ is its concentration on the surfaces once it has been adsorbed. Usually, one assumes that on the pore scale the adsorption rate - given as the normal flux in the Neumann boundary condition for the diffusion-convection process in the fluid - is a prescribed function of $u$ and/or $w$ on the boundary. But here we make the assumption that adsorption takes place only if the concentration $u$ exceeds certain thresholds and that the range of the concentration $w$ is bounded. In this way, one gets hysteresis phenomena of the kind discussed in this paper. In [7] this idea is applied to homogenization of reactive transport through porous media.

## 2 Numerical Examples

We consider a $\gamma$-multiple of the signum function

$$b = \gamma \text{ sgn}$$
(ε = 0) or a smooth approximation thereof, namely

\[ b_\varepsilon(z) = \gamma \frac{z}{\varepsilon + |z|} \]

(figure 2 shows \( b_\varepsilon \) with \( \gamma = \frac{1}{2} \) and \( \varepsilon = 0.1 \)), and the inverse of the scaled signum function

\[ c(z) = \text{sgn}^{-1}(\frac{z}{\delta}) \]

(see figure 1). For the following examples we simplify by using \( a(u) = u \), and \( f, g = 0 \). We are going to use the function

\[ h(t) = \alpha 2^{-\frac{4}{3}} \sin(2\pi \omega t) \]

several times. For the examples we have chosen \( \gamma = \frac{1}{2} \) and \( \delta = 1 \). The initial values are all zero in the examples. As a numerical method we have used the standard time-explicit difference scheme with constant step-sizes in \( x \) and \( t \).

1. As a one-dimensional example, let \( \Omega = (0, 1), \Gamma_H = \{0\}, \Gamma_D = \{1\} \). We assume \( uD(t) = h(t) \) with \( \alpha = 4, \beta = 10, \) and \( \omega = \frac{1}{5} \). We have used step sizes \( \Delta x = 0.02 \) and \( \Delta t = 8 \cdot 10^{-6} \). Figures 3 and 5 show \( u \) and \( w \) at \( x = 1 \) as functions of time with \( \varepsilon = 0.1, 0.0 \), resp.; the dotted line is the function \( h \). Figures 4 and 6 show \( w \) versus \( u \); the oblique lines that cut the corners are due to the discretization of time. Figure 4 shows the typical form of a play, whereas figure 6 has the form of a perfect relay. The fact that the boundary of the loops differ in both cases can be seen in figures 4 and 6 and also in figures 3 and 5. Not only has \( w \) upper and lower bounds, but the change between the regime \( -\delta < w < \delta \) and \( w = -\delta \) or \( w = \delta \) is abrupt for \( \varepsilon = 0 \). Therefore
the curves in figure 5 have sharp corners. In this sense, a relay is a degenerate case. On the other hand, a comparison of figures 3 and 5 shows that the relay ($\epsilon = 0$) can be approximated by a play ($\epsilon$ small) quite well.

2. The following is an example in 2D. We take $\Omega = \{(x_1, x_2) : 0 < x_1, x_2 < 1\}$ as the unit square in $\mathbb{R}^2$. Here we assume $\Gamma_D = \{(x_1, x_2) : x_1 = 0 \text{ or } x_1 = 1\}$ and $\Gamma_H = \{(x_1, x_2) : x_2 = 0 \text{ or } x_2 = 1\}$. We assume

$$u_D(t, x) = \begin{cases} h(t), & x_1 = 0 \\ -h(t), & x_1 = 1 \end{cases}$$

for $x \in \Gamma_D$ with $\alpha = 4$, $\beta = 2$, and $\omega = 1$. We have used step sizes $\Delta x = 0.03125$ and $\Delta t = 5 \cdot 10^{-5}$. Figure 7 shows the profile of the solution $u$ at time $t = 1.25$ with $\epsilon = 0$.

3. For another example in 2D we take again $\Omega = \{(x_1, x_2) : 0 < x_1, x_2 < 1\}$. Now we assume $\Gamma_D = \{(x_1, x_2) : x_1 = 0\}$ and $\Gamma_H = \partial \Omega \setminus \Gamma_D$ as in the previous example. Again we assume $u_D(t) = h(t)$ for $x \in \Gamma_D$ with the same parameters as in example 2. Figure 8 shows the profile of the solution $u$ at time $t = 1.25$ with $\epsilon = 0.1$. As for the one-dimensional examples, one sees easily that there is a similarity and a difference between the two cases $\epsilon > 0$ and $\epsilon = 0$: Obviously, the first case is an approximation of the latter. But for $\epsilon = 0$ there is a range of points on the boundary where $u$ is constant; here $u$ equals to the threshold values $\pm \delta$. Such a phenomenon does not occur for $\epsilon > 0$.  

Figure 3: $u$ and $w$ as functions of $t$ at $x = 1$ for example 1 with $\epsilon = 0.1$, $u$ at $x = 1$ : solid line, $u$ at $x = 0$ : dotted line, $w$ at $x = 1$ : dashed line.
Figure 4: Play: $w$ versus $u$ at $x = 1$ for example 1 with $\varepsilon = 0.1$.

Figure 5: $u$ and $w$ as functions of $t$ at $x = 1$ for example 1 with $\varepsilon = 0$. $u$ at $x = 1$: solid line, $u$ at $x = 0$: dotted line, $w$ at $x = 1$: dashed line.
Figure 6: Relay: $w$ versus $u$ at $x = 1$ for example 1 with $\varepsilon = 0$

Figure 7: Profile at $t = 1.25$ for example 2 with $\varepsilon = 0$
Figure 8: Profile at $t = 1.25$ for example 3 with $\varepsilon = 0.1$

It is obvious from the examples and their plots that parabolic equations with boundary conditions of hysteresis type can be treated numerically in a natural way that is similar to those with the usual Dirichlet or Neumann conditions. On the other hand the solutions of initial boundary value problems may have properties that are known for phase change problems. In special cases the boundary condition of hysteresis type results in switching back and forth from Dirichlet to Neumann or unilateral conditions on the boundary. The numerical experiments presented in this paper demonstrate these special features of the solutions of initial boundary value problems for partial differential equation.

References


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