1 Boundary-Value Problems

Neumann Boundary-Value Problem

Let G be a domain in $\mathbb{R}^N$ with boundary ∂G on which $\mathbf{n}$ is the unit outward normal. Let $a(\cdot) \in L^\infty(G)$ be uniformly positive: $a(x) \geq a_0 > 0,\ x \in G$.
Consider the *Neumann boundary-value problem*

$$
\begin{aligned}
-\boldsymbol{\nabla} \cdot a(x)\boldsymbol{\nabla} p(x) &= F(x), \quad x \in G, &\text{(1a)}\\
a(\cdot)\boldsymbol{\nabla} p \cdot \mathbf{n} &= g \text{ on } \partial G. &\text{(1b)}
\end{aligned}
$$

The Weak Solution

Set $V = H^1(G)$. If $p \in V$ is a solution of (1), then for each $q \in V$ we have

$$
\begin{aligned}
\int_G F(x)q(x)\,dx &= \int_G a(x)\boldsymbol{\nabla} p(x) \cdot \boldsymbol{\nabla} q(x)\,dx - \int_{\partial G} a\boldsymbol{\nabla} p \cdot \mathbf{n} q\,dS \\
&= \int_G a(x)\boldsymbol{\nabla} p(x) \cdot \boldsymbol{\nabla} q(x)\,dx - \int_{\partial G} g(s)\,q(s)\,dS\,,
\end{aligned}
$$

so we obtain

$$
p \in V: \quad \int_G a(x)\boldsymbol{\nabla} p(x) \cdot \boldsymbol{\nabla} q(x)\,dx = \int_G F(x)q(x)\,dx + \int_{\partial G} g(s)\,q(s)\,dS \text{ for all } q \in V. \quad (2)
$$

Conversely, we can show that any appropriately smooth solution of (2) is a solution of (1).

Notes

- Any two solutions of (2) differ by a constant in V, so we have uniqueness only up to constants.
- By taking $q(x) = 1$ in (2) we find a *necessary* condition for existence of a solution:

$$\int_G F(x)\,dx + \int_{\partial G} g(s)\,dS = 0\,. \tag{3}$$

It is clear that the constant functions in V play a prominent role here. Uniqueness holds up to them, and the right side of (2) must vanish on them. We define the unit constant function on G by $\chi(x) = 1,\ x \in G$. Constant functions are those in the linear span $< \chi >= \mathbb{R}\chi$.

Hereafter we assume that (3) holds. We define the subspace $V_0 = \{q \in V : \int_G q(x)\,dx = 0\} = \{\chi\}^{\perp}$. These are the functions of V with mean-value equal to zero. Then (2) is equivalent to

$$\tilde{p} \in V_0 : \quad \int_G a(x)\boldsymbol{\nabla}\tilde{p}(x) \cdot \boldsymbol{\nabla} q(x)\,dx = \int_G F(x)q(x)\,dx + \int_{\partial G} g(s)q(s)\,dS \text{ for all } q \in V_0. \tag{4}$$

where $\tilde{p}(x) = p(x) - \frac{1}{|G|}\int_G p(y)\,dy$. Thus we obtain an alternative weak formulation for which we have uniqueness. What remains is to show that the bilinear form $\int_G \boldsymbol{\nabla} p(x) \cdot \boldsymbol{\nabla} q(x)\,dx$ is equivalent to the $H^1(Q)$-scalar product on V_0. But this follows from the estimate

$$\|q\|^2_{L^2(G)} \leq \left(\int_G q(x)\,dx\right)^2 + \frac{N}{2}\,\|\boldsymbol{\nabla} q\|^2_{L^2(G)}\,.$$

Summary

- The equations (1) are the strong form, and (2) and (4) are equivalent weak forms of the Neumann boundary-value problem.
- The bilinear form $(\boldsymbol{\nabla} p,\, \boldsymbol{\nabla} q)_{L^2(G)}$ is equivalent to the $H^1(G)$ scalar product on V_0.

Theorem 1.1. *Assume $a(\cdot) \in L^\infty(G)$ is uniformly positive, $a(x) \geq a_0 > 0,\ x \in G$ and that $\int_G F(x)\,dx + \int_{\partial G} g(s)\,dS = 0$. Then the Neumann boundary-value problem (4) has a unique solution. That is, there exists a solution of (2), and any two solutions of (2) differ by a constant.*

A Mixed Formulation

The formulation (2) has the form

$$p \in V : \ \mathcal{A}p(q) = f(q), \ q \in V, \tag{5}$$

where the bilinear form

$$\mathcal{A}p(q) = \int_G a(x) \boldsymbol{\nabla} p(x) \cdot \boldsymbol{\nabla} q(x) \, dx, \quad p, \ q \in V,$$

and the linear functional

$$f(q) = \int_G F(x) q(x) \, dx + \int_{\partial G} g(s) \, q(s) \, dS, \quad q \in V,$$

are defined as indicated. That is, $\mathcal{A}p = f$ in V', where $\mathcal{A} : V \to V'$ is the indicated linear operator.

Now define another linear operator $\mathcal{B} : V \to \mathbb{R}'$ by

$$\mathcal{B}p = \int_G p(y) \, dy \, ,$$

and note that the dual operator $\mathcal{B}' : \mathbb{R} \to V'$ is given by

$$\mathcal{B}' r(\varphi) = r(\mathcal{B}\varphi) = \int_G r \, \varphi(x) \, dx \, ,$$

so we have $\mathrm{Rg}(\mathcal{B}') = \mathbb{R} \subset L^2(G) \subset V'$, the *constant functions*.

Then the formulation (4) is of the form

$$p \in \mathrm{Ker}(\mathcal{B}) : \ \mathcal{A}p - f \in \mathrm{Ker}(\mathcal{B})^0 = \mathrm{Rg}(\mathcal{B}'). \tag{6}$$

That is, we have

$$p \in V : \ \mathcal{A}p \ + \mathcal{B}' r \ = f \text{ in } V' \tag{7a}$$

$$r \in \mathbb{R} : \ \mathcal{B}p \qquad = 0 \text{ in } \mathbb{R}. \tag{7b}$$

In fact, we find that $\mathcal{B}' r = f(1)$, so $f - \mathcal{B}' r \in V_0'$.

Trace and Normal Derivative on ∂G

We use the Sobolev space $H^1(G) = \{q \in L^2(G) : \ \boldsymbol{\nabla} q \in \mathbf{L}^2(G)\}$. This is a Hilbert space with the scalar product

$$(p, q)_{H^1(G)} = (p, q)_{\mathbf{L}^2(G)} + (\boldsymbol{\nabla} p, \boldsymbol{\nabla} q)_{\mathbf{L}^2(G)}.$$

For $q \in H^1(G)$ there is a well-defined restriction to the boundary, the *trace* $\gamma(q) = q|_{\partial G}$ in $L^2(\partial G)$, and this map $q \mapsto \gamma(q)$ is linear and continuous: $\gamma \in \mathcal{L}(H^1(G), L^2(\partial G))$. The kernel of the trace map is denoted by $H_0^1(G)$. Define $\mathbb{B} \subset L^2(\partial G)$ to be the range of the trace map, the set of boundary-values of functions in $H^1(G)$. If we define the norm on $\mathbb{B}$ by

$$\|\mu\|_{\mathbb{B}} = \inf_{q \in H^1(G): \ \gamma(q) = \mu} \|q\|_{H^1(G)}, \tag{8}$$

then $\gamma \in \mathcal{L}(H^1(G), \mathbb{B})$ is *onto* $\mathbb{B}$. We denote the range of the trace map by $\mathbb{B} = H^{1/2}(\partial G)$ and its dual space by $\mathbb{B}' = H^{-1/2}(\partial G)$.

We shall also use the space $\mathbf{L}^2_{div}(G) = \{\mathbf{v} \in \mathbf{L}^2(G) : \ \boldsymbol{\nabla} \cdot \mathbf{v} \in L^2(G)\}$. This is a Hilbert space with the scalar product

$$(\mathbf{u}, \mathbf{v})_{\mathbf{L}^2_{div}(G)} = (\mathbf{u}, \mathbf{v})_{\mathbf{L}^2(G)} + (\boldsymbol{\nabla} \cdot \mathbf{u}, \boldsymbol{\nabla} \cdot \mathbf{v})_{L^2(G)},$$

and smooth functions are dense in this space. For $\mathbf{u} \in \mathbf{L}^2_{div}(G)$ there is a well-defined *normal trace* $\mathbf{u} \cdot \mathbf{n} \in H^{-1/2}(\partial G)$ for which

$$\int_G (\mathbf{u} \cdot \boldsymbol{\nabla} q + \boldsymbol{\nabla} \cdot \mathbf{u}\, q)\, dx = \mathbf{u} \cdot \mathbf{n}(\gamma q), \quad q \in H^1(G). \tag{9}$$

In fact, the left side of (9) defines a functional in $H^1(G)'$ which vanishes on $H_0^1(G)$, that is, an element of the annihilator $H_0^1(G)^0$. By the closed range theorem, this is equal to $\mathrm{Rg}(\gamma')$, so the functional equals $\gamma'(\mathbf{u} \cdot \mathbf{n})$ for a unique $\mathbf{u} \cdot \mathbf{n} \in \mathbb{B}'$. The right side of (9) is $\gamma'(\mathbf{u} \cdot \mathbf{n})(q)$. For smoother $\mathbf{u} \in \mathbf{H}^1(G)$ the (componentwise) trace has normal component $\gamma(\mathbf{u}) \cdot \mathbf{n} \in L^2(\partial G) \subset H^{-1/2}(\partial G)$, and the Stokes theorem shows

$$\mathbf{u} \cdot \mathbf{n}(\gamma q) = \int_{\partial G} \gamma(\mathbf{u}) \cdot \mathbf{n}\, q\, dS,$$

so we have denoted this functional by $\mathbf{u} \cdot \mathbf{n}$ also for $\mathbf{u} \in \mathbf{L}^2_{div}(G)$.

Finally we note the special case of (9) with $\mathbf{u} = \boldsymbol{\nabla} p$:

$$\int_G (\boldsymbol{\nabla} p \cdot \boldsymbol{\nabla} q + \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} p\, q)\, dx = \int_{\partial G} \boldsymbol{\nabla} p \cdot \mathbf{n}(\gamma q)\, dS$$
$$\text{for all } q \in H^1(G),\ p \in H^1(G) \text{ with } \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} p \in L^2(G). \quad (10)$$

The normal component of the gradient on ∂G is the *normal derivative* $\frac{\partial p}{\partial n} = \boldsymbol{\nabla} p \cdot \mathbf{n} \in H^{-1/2}(\partial G)$. The equation (10) is *Green's identity* for the Laplacean $\Delta = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$, and it was precisely this equation that we needed for the weak formulation of the Neumann problem (1).

Dirichlet-Neumann Boundary-Value-Problem

Set $Q = \{q \in H^1(G) :\ q|_{\Gamma_1} = 0\}$ where the boundary of G is the disjoint union $\partial G = \Gamma_1 \cup \Gamma_2$.

Direct Variational Form

$$p \in Q : \quad \int_G (\lambda c(x) pq + \kappa(x) \boldsymbol{\nabla} p \cdot \boldsymbol{\nabla} q)\, dx = \int_G fq\, dx + \int_{\Gamma_2} h\, q\, dS\ \forall q \in Q.$$

The strong form is, respectively,

$$p|_{\Gamma_1} = 0,\ \lambda c(x) p - \boldsymbol{\nabla} \cdot \kappa(x) \boldsymbol{\nabla} p = f \text{ in } G,$$
$$\text{and } \int_{\partial G} \kappa \boldsymbol{\nabla} p \cdot \mathbf{n}\, q\, dS = \int_{\Gamma_2} h\, q\, dS\ \forall q \in Q, \text{ that is, } \kappa \boldsymbol{\nabla} p \cdot \mathbf{n}|_{\Gamma_2} = h$$

A Gradient Mixed Formulation

Set $a(x) = \frac{1}{\kappa(x)}$ and $\mathbf{u}(x) \equiv -\kappa(x) \boldsymbol{\nabla} p$.

$$\mathbf{u} \in \mathbf{L}^2(G),\ p \in Q : \int_G a(x) \mathbf{u} \cdot \mathbf{v}\, dx + \int_G \boldsymbol{\nabla} p \cdot \mathbf{v}\, dx = 0\ \forall \mathbf{v} \in \mathbf{L}^2(G),$$
$$\lambda \int_G c(x) pq\, dx - \int_G \mathbf{u} \cdot \boldsymbol{\nabla} q\, dx = \int_G fq\, dx + \int_{\Gamma_2} h\, q\, dS\ \forall q \in Q.$$

The strong form is

$$p|_{\Gamma_1} = 0,\ a(x) \mathbf{u} + \boldsymbol{\nabla} p = 0,\ \lambda c(x) p + \boldsymbol{\nabla} \cdot \mathbf{u} = f \text{ in } G,$$
$$\text{and } -\int_{\partial G} \mathbf{u} \cdot \mathbf{n}\, q\, dS = \int_{\Gamma_2} h\, q\, dS\ \forall q \in Q, \text{ that is } -\mathbf{u} \cdot \mathbf{n}|_{\Gamma_2} = h.$$

Here we have set

$$\mathcal{A}\mathbf{u}(\mathbf{v}) = \int_G a(x)\mathbf{u}\cdot\mathbf{v}\,dx,\;\; \mathcal{B}\mathbf{v}(q) = \int_G \mathbf{v}\cdot\boldsymbol{\nabla} q\,dx,\;\; \mathcal{C}p(q) = \int_G c(x)pq\,dx.$$

Note: This mixed formulation is equivalent to the Direct Variational Form above. The constraint on $p|_{\Gamma_1}$ is a strong (prescribed) boundary condition; the constraint on $\mathbf{u}\cdot\mathbf{n}|_{\Gamma_2}$ is the corresponding weak or dual boundary condition. A non-homogeneous value for $p|_{\Gamma_1}$ can be obtained by translation.

A Divergence Mixed Formulation

Set $\mathbf{V} = \{\mathbf{v}\in \mathbf{L}^2_{div}(G):\; \mathbf{v}\cdot\mathbf{n}|_{\Gamma_2} = 0\}$.

$$\mathbf{u}\in\mathbf{V},\; p\in L^2(G):\; \int_G a(x)\mathbf{u}\cdot\mathbf{v}\,dx - \int_G p\boldsymbol{\nabla}\cdot\mathbf{v}\,dx = \int_{\Gamma_1} g_0\mathbf{v}\cdot\mathbf{n}\;\forall\mathbf{v}\in\mathbf{V},$$
$$\lambda\int_G c(x)pq\,dx + \int_G \boldsymbol{\nabla}\cdot\mathbf{u}\,q\,dx = \int_G fq\,dx\;\forall q\in L^2(G).$$

Although we write it as an integral, the right side of the first equation is actually $\mathbf{v}\cdot\mathbf{n}(g_0)$ for an appropriate g_0. (See Remark 1.1 below.) The strong form of this problem is

$$\mathbf{u}\cdot\mathbf{n}|_{\Gamma_2} = 0,\;\; a(x)\mathbf{u} + \boldsymbol{\nabla} p = 0,\;\; \lambda c(x)p + \boldsymbol{\nabla}\cdot\mathbf{u} = f,$$
$$\text{and}\;\; -\int_{\partial G} p\mathbf{v}\cdot\mathbf{n} = \int_{\Gamma_1} g_0\mathbf{v}\cdot\mathbf{n}\;\forall\mathbf{v}\in\mathbf{V},\;\text{ that is, }\; -p|_{\Gamma_1} = g_0$$

In this formulation we set $\mathcal{A}$ and $\mathcal{C}$ as above, but

$$\mathcal{B}\mathbf{v}(q) = -\int_G \boldsymbol{\nabla}\cdot\mathbf{v}\cdot q\,dx$$

Note: The constraint on $\mathbf{u}\cdot\mathbf{n}|_{\Gamma_2}$ is a strong (prescribed) boundary condition; the constraint on $p|_{\Gamma_1}$ is the corresponding weak or dual boundary condition. A non-homogeneous value for $\mathbf{u}\cdot\mathbf{n}|_{\Gamma_2}$ can be obtained by translation.

The non-homogeneous Poisson system

The fully non-homogeneous case can be handled in other ways. The boundary conditions or other constraints can be moved from the space to the operators. Thus set $\mathbf{V} = \mathbf{L}^2_{div}(G),\ Q = L^2(G) \times \mathbb{B}'$ and seek

$$\mathbf{u} \in \mathbf{V},\ p = [p_1, p_2] \in Q :$$
$$\int_G a(x)\mathbf{u} \cdot \mathbf{v}\, dx - \int_G p_1 \boldsymbol{\nabla} \cdot \mathbf{v}\, dx + \int_{\Gamma_2} p_2\, \mathbf{v} \cdot \mathbf{n}\, dS$$
$$= \int_G \mathbf{f}_0 \cdot \mathbf{v}\, dx + \int_{\Gamma_1} g_0\, \mathbf{v} \cdot \mathbf{n}\, dS \quad \forall \mathbf{v} \in \mathbf{V}, \tag{11a}$$

$$\lambda \int_G c(x) p_1 q_1\, dx + \int_G \boldsymbol{\nabla} \cdot \mathbf{u}\, q_1\, dx - \int_{\Gamma_2} \mathbf{u} \cdot \mathbf{n}\, q_2\, dS$$
$$= \int_G f q_1\, dx \ + \int_{\Gamma_2} h q_2\, dS \quad \forall q \in L^2(G). \tag{11b}$$

In this formulation we define $\mathcal{A}$ and $\mathcal{C}$ as above, but

$$\mathcal{B}\mathbf{v}(q) = -\int_G \boldsymbol{\nabla} \cdot \mathbf{v} \cdot q_1\, dx + \int_{\Gamma_2} \mathbf{v} \cdot \mathbf{n} q_2\, dS.$$

Here $\mathcal{B} : \mathbf{L}^2_{div}(G) \to L^2(G) \times \mathbb{B}'$ and $\mathcal{B}' : L^2(G) \times \mathbb{B} \to \mathbf{L}^2_{div}(G)'$, and we have a *pair* of Lagrange multipliers p_1, p_2. These are independent, so we have equivalently a pair of constraint operators $\mathcal{B}_1 : \mathbf{L}^2_{div}(G) \to L^2(G)$ and $\mathcal{B}_2 : \mathbf{L}^2_{div}(G) \to \mathbb{B}'$. The strong form is the non-homogeneous system

$$a(x)\mathbf{u} + \boldsymbol{\nabla} p_1 = \mathbf{f}_0,\ \lambda c(x) p_1 + \boldsymbol{\nabla} \cdot \mathbf{u} = f, \tag{12a}$$
$$-p_1|_{\Gamma_1} = g_0,\ p_1|_{\Gamma_2} = p_2,\ -\mathbf{u} \cdot \mathbf{n}|_{\Gamma_2} = h. \tag{12b}$$

The trace values of p_1 are meaningful because (11a) first shows that $\boldsymbol{\nabla} p_1 \in \mathbf{L}^2(G)$, and then it gives

$$-\int_{\partial G} p_1 \mathbf{v} \cdot \mathbf{n}\, dS + \int_{\Gamma_2} p_2 \mathbf{v} \cdot \mathbf{n}\, dS = \int_{\Gamma_1} g_0\, \mathbf{v} \cdot \mathbf{n}\, dS \quad \forall \mathbf{v} \in \mathbf{V}.$$

In particular, the Lagrange multipliers of a solution are *not* independent.

Remark 1.1. *The second term of $\mathcal{B}$ is delicate because $\mathbf{v} \cdot \mathbf{n} \in \mathbb{B}'$ so q_2 needs to be extended to $\mathbb{B}$. This same issue is implicit in the preceding formulations of the Dirichlet-Neumann problem as well. This leads to the special spaces of* local *boundary values, $H^{1/2}_{00}(\Gamma_2)$.*

The non-homogeneous boundary-value problem (12) can be also be resolved by using non-zero constraints in the convex set. For example, the second component p minimizes the functional ($\kappa(x) = 1/a(x)$)

$$J_1(q) = \tfrac{1}{2}\int_G \kappa(x)|\boldsymbol{\nabla} q|^2 dx - \int_G \kappa(x)\mathbf{f}_0 \cdot \boldsymbol{\nabla} q \\ - \int_{\Gamma_2} h\, q\, dS - \int_G f\, q\, dS + \tfrac{\lambda}{2}\int_G c(x) q^2\, dx$$

on the convex set $K_1 = \{q \in H^1(G) : \gamma(q)|_{\Gamma_1} = -g_0\}$. This follows directly from the calculations

$$\begin{aligned} J_1(q) &= \tfrac{1}{2}\int_G \kappa(|\boldsymbol{\nabla} q|^2 - 2\boldsymbol{\nabla} p \cdot \boldsymbol{\nabla} q)\, dx - \int_G \mathbf{u}\boldsymbol{\nabla} q\, dx \\ &\quad - \int_{\Gamma_2} h\, q\, dS - \int_G f\, q\, dS + \tfrac{\lambda}{2}\int_G c(x) q^2\, dx \\ &= \tfrac{1}{2}\int_G \kappa(|\boldsymbol{\nabla} q|^2 - 2\boldsymbol{\nabla} p \cdot \boldsymbol{\nabla} q)\, dx + \int_G \boldsymbol{\nabla} \cdot \mathbf{u} q\, dx - \int_{\partial G} \mathbf{u} \cdot \mathbf{n}\, q\, dS \\ &\quad - \int_{\Gamma_2} h\, q\, dS - \int_G f\, q\, dS + \tfrac{\lambda}{2}\int_G c(x) q^2\, dx \\ &= \tfrac{1}{2}\int_G \kappa(|\boldsymbol{\nabla} q|^2 - 2\boldsymbol{\nabla} p \cdot \boldsymbol{\nabla} q)\, dx - \int_{\Gamma_1} \mathbf{u} \cdot \mathbf{n}\, q\, dS + \tfrac{\lambda}{2}\int_G c(x)(q^2 - 2pq)\, dx, \end{aligned}$$

$$J_1(p) = -\tfrac{1}{2}\int_G |\boldsymbol{\nabla} p|^2\, dx + \int_{\Gamma_1} \mathbf{u} \cdot \mathbf{n}\, g_0\, dS - \tfrac{\lambda}{2}\int_G c(x) p^2\, dx,$$

from which we obtain

$$J_1(q) - J_1(p) = \tfrac{1}{2}\int_G \kappa(x)|\boldsymbol{\nabla}(q - p)|^2\, dx + \tfrac{\lambda}{2}\int_G c(x)(p - q)^2\, dx \geq 0$$

for all $q \in K_1$. The minimal point p is obtained as the solution of a variational inequality.

Likewise, for the case $\lambda = 0$, the first component $\mathbf{u}$ minimizes the functional

$$J_2(\mathbf{v}) = \tfrac{1}{2}\int_G a(x)|\mathbf{v}|^2\, dx - \int_G \mathbf{v} \cdot \mathbf{f}_0\, dx - \int_{\Gamma_1} g_0\, \mathbf{v} \cdot \mathbf{n}\, dS$$

on the set $K_2 = \{\mathbf{v} \in L^2_{div}(G) : \boldsymbol{\nabla} \cdot \mathbf{v} = f \text{ in } G, \mathbf{v} \cdot \mathbf{n}|_{\Gamma_2} = -h\}$:

$$J_2(\mathbf{v}) = \tfrac{1}{2} \int_G a(x)(|\mathbf{v}|^2 - 2\mathbf{v} \cdot \mathbf{u})\, dx - \int_G \mathbf{v} \cdot \boldsymbol{\nabla} p\, dx - \int_{\Gamma_1} g_0\, \mathbf{v} \cdot \mathbf{n}\, dS$$

$$= \tfrac{1}{2} \int_G a(x)(|\mathbf{v}|^2 - 2\mathbf{v} \cdot \mathbf{u})\, dx + \int_G \boldsymbol{\nabla} \cdot \mathbf{v}\, p\, dx - \int_{\partial G} \mathbf{v} \cdot \mathbf{n}\, p\, dS - \int_{\Gamma_1} g_0\, \mathbf{v} \cdot \mathbf{n}\, dS$$

$$= \tfrac{1}{2} \int_G a(x)(|\mathbf{v}|^2 - 2\mathbf{v} \cdot \mathbf{u})\, dx + \int_G f\, p\, dx + \int_{\Gamma_2} h\, p\, dS \text{ for all } \mathbf{v} \in K_2,$$

$$J_2(\mathbf{u}) = -\tfrac{1}{2} \int_G a(x)|\mathbf{u}|^2\, dx + \int_G f\, p\, dx + \int_{\Gamma_2} h\, p\, dS,$$

so that we find

$$J_2(\mathbf{v}) - J_2(\mathbf{u}) = \tfrac{1}{2} \int_G a(x)|\mathbf{v} - \mathbf{u}|^2\, dx \geq 0 \text{ for all } \mathbf{v} \in K_2,$$

and $\mathbf{u}$ is characterized as the solution of a variational inequality.

When $\lambda > 0$ and $c(x) \geq c_0 > 0$, we can show directly that the first component $\mathbf{u}$ minimizes the functional

$$J_3(\mathbf{v}) = \tfrac{1}{2} \int_G a(x)|\mathbf{v}|^2\, dx + \int_G \tfrac{1}{2\lambda c(x)} (\boldsymbol{\nabla} \cdot \mathbf{v} - f)^2\, dx - \int_G \mathbf{v} \cdot \mathbf{f}_0\, dx - \int_{\Gamma_1} g_0\, \mathbf{v} \cdot \mathbf{n}\, dS$$

on the set $K_2 = \{\mathbf{v} \in L^2_{div}(G) : \mathbf{v} \cdot \mathbf{n}|_{\Gamma_2} = -h\}$.

Interface Boundary-Value-Problems

Let the domain G be split into subdomains G_1 and G_2 by an interface Γ. The normal $\mathbf{n}$ on Γ is directed out of G_1 and into G_2. For $q \in L^2(G)$, we denote with subscripts the restrictions to the respective regions, $q_j = q|_{G_j},\ j = 1, 2$. Similarly we denote restrictions of vector-valued $\mathbf{v} \in \mathbf{L}^2(G)$ by $\mathbf{v}_j,\ j = 1, 2$.

- If we choose a **gradient** mixed formulation on G ($p \in L^2(G)$) the strong interface condition is $p_1 = p_2$ on Γ and there is a corresponding weak or dual interface condition on $\mathbf{u}_1 \cdot \mathbf{n} - \mathbf{u}_2 \cdot \mathbf{n}$.
- If we choose a **divergence** mixed formulation on G ($\mathbf{u} \in L^2_{div}(G)$) the strong interface condition is $\mathbf{u}_1 \cdot \mathbf{n} = \mathbf{u}_2 \cdot \mathbf{n}$ on Γ and there is a corresponding weak or dual interface condition on $p_1 - p_2$.

These are the most frequently appropriate conditions, but there can be reasons to consider more general situations, such as modeling over multiple scales. See [1].

A Neumann Interface problem in Direct Formulation

$Q = \{q \in L^2(G) : \ \boldsymbol{\nabla} q_1 \in L^2(G_1), \ \boldsymbol{\nabla} q_2 \in L^2(G_2)\}$ so traces on

$$\Gamma = \partial G_1 \cap \partial G_2, \ \Gamma_1 = \partial G_1 - \Gamma, \ \Gamma_2 = \partial G_2 - \Gamma$$

are all defined. Define the convex set

$$K_g = \{q \in Q : \ \beta_1 q_1 - \beta_2 q_2 = g \text{ on } \Gamma\}$$

The variational inequality

$$p \in K_g : \ \int_G (cp(q-p) + k\boldsymbol{\nabla} p \cdot \boldsymbol{\nabla}(q-p))\, dx \geq \int_G F(q-p)\, dx + \int_{\Gamma\cup\Gamma_1\cup\Gamma_2} h(q-p)\, dS, \ q \in K_g$$

is equivalent to

$$p \in K_g : \ \int_G (cpq + k\boldsymbol{\nabla} p \cdot \boldsymbol{\nabla} q)\, dx = \int_G Fq\, dx + \int_{\Gamma\cup\Gamma_1\cup\Gamma_2} hq\, dS, \ q \in K_0.$$

This is a weak formulation of the *Neumann-interface problem*

$$c_1 p_1 - \boldsymbol{\nabla} \cdot (k_1 \boldsymbol{\nabla} p_1) = F_1 \text{ in } G_1, \quad k_1 \boldsymbol{\nabla} p_1 \cdot \mathbf{n} = h_1 \text{ on } \Gamma_1, \tag{13a}$$

$$c_2 p_2 - \boldsymbol{\nabla} \cdot (k_2 \boldsymbol{\nabla} p_2) = F_2 \text{ in } G_2, \quad k_2 \boldsymbol{\nabla} p_2 \cdot \mathbf{n} = h_2 \text{ on } \Gamma_2, \tag{13b}$$

$$\beta_1 p_1 - \beta_2 p_2 = g, \quad \beta_2 k_1 \boldsymbol{\nabla} p_1 \cdot \mathbf{n} - \beta_1 k_2 \boldsymbol{\nabla} p_2 \cdot \mathbf{n} = \beta_2 h \text{ on } \Gamma. \tag{13c}$$

Note that the coefficients β_1, β_2 were defined in the convex set K, and those same coefficients appear in *both* of the interface conditions.

A Mixed-Mixed Formulation

Here we 'mix' the gradient and divergence formulations to get **both** interface conditions **weak**. In particular, there are **no constraints** to couple the spaces.

$$\mathbf{V} \equiv \{\mathbf{u} \in \mathbf{L}^2(G_1) \times \mathbf{L}^2_{div}(G_2) : \ \alpha^{1/2}\mathbf{u}_2 \cdot \mathbf{n} \in L^2(\Gamma)\} \quad Q \equiv H^1(G_1) \times L^2(G_2)$$
$$\mathcal{A}\mathbf{u}(\mathbf{v}) = \int_{G_1} a_1\mathbf{u}_1 \cdot \mathbf{v}_1 \, dx + \int_{G_2} a_2\mathbf{u}_2 \cdot \mathbf{v}_2 \, dx + \int_\Gamma \alpha\mathbf{u}_2 \cdot \mathbf{n} \, \mathbf{v}_2 \cdot \mathbf{n} \, dS$$
$$\mathcal{B}\mathbf{u}(q) = \int_{G_1} \beta_1\mathbf{u}_1 \cdot \boldsymbol{\nabla} q_1 \, dx - \int_{G_2} \beta_2 \boldsymbol{\nabla} \cdot \mathbf{u}_2 q_2 - \int_\Gamma \beta_3\mathbf{u}_2 \cdot \mathbf{n} \, q_1 \, dS$$
$$\mathcal{C}p(q) = \int_{G_1} c_1 p_1 q_1 \, dx + \int_{G_2} c_2 p_2 q_2 \, dx + \int_\Gamma c_3 p_1 q_1 \, dS$$

The mixed problem is

$$\begin{aligned} \mathbf{u} \in \mathbf{V}, \ p \in Q : \ \mathcal{A}\mathbf{u} + \mathcal{B}'p &= f \text{ in } \mathbf{V}', \\ -\mathcal{B}\mathbf{u} + \mathcal{C}p &= g \text{ in } Q', \end{aligned} \tag{14}$$

and it has the form

$$\begin{aligned} \mathbf{u}_1 \in \mathbf{L}^2(G_1), \ \mathbf{u}_2 \in \mathbf{L}^2_{div}(G_2), \ \mathbf{u}_2 \cdot \mathbf{n} \in L^2(\Gamma), \\ p_1 \in H^1(G_1), \ p_2 \in L^2(G_2) : \\ \int_{G_1} (a_1\mathbf{u}_1 \cdot \mathbf{v}_1 \, dx + \beta_1 \boldsymbol{\nabla} p_1 \cdot \mathbf{v}_1 - \beta_1\mathbf{u}_1 \cdot \boldsymbol{\nabla} q_1 + c_1 p_1 q_1) \, dx \\ + \int_{G_2} (a_2\mathbf{u}_2 \cdot \mathbf{v}_2 - \beta_2 p_2 \boldsymbol{\nabla} \cdot \mathbf{v}_2 + \beta_2 \boldsymbol{\nabla} \cdot \mathbf{u}_2 \, q_2 + c_2 p_2 q_2) \, dx \\ + \int_\Gamma (\alpha\mathbf{u}_2 \cdot \mathbf{n}\mathbf{v}_2 \cdot \mathbf{n} - \beta_3 p_1 \mathbf{v}_2 \cdot \mathbf{n} + \beta_3\mathbf{u}_2 \cdot \mathbf{n} \, q_1 + c_3 p_1 q_1) \, dS \\ = \int_G (\mathbf{f} \cdot \mathbf{v} + g \, q) \, dx + \int_\Gamma (f_0 \mathbf{v}_2 \cdot \mathbf{n} + g_0 q_1) \, dS \end{aligned}$$

for all $\mathbf{v}_1 \in \mathbf{L}^2(G_1), \ \mathbf{v}_2 \in \mathbf{L}^2_{div}(G_2)$ with $\mathbf{v}_2 \cdot \mathbf{n} \in L^2(\Gamma), \ q_1 \in H^1(G_1), \ q_2 \in L^2(G_2)$.

The **strong form** of the system is

$$\begin{aligned} a_1\mathbf{u}_1 + \beta_1 \boldsymbol{\nabla} p_1 = \mathbf{f}_1, \ c_1 p_1 + \boldsymbol{\nabla} \cdot \beta_1 \mathbf{u}_1 = g_1 \text{ in } G_1, \\ a_2\mathbf{u}_2 + \boldsymbol{\nabla} \beta_2 p_2 = \mathbf{f}_2, \ c_2 p_2 + \beta_2 \boldsymbol{\nabla} \cdot \mathbf{u}_2 = g_2 \text{ in } G_2, \end{aligned}$$

the boundary conditions

$$\mathbf{u}_1 \cdot \mathbf{n} = 0 \text{ on } \partial G_1 - \Gamma, \quad p_2 = 0 \text{ on } \partial G_2 - \Gamma,$$

and

$$\int_\Gamma (-\beta_1 \mathbf{u}_1 \cdot \mathbf{n}\, q_1 + \beta_2 p_2 \mathbf{v}_2 \cdot \mathbf{n} + \alpha \mathbf{u}_2 \cdot \mathbf{n} \mathbf{v}_2 \cdot \mathbf{n}$$
$$- \beta_3 p_1 \mathbf{v}_2 \cdot \mathbf{n} + \beta_3 \mathbf{u}_2 \cdot \mathbf{n}\, q_1 + c_3 p_1 q_1) dS = \int_\Gamma (f_0 \mathbf{v}_2 \cdot \mathbf{n} + g_0 q_1)\, dS$$

for all $\mathbf{v}_2 \in \mathbf{L}^2_{div}(G_2)$ with $\mathbf{v}_2 \cdot \mathbf{n} \in L^2(\Gamma),\ q_1 \in H^1(G_1)$. That is,

$$\beta_2 p_2 - \beta_3 p_1 + \alpha \mathbf{u}_2 \cdot \mathbf{n} = f_0 \text{ and}$$
$$-\beta_1 \mathbf{u}_1 \cdot \mathbf{n} + \beta_3 \mathbf{u}_2 \cdot \mathbf{n} + c_3 p_1 = g_0 \text{ on } \Gamma.$$

Note that the first term in each of these equations is defined only for *solutions* of the problem. Both equations are *dual statements*. Moreover, we can prescribe the ratios of p_1, p_2 and $\mathbf{u}_1 \cdot \mathbf{n}$, $\mathbf{u}_2 \cdot \mathbf{n}$ independently with the three numbers β_1, β_2, β_3.

References

[1] Vincent Martin, Jérôme Jaffré, and Jean E. Roberts. Modeling fractures and barriers as interfaces for flow in porous media. *SIAM J. Sci. Comput.*, 26(5):1667–1691 (electronic), 2005.