Problems

Assume $V = \mathbb{R}^M$ and $W = \mathbb{R}^N$ in Prob 1 and Prob 2.

1. Let $\mathcal{E} \in \mathcal{L}(V, V')$ and $h \in V'$.
The *Linear Complementarity Problem* is to find

$$\mathbf{z} \in V \text{ such that } \mathbf{z} \geq \mathbf{0},\ \mathcal{E}\mathbf{z} \geq h,\ (\mathcal{E}\mathbf{z} - h)(\mathbf{z}) = 0. \tag{1}$$

(a) Show that (1) is equivalent to a variational inequality,

$$\mathbf{z} \in K:\ (\mathcal{E}\mathbf{z} - h)(\mathbf{w} - \mathbf{z}) \geq 0 \text{ for all } \mathbf{w} \in K \tag{2}$$

with $K = \{\mathbf{z} \in V : \mathbf{z} \geq \mathbf{0}\}$.
(b) If $\mathcal{E}$ is symmetric and nonnegative ($\mathcal{E} = \mathcal{E}' \geq 0$), show that a solution $\mathbf{z}$ of (1) is characterized by a minimization problem.

2. Let $\mathcal{A} \in \mathcal{L}(V, V')$ be symmetric and nonnegative, $f \in V'$, $\mathcal{B} \in \mathcal{L}(V, W')$, $g \in W'$, and set $J(\mathbf{u}) = \frac{1}{2}\mathcal{A}\mathbf{u}(\mathbf{u}) - f(\mathbf{u}),\ \mathbf{u} \in V$.
The *Constrained Quadratic Problem* is to find $\mathbf{x} \in V$ which minimizes $J(\mathbf{u})$ subject to the constraint $\mathcal{B}\mathbf{u} \leq g$. Assume that $\mathcal{B}'$ is injective.
Show that a solution $\mathbf{x}$ is characterized by the system

$$\mathcal{A}\mathbf{x} + \mathcal{B}'\mathbf{y} = f, \tag{3a}$$

$$\mathcal{B}\mathbf{x} - g \leq 0,\ \mathbf{y} \geq \mathbf{0},\ (\mathcal{B}\mathbf{x} - g)(\mathbf{y}) = 0. \tag{3b}$$

Hint: Set $K = \{u \in V : \mathcal{B}u - g \leq 0\}$ and show (3) is equivalent to $x \in K : (\mathcal{A}x - f)(u - x) \geq 0$ for all $u \in K$.

Assume that V and W are Hilbert spaces. Let $\mathcal{A} \in \mathcal{L}(V, V')$ be symmetric and nonnegative, $f \in V'$, $\mathcal{B} \in \mathcal{L}(V, W')$ have closed range, and $g \in W'$. Let the symmetric and positive $\mathcal{C} \in \mathcal{L}(W, W')$ be the Riesz isomorphism: $\mathcal{C}p(q) = (p, q)_W = (\mathcal{C}p, \mathcal{C}q)_{W'}$ for all $p,\ q \in W$.

3. To minimize $\frac{1}{2}\mathcal{A}\mathbf{v}(\mathbf{v}) - f(\mathbf{v})$ and approximate the constraint, $\mathcal{B}\mathbf{v} = g$, choose a small $\varepsilon > 0$ and minimize

$$J_\varepsilon(\mathbf{v}) = \tfrac{1}{2}\mathcal{A}\mathbf{v}(\mathbf{v}) - f(\mathbf{v}) + \tfrac{1}{2\varepsilon}\|\mathcal{B}\mathbf{v} - g\|^2_{W'}, \quad \mathbf{v} \in V. \tag{4}$$

(a) Show that a minimum is obtained at $\mathbf{u}$ if and only if

$$\mathbf{u} \in V:\ \mathcal{A}\mathbf{u} + \tfrac{1}{\varepsilon}\mathcal{B}'\mathcal{C}^{-1}(\mathcal{B}\mathbf{u} - g) = f \text{ in } V'. \tag{5}$$

This is equivalent to the system

$$\mathbf{u} \in V,\ p \in W : \quad \mathcal{A}\mathbf{u} + \mathcal{B}'p = f \text{ in } V', \tag{6}$$
$$\mathcal{B}\mathbf{u} - \varepsilon\mathcal{C}p = g \text{ in } W'.$$

A Lagrangian for the system (6) is

$$\mathcal{L}(\mathbf{v}, q) = \tfrac{1}{2}\mathcal{A}\mathbf{v}(\mathbf{v}) - f(\mathbf{v}) + (\mathcal{B}\mathbf{v} - g)q - \tfrac{\varepsilon}{2}\mathcal{C}q(q), \mathbf{v} \in V,\ q \in W \tag{7}$$

(b) Show that stable points of (7) correspond to solutions of the system (6).
(c) Under the additional assumptions of either Theorem 1.9 or Theorem 1.10, compute each of

$$\overline{\varphi}(v) = \sup_{q \in W} \mathcal{L}(v, q), \quad \underline{\varphi}(q) = \inf_{v \in V} \mathcal{L}(v, q).$$

(d) Find the dual problem to (5).

4. The norm on $H^{1/2}(\partial G)$ is defined by a constrained minimization problem

$$\tfrac{1}{2}\|\mu\|_{\mathbb{B}}^2 = \inf_{v \in H^1(G):\ \gamma(v) = \mu} \tfrac{1}{2}\|v\|_{H^1(G)}^2. \tag{8}$$

Characterize this problem with a mixed system and display the dual problem.

5. The Dirichlet-Neumann boundary-value-problem was formulated as a mixed system (6) in two ways.
(a) For the gradient mixed formulation of the Dirichlet-Neumann boundary-value-problem, find the primal and dual problems. (Use Problems above or take case $\mathcal{C} = 0$.)
(b) Repeat for the divergence mixed formulation of the Dirichlet-Neumann boundary-value-problem.