1. Evolution Equations

An unbounded linear operator $L : D \to H'$ with domain D in the Hilbert space H is *monotone* if

$$Lx(x) \geq 0 \ , \qquad x \in D \ ,$$

and it is *maximal monotone* if, in addition, $\mathcal{R} + L$ maps D onto H', where $\mathcal{R}$ is the Riesz map $\mathcal{R} : H \to H'$, $\mathcal{R}x(y) = (x, y)_H$. Sufficient conditions for an initial-value problem to be well-posed are provided by the Hille-Yosida theorem.

Theorem 1.1. *Let the operator $L : D \to H'$ be maximal monotone on the Hilbert space H. Then for every $w^0 \in D(A)$ and $f \in C^1([0, \infty), H')$ there is a unique solution $w \in C^1([0, \infty), H)$ of the initial-value problem*

$$\frac{d}{dt}\mathcal{R}w(t) + Lw(t) = f(t) \ , \qquad t > 0 \ , \quad w(0) = w^0. \tag{1}$$

If additionally L is self-adjoint, then for each $w^0 \in H$ and Hölder continuous $f \in C^\beta([0, \infty), H')$, $0 < \beta < 1$, there is a unique solution $w \in C([0, \infty), H) \cap C^1((0, \infty), H)$ of (1)*.*

Example 2. *Let $H = L^2(0, 1)$ with the usual scalar product, so $\mathcal{R}$ is the identity, $D = \{v \in H^1(0, 1) : \ v(0) = 0\}$, and $Lv = \partial v$ for $v \in D$. Then L is maximal monotone. The equation*

$$v + Lv = f \ \text{in} \ L^2(0, 1)$$

corresponds to the problem

$$v(x) + \partial v(x) = f(x), \ x \in (0, 1), \ v(0) = 0,$$

and the equation (1) *corresponds to the problem*

$$\partial_t w(x, t) + \partial_x w(x, t) = f(x, t),$$
$$w(x, 0) = w^0(x), \ w(0, t) = 0, \ t > 0, \ 0 < x < 1,$$

for the advection equation*.*

Example 3. *Let $\rho \in L^\infty(0, 1)$ with $\rho(x) \geq \rho_0 > 0$. Set $H_\rho = L^2(0, 1)$ with the scalar product $(u, v)_{H_\rho} = \int_0^1 \rho(x)u(x)v(x)\,dx$. Then the Riesz map is given by $\mathcal{R}u = \rho u$. Set $D = \{v \in H^2(0, 1) : \ v(0) = 0, \ v'(1) = 0\}$, and $Lv = -\partial^2 v$ for $v \in D$. Then L is maximal monotone. The equation*

$$\mathcal{R}v + Lv = f \ \text{in} \ L^2(0, 1)$$

corresponds to the boundary-value problem

$$\rho(x)\, v(x) - \partial^2 v(x) = f(x), \ x \in (0, 1), \ v(0) = 0, \ v'(1) = 0,$$

and the equation (1) *corresponds to the initial-boundary-value problem*

$$\rho(x)\, \partial_t w(x, t) - \partial_x^2 w(x, t) = f(x, t),$$
$$w(x, 0) = w^0(x), \ w(0, t) = 0, \partial_x w(1, t) = 0, \ t > 0, \ 0 < x < 1,$$

for the diffusion equation*. This L is self-adjoint.*

Mixed Problems, I. Consider first the evolutionary Stokes system. We defined the subspace $\mathbf{V}_0 = \{\mathbf{w} \in \mathbf{H}_0^1(\Omega) : \ \boldsymbol{\nabla} \cdot \mathbf{w} = 0 \text{ in } \Omega\}$ of $\mathbf{H}_0^1(\Omega)$ and showed the first component $\mathbf{u}(t) = \mathbf{u}(\cdot, t)$ (velocity) of the solution is characterized by the *Stokes equation*

$$\text{(2)} \quad \mathbf{u}(t) \in \mathbf{V}_0 : \int_\Omega \rho(x) \tfrac{\partial}{\partial t}\mathbf{u}(t) \cdot \mathbf{w}\, dx + \mu \int_\Omega \boldsymbol{\nabla} u_i(t) \cdot \boldsymbol{\nabla} w_i \, dx$$
$$= \int_\Omega \mathbf{f}(t) \cdot \mathbf{w}\, dx \text{ for all } \mathbf{w} \in \mathbf{V}_0 \;\; t > 0, \;\; \mathbf{u}(\cdot, 0) = \mathbf{u}_0(\cdot).$$

Let $\rho \in L^\infty(\Omega)$ satisfy $\rho(x) \geq \rho_0 > 0$, and define $\mathbf{H}_\rho$ to be the closure of $\mathbf{V}_0$ with the scalar product $(\mathbf{u}, \mathbf{v})_{\mathbf{H}_\rho} = \int_\Omega \rho(x) \mathbf{u}(x) \cdot \mathbf{v}(x)\, dx$. Then the Riesz map is given by $\mathcal{R}\mathbf{u} = \rho \mathbf{u}$ and $\mathbf{V}_0 \hookrightarrow \mathbf{H}_\rho$ is continuous with $\mathbf{V}_0$ dense in $\mathbf{H}_\rho$, hence, $\mathbf{H}_\rho' \subset \mathbf{H}_0^1(\Omega)' = \mathbf{H}^{-1}(\Omega)$. Note also that $\mathbf{V}_0 \subset \mathbf{H}_0^1(\Omega)$.

Define the operator $\mathcal{A} : \mathbf{H}_0^1(\Omega) \to \mathbf{H}^{-1}(\Omega)$ by

$$\mathcal{A}\mathbf{u}(\mathbf{w}) = \mu \int_\Omega \boldsymbol{\nabla} u_i(t) \cdot \boldsymbol{\nabla} w_i \, dx, \;\; \mathbf{u}, \, \mathbf{w} \in \mathbf{H}_0^1(\Omega),$$

and let $f(t) \in \mathbf{H}_0^1(\Omega)'$ be given for each $t > 0$. Then (2) takes the form

$$\text{(3)} \qquad \mathbf{u}(t) \in \mathbf{V}_0, \;\; \tfrac{d}{dt}\mathcal{R}\mathbf{u}(t)(\mathbf{w}) + \mathcal{A}\mathbf{u}(t)(\mathbf{w}) = f(t)(\mathbf{w}), \;\; \mathbf{w} \in \mathbf{V}_0, \;\; t > 0 \;, \quad \mathbf{u}(0) = \mathbf{u}_0.$$

Define $D = \{\mathbf{v} \in \mathbf{V}_0 : \mathcal{A}\mathbf{v} \in \mathbf{H}_\rho'\}$ and set $L\mathbf{u} = \mathcal{A}\mathbf{u}$ for $\mathbf{u} \in D$. Then L is maximal monotone and symmetric, so Theorem 1.1 shows that for each $\mathbf{u}_0 \in \mathbf{H}_\rho$ and Hölder continuous $f \in C^\beta([0, \infty), \mathbf{H}_\rho')$, $0 < \beta < 1$, there is a unique solution $\mathbf{u} \in C([0, \infty), \mathbf{H}_\rho) \cap C^1((0, \infty), \mathbf{H}_\rho)$ of (3) with $\mathbf{u}(t) \in \mathbf{V}_0$ and $\Delta \mathbf{u}(t) \in \mathbf{H}_\rho'$ for each $t > 0$. Since $\mathbf{V}_0$ is the kernel of the divergence, we then obtain the pressure $p(t) \in L^2(\Omega)$ for $t > 0$ to complete the system.

The Stokes system is an example of a mixed formulation in which the evolution term (time derivative) occurs in the first equation. We shall call this a mixed evolution system of *Type I*. More generally, suppose we have a pair of spaces $V, \; W$ and operators

$$\mathcal{A} : V \to V', \;\; \mathcal{B} : V \to W', \;\; \mathcal{C} : W \to W'$$

which determine an evolution system in mixed form

$$u(t) \in V, \;\; p(t) \in W : \;\; \rho \tfrac{\partial}{\partial t} u(t) + \mathcal{A}u(t) + \mathcal{B}'p(t) = f \text{ in } V',$$
$$\text{(4)} \qquad -\mathcal{B}u(t) + \mathcal{C}p(t) = 0 \text{ in } W'.$$

Assume that $V \hookrightarrow H_\rho$ is dense, where $H_\rho = L^2(\Omega)$ is given as in the preceding example, so that $H_\rho' \hookrightarrow V'$.

Define an operator $L : D \to H_\rho'$ on the domain $D \subset H_\rho$ by $D = \{\mathbf{v} \in V : \mathcal{A}\mathbf{v} + \mathcal{B}'p \in H_\rho'$ and $-\mathcal{B}\mathbf{v} + \mathcal{C}p = 0$ for some $p \in W\}$. Then set $L\mathbf{v} = \mathcal{A}\mathbf{v} + \mathcal{B}'p$ for $\mathbf{v} \in D$. The operator L is monotone, since $L\mathbf{v}(\mathbf{v}) = \mathcal{A}\mathbf{v}(\mathbf{v}) + \mathcal{C}p(p) \geq 0$, and maximal if the system

$$v \in V, \;\; p \in W : \;\; \rho\, v + \mathcal{A}v + \mathcal{B}'p = f \text{ in } V',$$
$$-\mathcal{B}v + \mathcal{C}p = 0 \text{ in } W'.$$

always has a unique solution with $\mathcal{B}'p$ unique. Then $u(t), \; p(t)$ is a solution of (4) if and only if

$$\text{(5)} \qquad \rho \tfrac{\partial}{\partial t} u(t) + Lu(t) = f(t) \text{ in } H_\rho'.$$

The operator L is symmetric, so the initial-value problem can be solved for each $\mathbf{u}(0) = \mathbf{u}_0 \in H_\rho$. Note that if the second equation in (4) is non-homogeneous, then we can use a translation to reduce the problem back to (4). This will be illustrated in the next example.

Mixed Problems, II. Porous media flow provides an example of a mixed evolution system of *Type II* in which the evolution term is in the second equation. The constitutive law of Darcy is

$$a(x)\,\mathbf{u} + \nabla p = \mathbf{g}(x), \tag{6a}$$

where $\mathbf{u}$ represents the flux, p the pressure, and $a(\cdot)$ the resistance of the porous medium, *i.e.*, viscosity times the reciprocal of permeability, and $\mathbf{g}$ represents the gravity force. After dividing by the (constant) density, the fluid conservation law is

$$c(x)\,\tfrac{\partial}{\partial t} p + \nabla \cdot \mathbf{u} = f(x,t). \tag{6b}$$

We shall show that these equations together with an initial condition and boundary condition

$$p(x,0) = p_0(x), x \in \Omega, \quad \mathbf{u}(s,t)\cdot\mathbf{n} = 0,\ s \in \partial\Omega, \tag{6c}$$

constitute a well-posed problem.

Define the operators

$$\begin{aligned}
&\mathcal{A} : V \to V',\ \mathcal{B} : V \to W',\ \mathcal{C} : W \to W' \\
&\mathcal{A}\mathbf{u}(\mathbf{v}) = \int_\Omega a(x)\mathbf{u}\cdot\mathbf{v}\,dx, \quad \mathbf{u},\mathbf{v} \in V, \\
&\mathcal{B}\mathbf{u}(q) = \int_\Omega \mathbf{u}\cdot\boldsymbol{\nabla} q\,dx, \quad \mathbf{u} \in V,\ q \in W, \\
&\mathcal{C}p(q) = \int_\Omega c(x)pq\,dx, \quad p,q \in W
\end{aligned}$$

on the spaces $V \equiv \mathbf{L}^2(\Omega)$ and $W = H^1(\Omega)$.

The mixed formulation of the *initial-boundary-value problem* (6) takes the form

$$\begin{aligned}
&\mathbf{u}(t) \in V,\ p(t) \in W :\ \mathcal{A}\mathbf{u}(t)(\mathbf{v}) + \mathcal{B}'p(t)(\mathbf{v}) = \mathbf{g}(\mathbf{v}), \\
&\mathcal{C}\tfrac{\partial}{\partial t}p(t)(q) - \mathcal{B}\mathbf{u}(t)(q) = f(t)(q),\ \mathbf{v} \in V,\ q \in W.
\end{aligned} \tag{7}$$

Remark 3.1. *The backward-difference approximation* $\frac{\partial}{\partial t}p(t) \approx (p(t) - p(t-h))/h$ *leads to the* stationary system

$$\begin{aligned}
&\mathbf{u}(t) \in V,\ p(t) \in W :\ \mathcal{A}\mathbf{u}(t)(\mathbf{v}) + \mathcal{B}'p(t)(\mathbf{v}) = \mathbf{g}(\mathbf{v}), \\
&-\mathcal{B}\mathbf{u}(t)(q) + \lambda\mathcal{C}p(t)(q) = \lambda\mathcal{C}p(t-h)(q) + f(q),\ \mathbf{v} \in V,\ q \in W,
\end{aligned} \tag{8}$$

where $\lambda = h^{-1}$ *is the reciprocal of the time increment,* $h > 0$.

Let's resolve (7). First find a pair $u_g,\ p_g$ which is a solution of a stationary problem,

$$\begin{aligned}
\mathbf{u}_g \in V,\ p_g \in W :\ \mathcal{A}\mathbf{u}_g + \mathcal{B}'p_g &= \mathbf{g}, \\
-\mathcal{B}\mathbf{u}_g &= 0.
\end{aligned}$$

Subtract u_g from $\mathbf{u}(t)$ to get the problem (7) but with $\mathbf{g} = \mathbf{0}$ and the initial condition $\mathcal{C}p(0) = \mathcal{C}p_0 - \mathcal{C}p_g$ for the translates, $\mathbf{u}(t) - \mathbf{u}_g,\ p(t) - p_g$. Thus, we want to solve the reduced evolution system

$$\begin{aligned} \mathbf{u}(t) \in V,\ p(t) \in W:\ & \mathcal{A}\mathbf{u}(t) + \mathcal{B}'p(t) = \mathbf{0} \text{ in } V', \\ & -\mathcal{B}\mathbf{u}(t) + \mathcal{C}\tfrac{\partial}{\partial t}p(t) = f(t) \text{ in } W'. \end{aligned} \tag{9}$$

Let $c \in L^\infty(\Omega),\ c(x) \geq c_0 > 0$ and W_c denote the Hilbert space $L^2(\Omega)$ with the scalar product $(u, v)_c = \int_\Omega c(x)u(x)v(x)\,dx$. Note that $W \hookrightarrow W_c$ and $W_c' \hookrightarrow W'$.

Define L as follows: $Lp = f$ if $f \in W_c'$ and

$$\mathbf{u} \in V,\ p \in W:\ \mathcal{A}\mathbf{u} + \mathcal{B}'p = \mathbf{0},\ -\mathcal{B}\mathbf{u} = f. \tag{10}$$

Then $u(t),\ p(t)$ is a solution of (9) if and only if

$$\mathcal{C}\tfrac{\partial}{\partial t}p(t) + Lp(t) = f(t) \text{ in } W_c',\ \mathcal{C}p(0) = \mathcal{C}p_0. \tag{11}$$

Note that L is

- monotone: $Lp(p) = -\mathcal{B}\mathbf{u}(p) = -\mathcal{B}'p(\mathbf{u}) = \mathcal{A}\mathbf{u}(\mathbf{u}) \geq 0$
- maximal: $Cp + Lp = f$ is solvable for $f \in W_c' = L^2(\Omega)$

because we can solve the equivalent mixed system

$$\begin{aligned} \mathbf{u} \in V,\ p \in W:\ & \mathcal{A}\mathbf{u} + \mathcal{B}'p = \mathbf{0}, \\ & -\mathcal{B}\mathbf{u} + \mathcal{C}p = f. \end{aligned} \tag{12}$$

This corresponds to

$$\begin{aligned} \mathbf{u} \in \mathbf{L}^2(\Omega),\ p \in H^1(\Omega):\ & a(x)\,\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{0} \text{ and} \\ & \boldsymbol{\nabla}\cdot\mathbf{u} + c(x)p = f \text{ in } \mathbf{L}^2(\Omega), \\ & \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega. \end{aligned} \tag{13}$$

Remark 3.2. *More generally, if $\mathcal{C}(\cdot)(\cdot)$ is a scalar product on W and we denote the completion of this space by W_c, then the corresponding extension $\mathcal{C} : W_c \to W_c'$ is the Riesz map and the preceding construction applies.*

Mixed Problems, III. If the evolution terms appear in both equations, we call it a mixed evolution system of *Type III*. These take the form

$$u(t) \in V,\ p(t) \in W:\ \rho\tfrac{\partial}{\partial t}u(t) + \mathcal{A}u(t)(v) + \mathcal{B}'p(t)(v) = g(t)(v), \tag{14}$$

$$\mathcal{C}\tfrac{\partial}{\partial t}p(t)(q) - \mathcal{B}u(t)(q) = f(t)(q),\ v \in V,\ q \in W,\ t > 0. \tag{15}$$

Write this system in the form

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho u(t) \\ \mathcal{C}p(t) \end{pmatrix} + \begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & 0 \end{pmatrix}\begin{pmatrix} u(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} g(t) \\ f(t) \end{pmatrix}. \tag{16}$$

Choose $\mathcal{R}$ on the space H obtained by completing $V \times W$ with the scalar product

$$\left(\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix}\right)_H = (u, v)_{H_\rho} + \mathcal{C}p(q)$$

so $\mathcal{R} = \begin{pmatrix} \rho & 0 \\ 0 & \mathcal{C} \end{pmatrix}$. Construct L from the matrix $\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & 0 \end{pmatrix}$ restricted to H' as before.

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