1 The Stokes' System

The motion of a (possibly compressible) homogeneous fluid is described by its density $\rho(x,t)$, pressure $p(x,t)$ and velocity $\mathbf{v}(x,t)$. Assume that the fluid is *barotropic*, i.e., the density and pressure are related by a *state equation*

$$\rho = s(p) \tag{1a}$$

in which the constitutive function $s(\cdot)$ is non-decreasing and characterizes the type of fluid. The *conservation of mass* of fluid is expressed by

$$\dot\rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = \rho\, g(x)\,, \tag{1b}$$

and the *conservation of momentum* has the form

$$\rho\dot{\mathbf{v}} - (\lambda_1+\mu_1)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \mu_1\Delta\mathbf{v} + \boldsymbol{\nabla}p = \rho\,\mathbf{f}(x) \text{ in } \Omega. \tag{1c}$$

Here $\mathbf{f}(x)$ is the mass-distributed force density over Ω and $g(x)$ is the mass-distributed fluid source. These three nonlinear equations comprise the system for a general *compressible fluid.*

In order to obtain a linear model to approximate the solutions of this system, we consider small oscillations about a *rest state* at which $\mathbf{v} = \mathbf{0}$ and, hence, the functions $\rho_0(x)\,,\ p_0(x)\,,\ \mathbf{f}(x)\,,\ g(x)$ satisfy

$$\rho_0 = s(p_0)\,,\quad g(x) = 0,\quad \boldsymbol{\nabla}p_0 = \rho_0\,\mathbf{f}(x)\,.$$

The quantity $c_0(x) \equiv \frac{1}{\rho_0(x)}\frac{\partial\rho}{\partial p}(p_0(x))$ denotes the *compressibility* of the fluid at the rest state. Using the chain rule with the state equation (1a) yields

$$\boldsymbol{\nabla}\rho_0(x) = \frac{\partial\rho}{\partial p}\boldsymbol{\nabla}p_0(x) = \rho_0(x)c_0(x)\rho_0(x)\mathbf{f}(x)\,.$$

Introduce a small parameter $\varepsilon > 0$ to characterize the size of the oscillations and the deviations from the rest state, and consider the corresponding asymptotic expansions

$$\begin{aligned}
\rho &= \rho_0(x) + \varepsilon\rho^1(x,t) + \mathcal{O}(\varepsilon^2)\,,\\
p &= p_0(x) + \varepsilon p^1(x,t) + \mathcal{O}(\varepsilon^2)\,,\\
\mathbf{v} &= \varepsilon\mathbf{v}^1(x,t) + \mathcal{O}(\varepsilon^2)\,,\\
g &= \varepsilon g^1(x) + \mathcal{O}(\varepsilon^2)\,.
\end{aligned}$$

Again from the chain rule we obtain

$$\dot\rho^1(x,t) = \rho_0(x)c_0(x)\dot p^1(x,t)\,,\quad \rho^1(x,t) = \rho_0(x)c_0(x)p^1(x,t)\,.$$

Conservation of mass (1b) implies to first order in ε that

$$\dot\rho^1 + \boldsymbol{\nabla}\cdot(\rho_0\mathbf{v}^1) = \rho_0(x)\,g^1(x)\,.$$

From conservation of momentum (1c) we get

$$\varepsilon(\rho_0 + \varepsilon\rho^1)\dot{\mathbf{v}}^1 - \varepsilon(\lambda_1 + \mu_1)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}^1) - \varepsilon\mu_1\Delta\mathbf{v}^1 + \boldsymbol{\nabla}(p_0 + \varepsilon p^1) = (\rho_0 + \varepsilon\rho^1)\mathbf{f}(x)\,,$$

and then the definitions of rest state and compressibility give the linear system

$$\rho_0(x)\dot{\mathbf{v}}^1 - (\lambda_1 + \mu_1)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}^1) - \mu_1\Delta\mathbf{v}^1 + \boldsymbol{\nabla}p^1 = c_0(x)\rho_0(x)\mathbf{f}(x)\,p^1\,, \tag{2a}$$
$$c_0(x)\dot{p}^1 + \boldsymbol{\nabla}\cdot\mathbf{v}^1 + c_0(x)\rho_0(x)\mathbf{f}(x)\cdot\mathbf{v}^1 = g^1(x)\,, \tag{2b}$$
$$\mathbf{v}^1 = \mathbf{0} \text{ on } \Gamma_0\,,\ \lambda_1(\boldsymbol{\nabla}\cdot\mathbf{v}^1)\mathbf{n} + 2\mu_1\,\varepsilon(\mathbf{v}^1,\mathbf{n}) - p\,\mathbf{n} = 0 \text{ on } \Gamma_1 \tag{2c}$$

for small variations of a *compressible fluid*. Here the two sets $\Gamma_0,\ \Gamma_1$ comprise a partition of the boundary $\Gamma = \partial\Omega$. In the incompressible case, $c_0(x) = 0$, we obtain the *Stokes' system*,

$$\rho_0(x)\dot{\mathbf{v}}^1 - \mu_1\Delta\mathbf{v}^1 + \boldsymbol{\nabla}p^1 = (\lambda_1 + \mu_1)\boldsymbol{\nabla}g^1(x)\,,$$
$$\boldsymbol{\nabla}\cdot\mathbf{v}^1 = g^1(x)\,,$$
$$\mathbf{v}^1 = \mathbf{0} \text{ on } \Gamma_0\,,\ 2\mu_1\,\varepsilon(\mathbf{v}^1,\mathbf{n}) - p\,\mathbf{n} = 0 \text{ on } \Gamma_1\,.$$

Navier-Stokes System

The *material derivative* of velocity has been approximated here by the acceleration. For the calculation of the acceleration of a fluid element, the displacement of that element along with the points must be considered. The momentum of the small subdomain $B \subset \Omega$ travelling with the fluid is $\int_B \rho\mathbf{v}(x+\mathbf{u}(x,t),t)\,dx$, and its derivative is given by the chain rule as

$$\int_B \rho\left(\frac{\partial\mathbf{v}(x+\mathbf{u}(x,t),t)}{\partial t} + \partial_j\mathbf{v}(x+\mathbf{u}(x,t),t)\,v_j(x+\mathbf{u}(x,t),t)\right)dx\,.$$

Thus, the momentum equation for the fluid includes the additional term $(\mathbf{v}\cdot\nabla)\mathbf{v} = v_j\partial_j\mathbf{v}$, and the corresponding system is the *Navier-Stokes system*

$$\rho\dot{\mathbf{v}} - \mu_1\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \boldsymbol{\nabla}p = \mathbf{f}\,,\quad \boldsymbol{\nabla}\cdot\mathbf{v} = g \text{ in } \Omega\,,$$
$$\mathbf{v} = 0 \text{ on } \Gamma_0\,,\quad -p\,\mathbf{n} + 2\mu_1\,\varepsilon(\mathbf{v},\mathbf{n}) = \mathbf{g} \text{ on } \Gamma_1\,,$$

for a viscous incompressible fluid. Note that the quadratic nonlinearity arises from the *geometry* of the motion, and it is *not* based on any independent assumptions.

The Stokes Equation

The (slow) flow of an incompressible homogeneous fluid is described by its pressure $p(x,t)$ and velocity $\mathbf{v}(x,t)$. The (evolutionary) *Stokes system* is to find such a pair of functions on the smoothly bounded region Ω in $\mathbb{R}^n$ for $t > 0$ which satisfy the initial-boundary-value problem

$$\rho_0(x)\dot{\mathbf{v}} - \mu\Delta\mathbf{v} + \boldsymbol{\nabla}p = \mathbf{f}\,,\qquad \boldsymbol{\nabla}\cdot\mathbf{v} = 0 \text{ in } \Omega\times\mathbb{R}^+\,,$$
$$\mathbf{v} = \mathbf{0} \text{ on } \Gamma\times\mathbb{R}^+\,,$$
$$\mathbf{v}(0) = \mathbf{v}_0 \text{ in } \Omega\,,$$

where $\Gamma = \partial\Omega$ is the boundary of Ω. The external force gives an acceleration of the flow $\mathbf{v}(x,t)$. In order to maintain the null-divergence condition of incompressibility, a pressure gradient arises to avoid a change in density. That is, a pressure builds up to prevent the development of sources or sinks. This is the Lagrange multiplier that modifies the momentum equation in order to maintain the divergence-free constraint.

If we separate variables, i.e., look for a solution in the form $\mathbf{v}(x,t) = e^{\lambda t}\mathbf{u}(x)$ for some number λ, we are led to the *stationary Stokes system*

$$\rho_0(x)\lambda\mathbf{u} - \mu\Delta\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f}\,, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \text{ in } \Omega\,, \tag{3a}$$

$$\mathbf{u} = \mathbf{0} \text{ on } \Gamma\,, \tag{3b}$$

for the pair $\mathbf{u}(x)$, $p(x)$. We focus on this stationary system, but the results apply as well to the evolutionary system.

Remark 1.1. *For a pair of functions* $\mathbf{u} \in \mathbf{H}^1(\Omega)$, $q \in H^1(\Omega)$, *we have*

$$\int_\Omega \boldsymbol{\nabla}\cdot\mathbf{u}\, q\, dx = -\int_\Omega \mathbf{u}\cdot\boldsymbol{\nabla} q\, dx + \int_\Gamma \mathbf{u}\cdot\mathbf{n}\, q\, ds$$

With appropriate boundary conditions, this shows that $\mathrm{Ker}(\boldsymbol{\nabla}\cdot) = \mathrm{Rg}(\boldsymbol{\nabla})^\perp$ *so we expect* $\mathrm{Rg}(\boldsymbol{\nabla}) = \mathrm{Ker}(\boldsymbol{\nabla}\cdot)^\perp$ *up to closure in appropriate spaces. Eventually, we will need to construct carefully the gradient and divergence operators.*

Now, if the pair $\mathbf{u}(x)$, $p(x)$ is a solution to the stationary Stokes system (3), then for every $\mathbf{w} \in \mathbf{H}^1(\Omega)$ we have

$$\lambda\int_\Omega \rho_0(x)\mathbf{u}\cdot\mathbf{w}\, dx + \mu\int_\Omega \boldsymbol{\nabla} u_i\cdot\boldsymbol{\nabla} w_i\, dx - \int_\Gamma (\boldsymbol{\nabla} u_i\cdot\mathbf{n})w_i\, ds$$
$$-\int_\Omega (\boldsymbol{\nabla}\cdot\mathbf{w})p\, dx + \int_\Gamma (\mathbf{w}\cdot\mathbf{n})p\, ds = \int_\Omega \mathbf{f}\cdot\mathbf{w}\, dx\,.$$

We define the space $\mathbf{V}_0 = \{\mathbf{w} \in \mathbf{H}^1(\Omega) : \ \boldsymbol{\nabla}\cdot\mathbf{w} = 0 \text{ in } \Omega,\ \mathbf{w} = \mathbf{0} \text{ on } \Gamma\}$. The first component of the solution $\mathbf{u}$, p satisfies the *Stokes equation*

$$\mathbf{u} \in \mathbf{V}_0 : \ \lambda\int_\Omega \rho_0(x)\mathbf{u}\cdot\mathbf{w}\, dx + \mu\int_\Omega \boldsymbol{\nabla} u_i\cdot\boldsymbol{\nabla} w_i\, dx = \int_\Omega \mathbf{f}\cdot\mathbf{w}\, dx \text{ for all } \mathbf{w} \in \mathbf{V}_0\,. \tag{4}$$

It is the *projection* or *restriction* of $\mathcal{A}u - f$ to $\mathbf{V}_0 \subset \mathbf{H}^1_0(\Omega)$.

Define a continuous bilinear form $a(\cdot,\cdot)$ on $\mathbf{H}^1_0(\Omega)$ by

$$a(\mathbf{u},\mathbf{w}) = \int_\Omega (\lambda\rho_0(x)\mathbf{u}\cdot\mathbf{w} + \mu\boldsymbol{\nabla} u_i\cdot\boldsymbol{\nabla} w_i)\, dx\,, \quad \mathbf{u},\ \mathbf{w} \in \mathbf{H}^1_0(\Omega),$$

and the linear functional $f(\cdot)$ on $\mathbf{H}^1_0(\Omega)$ by

$$f(\mathbf{w}) = \int_\Omega \mathbf{f}\cdot\mathbf{w}\, dx\,, \quad \mathbf{w} \in \mathbf{H}^1_0(\Omega),$$

where $\mathbf{f} \in \mathbf{L}^2(\Omega)$ is given. Note that the principle part of $a(\cdot,\cdot)$ is a double sum

$$\boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} w_i = \sum_{i,j=1}^{n} \frac{\partial u_i}{\partial x_j} \frac{\partial w_i}{\partial x_j}$$

which contains *all* first-order derivatives. Thus, the space $\mathbf{H}_0^1(\Omega)$ is appropriate, and we have the coercivity estimate

$$a(\mathbf{w},\mathbf{w}) \geq \int_\Omega \sum_{i,j=1}^{n} \left(\frac{\partial w_i}{\partial x_j} \right)^2 dx \, .$$

This gives the *ellipticity* condition

$$a(\mathbf{w},\mathbf{w}) \geq c_0 \|\mathbf{w}\|^2, \qquad \text{for all } \mathbf{w} \in \mathbf{H}_0^1(\Omega) \, , \tag{5}$$

for some constant $c_0 > 0$. We noted already that $a(\cdot,\cdot)$ is continuous with respect to the $\mathbf{H}_0^1(\Omega)$ norm, and likewise the functional $f(\cdot)$ is continuous. We have the following result immediately.

Theorem 1.1. *There is exactly one solution of the stationary Stokes equation*

$$\mathbf{u} \in \mathbf{V}_0 : \quad a(\mathbf{u},\mathbf{w}) = f(\mathbf{w}) \textit{ for all } \mathbf{w} \in \mathbf{V}_0 \, . \tag{6}$$

There remains the issue of the sense in which the solution of the Stokes equation (4) satisfies the Stokes system (3).

The Mixed System

With the space $\mathbf{V}_0 = \{\mathbf{w} \in \mathbf{H}^1(\Omega) : \ \boldsymbol{\nabla} \cdot \mathbf{w} = 0 \text{ in } \Omega, \ \mathbf{w} = \mathbf{0} \text{ on } \Gamma\}$, the stationary Stokes equation is given by

$$\mathbf{u} \in \mathbf{V}_0 : \ \lambda \int_\Omega \rho_0(x) \mathbf{u} \cdot \mathbf{w} \, dx + \mu \int_\Omega \boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} w_i \, dx = \int_\Omega \mathbf{f} \cdot \mathbf{w} \, dx \text{ for all } \mathbf{w} \in \mathbf{V}_0 \, .$$

We need to characterize the annihilator of the space $\mathbf{V}_0$, the kernel of the divergence operator on $\mathbf{H}_0^1(\Omega)$. This will be used to show the equivalence of the weak form of the Stokes equation with the strong formulation, and it is related to the characterization of the range of the *gradient* operator from $L^2(\Omega)$ into $\mathbf{H}^{-1}(\Omega)$.

For background, we mention the following profound result on the annihilator of

$$\mathbb{V}_0 \equiv \{\mathbf{v} \in C_0^\infty(\Omega) = \mathcal{D}(\Omega) : \ \boldsymbol{\nabla} \cdot \mathbf{v} = 0\}.$$

Theorem 1.2 (de Rham (1955))**.** *Let Ω be a domain in $\mathbb{R}^n$. Then $\mathbf{f} = \boldsymbol{\nabla} p$ in $\mathcal{D}'(\Omega)$ if and only if $\mathbf{f}(\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbb{V}_0$.*

A related result is the following.

Theorem 1.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. If $p \in \mathcal{D}'(\Omega)$ satisfies $\boldsymbol{\nabla} p \in \mathbf{H}^{-1}(\Omega)$, then $p \in L^2(\Omega)$, and we have the estimate*

$$\|p\|_{L^2(\Omega)/\mathbb{R}} \leq C_\Omega \|\boldsymbol{\nabla} p\|_{\mathbf{H}^{-1}(\Omega)} \,. \tag{7}$$

Theorem 1.3 was proved by J.-L. Lions ca. 1958 for smoothly bounded Ω (published in Duvaut-Lions (1972)), by Magenes & Stampacchia (1958) if Ω has a C^1 boundary and by Neĉas (1967) for the case of a Lipschitz boundary.

Recall that the space $L^2(\Omega)/\mathbb{R}$ is just the quotient of $L^2(\Omega)$ with the constant functions. Estimates equivalent to (7) are

$$\|p\|_{L^2(\Omega)} \leq C_\Omega \left(\left| \int_\Omega p(x)\, dx \right| + \|\boldsymbol{\nabla} p\|_{\mathbf{H}^{-1}(\Omega)} \right),$$
$$\|p\|_{L^2(\Omega)} \leq C_\Omega \left(\|p\|_{\mathbf{H}^{-1}(\Omega)} + \|\boldsymbol{\nabla} p\|_{\mathbf{H}^{-1}(\Omega)} \right).$$

An immediate consequence of these estimates is the following result that is fundamental for the Stokes system.

Corollary 1.4. *Let Ω be bounded and open in $\mathbb{R}^n$ with Lipschitz boundary. The gradient operator $\boldsymbol{\nabla} : L^2(\Omega) \to \mathbf{H}^{-1}(\Omega)$ has* closed range. *If additionally Ω is connected, then the* Ker$(\boldsymbol{\nabla})$ *consists of constant functions and*

$$\|p\|_{L^2(\Omega)/\mathbb{R}} \leq C_\Omega \|\boldsymbol{\nabla} p\|_{\mathbf{H}^{-1}(\Omega)}. \tag{8}$$

The divergence operator $\boldsymbol{\nabla}\cdot : \mathbf{H}_0^1(\Omega) \to L^2(\Omega)$ is just the negative of the dual of $\boldsymbol{\nabla} : L^2(\Omega) \to \mathbf{H}^{-1}(\Omega)$, so it also has closed range, and we obtain a form of de Rham's theorem sufficient for our purposes.

Corollary 1.5. *If $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ satisfies $\mathbf{f}(\mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{V}_0$, then there is a $p \in L^2(\Omega)$ such that $\mathbf{f} = \boldsymbol{\nabla} p$. If Ω is connected, then p is unique up to a constant.*

This follows directly from the observation $\mathbf{V}_0 = \text{Ker}(\boldsymbol{\nabla}\cdot)$.

For the non-homogeneous problems, we shall use the following.

Corollary 1.6. *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. Then the divergence $\boldsymbol{\nabla}\cdot : \mathbf{H}_0^1(\Omega) \to L^2(\Omega)$ is an isomorphism of $\mathbf{V}_0^\perp$ onto $L_0^2(\Omega)$.*

For this we need only note that we can identify the annihiltor $(\mathbf{V}_0)^0$ with the orthogonal complement $\mathbf{V}_0^\perp$.

Of course we also obtain the corresponding inf-sup conditions along with the characterization of the closed ranges.

The Stokes equation (4) is expressed in terms of the operator $\mathcal{A} : \mathbf{H}_0^1(\Omega) \to \mathbf{H}^{-1}(\Omega)$ and $f \in \mathbf{H}^{-1}(\Omega)$. It takes the form $\mathbf{u} \in \mathbf{V}_0,\ \mathcal{A}\mathbf{u} - f \in (V_0)^0$. From Corollary 1.5, this means there is a $p \in L^2(\Omega)$ for which $\mathcal{A}\mathbf{u} - f = -\boldsymbol{\nabla} p \in \mathbf{H}^{-1}(\Omega)$. Thus, we have

Theorem 1.7. *There exists a pair of functions*

$$\mathbf{u} \in \mathbf{H}_0^1(\Omega),\ p \in L^2(\Omega) : \quad \mathcal{A}\mathbf{u} + \boldsymbol{\nabla} p = f \text{ in } \mathbf{H}^{-1}(\Omega), \quad \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \text{ in } L^2(\Omega). \tag{9}$$

This is precisely the form of our mixed system with $\mathcal{B} = \boldsymbol{\nabla}\cdot$.

The Non-Homogeneous Problem

The fully non-homogeneous Dirichlet problem for the Stokes system takes the form

$$\mathbf{u} \in \mathbf{H}^1(\Omega),\ p \in L^2(\Omega): \quad \mathcal{A}\mathbf{u} + \boldsymbol{\nabla} p = f \text{ in } \mathbf{H}^{-1}(\Omega), \tag{10a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = g \text{ in } L^2(\Omega), \quad \mathbf{u} = \mathbf{h} \text{ on } \Gamma. \tag{10b}$$

We can reduce this to the case of the stationaly Stokes system (3) given by Theorem 1.7 by a sequence of translations.

- Find $\mathbf{u}_h \in \mathbf{H}^1(\Omega):\ \gamma(\mathbf{u}_h) = \mathbf{h} \in \mathbf{H}^{1/2}(\Gamma)$.
  This is easy since the range of the trace operator is $\mathbf{H}^{1/2}(\Gamma)$.
- Find $\mathbf{u}_g \in \mathbf{H}_0^1(\Omega):\ -\mathcal{B}\mathbf{u}_g = g - \boldsymbol{\nabla} \cdot \mathbf{u}_h$
  This requires $g - \boldsymbol{\nabla} \cdot \mathbf{u}_h \in L_0^2(\Omega)$ to apply Corollary 1.6, so we need

$$\int_\Omega g\, dx + \int_\Gamma \mathbf{h} \cdot \mathbf{n}\, dS = 0. \tag{11}$$

- Find

$$\mathbf{u}_s \in \mathbf{H}_0^1(\Omega),\ p \in L^2(\Omega): \\ \mathcal{A}\mathbf{u}_s + \boldsymbol{\nabla} p = f - \mathcal{A}(\mathbf{u}_g + \mathbf{u}_h) \text{ in } \mathbf{H}^{-1}(\Omega), \quad \boldsymbol{\nabla} \cdot \mathbf{u}_s = 0 \text{ in } L^2(\Omega).$$

  This follows from Theorem 3

Now set $\mathbf{u} = \mathbf{u}_g + \mathbf{u}_h + \mathbf{u}_s$ to get the solution of (10).

Corollary 1.8. *Assume $f \in H^{-1}(\Omega)$, $g \in L^2(\Omega)$, $h \in \mathbf{H}^{1/2}(\Gamma)$ and that (11) holds. Then there exists a solution of (10), $\mathbf{u}$ is unique and p is determined up to a constant.*

The saddle-point problem

The solution of (10) with homogeneous boundary condition $\mathbf{h} = \mathbf{0}$ can be characterized by a saddle-point problem. For this we set

$$J(\mathbf{v}) = \tfrac{1}{2} \int_\Omega \left(\lambda \rho_0(x) \|\mathbf{v}\|^2 dx + \mu \sum_{i=1}^n \|\boldsymbol{\nabla} v_i\|^2 \right) dx - \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx \text{ for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega)$$

and $\mathcal{B} = -\boldsymbol{\nabla}\cdot :\ \mathbf{H}_0^1(\Omega) \to L^2(\Omega)$. Then (10) corresponds to minimization subject to a *constraint*,

$$\mathbf{u} \in \mathbf{H}_0^1(\Omega),\ \mathcal{B}u = g: \quad J(u) - f(u) \le J(v) - f(v)\ \forall v \in \mathbf{H}_0^1(\Omega) \text{ with } \mathcal{B}v = g.$$

We saw the solution is the first component of the saddle-point problem

$$[u, p] \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega):\ \mathcal{L}(u, q) \le \mathcal{L}(u, p) \le \mathcal{L}(v, p)\ \forall v \in \mathbf{H}_0^1(\Omega),\ q \in L_0^2(\Omega) \tag{12}$$

in which the *Lagrangian* $\mathcal{L} : \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \to \mathrm{I\!R}$ is

$$\mathcal{L}(v,q) = J(v) + \mathcal{B}v(q) - g(q), \quad [v,q] \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$$

and the second component p is the Lagrange multiplier. The dual problem involves a non-local pseudodiffeential operator of order 0. The non-homogeneous boundary condition $\gamma(\mathbf{v}) = \mathbf{h}$ could be obtained by a translation as before, and it could be obtained also by introducing a second Lagrange multiplier of the form $q_2(\gamma(\mathbf{v}) - \mathbf{h})$ into the operator $\mathcal{B}$ and adding a second space to its range.

The Mixed Problem

We shall consider the *stationary Stokes system* with mixed boundary conditions

$$\begin{gathered} \lambda\mathbf{u} - \mu\Delta\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f}\,, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \text{ in } \Omega\,, \\ \mathbf{u} = \mathbf{0} \text{ on } \Gamma_0\,, \quad \mu\frac{\partial\mathbf{u}}{\partial n} - p\,\mathbf{n} = \mathbf{g} \text{ on } \Gamma_1\,. \end{gathered}$$

Here the two sets $\Gamma_0,\ \Gamma_1$ comprise a partition of the boundary $\Gamma = \partial\Omega$, and we are given the functions $\mathbf{f} \in L^2(\Omega)$ and $\mathbf{g} \in L^2(\Gamma_1)$.

Now, if the pair $\mathbf{u}(x)$, $p(x)$ is a solution to the stationary Stokes system, then for every $\mathbf{w} \in \mathbf{H}^1(\Omega)$ we have

$$\begin{gathered} \int_\Omega \lambda\mathbf{u}\cdot\mathbf{w}\,dx + \mu\int_\Omega \boldsymbol{\nabla} u_i\cdot\boldsymbol{\nabla} w_i\,dx - \mu\int_\Gamma (\boldsymbol{\nabla} u_i\cdot\mathbf{n})w_i\,ds \\ - \int_\Omega p\,(\boldsymbol{\nabla}\cdot\mathbf{w})\,dx + \int_\Gamma (\mathbf{w}\cdot\mathbf{n})p\,ds = \int_\Omega \mathbf{f}\cdot\mathbf{w}\,dx\,. \end{gathered}$$

We define the space $\mathbf{V} = \{\mathbf{w} \in \mathbf{H}^1(\Omega) : \ \boldsymbol{\nabla}\cdot\mathbf{w} = 0 \text{ in } \Omega,\ \mathbf{w} = \mathbf{0} \text{ on } \Gamma_0\}$. Then our solution is chosen from this space, and if we also choose our test function $\mathbf{w} \in \mathbf{V}$ above, it follows that

$$\begin{gathered} \int_\Omega \lambda\mathbf{u}\cdot\mathbf{w}\,dx + \mu\int_\Omega \boldsymbol{\nabla} u_i\cdot\boldsymbol{\nabla} w_i\,dx - \mu\int_{\Gamma_1} (\boldsymbol{\nabla} u_i\cdot\mathbf{n})w_i\,ds \\ + \int_{\Gamma_1} (\mathbf{w}\cdot\mathbf{n})p\,ds = \int_\Omega \mathbf{f}\cdot\mathbf{w}\,dx\,. \end{gathered}$$

Thus, the weak form of the problem is now to find

$$\mathbf{u} \in \mathbf{V} : \ \int_\Omega (\lambda\mathbf{u}\cdot\mathbf{w} + \mu\boldsymbol{\nabla} u_i\cdot\boldsymbol{\nabla} w_i)\,dx = \int_\Omega \mathbf{f}\cdot\mathbf{w}\,dx + \int_{\Gamma_1} \mathbf{g}\cdot\mathbf{w}\,ds \text{ for all } \mathbf{w} \in \mathbf{V}\,.$$

As before, we obtain the following result.

Theorem 1.9. *There is exactly one weak solution* $\mathbf{u}$ *to the stationary Stokes system.*

Note also that the pressure p is determined up to a constant.

Conversely, let's assume that $\mathbf{u}$ is a solution of the weak form of the stationary Stokes system. Then from the inclusion $\mathbf{u} \in \mathbf{V}$ we obtain $\mathbf{u} \in \mathbf{H}^1(\Omega)$, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ in Ω, and $\mathbf{u} = \mathbf{0}$ on Γ_0 in the sense of boundary trace. Furthermore, by taking test functions $\mathbf{w} \in \mathbf{V}_0 = \mathbf{V} \cap \mathbf{H}_0^1(\Omega)$, we find that

$$\lambda \mathbf{u} - \mu \Delta \mathbf{u} - \mathbf{f} \in \mathbf{V}_0^{\perp} ,$$

the indicated annihilator of $\mathbf{V}_0$ in $\mathbf{H}^{-1}(\Omega)$, and so it is a *gradient*, and we have

$$\lambda \mathbf{u} - \mu \Delta \mathbf{u} - \mathbf{f} = -\boldsymbol{\nabla} p , \quad p \in L^2(\Omega) .$$

It follows that for each $i = 1, 2, ..., n$ we have $\mu \boldsymbol{\nabla} u_i - p\, \mathbf{e}_i \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{\nabla} \cdot (\mu \boldsymbol{\nabla} u_i - p\, \mathbf{e}_i)$ is the i-th component of $\mu \Delta \mathbf{u} - \boldsymbol{\nabla} p = \lambda \mathbf{u} - \mathbf{f} \in \mathbf{L}^2(\Omega)$. This shows that we have each $\mu \boldsymbol{\nabla} u_i - p\, \mathbf{e}_i \in \mathbf{L}^2_{div}(\Omega)$ and so there is a well-defined *normal trace* on the boundary. We use this equation to substitute for $\mathbf{f}$ in the weak form to obtain

$$\int_\Omega (\mu \boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} w_i + (\mu \Delta \mathbf{u} - \boldsymbol{\nabla} p) \cdot \mathbf{w}) \; dx = \int_{\Gamma_1} \mathbf{g} \cdot \mathbf{w} \, ds \text{ for all } \mathbf{w} \in \mathbf{V} .$$

Since $\boldsymbol{\nabla} \cdot \mathbf{w} = 0$, the i-th component of the integrand on the left side is given by $\boldsymbol{\nabla} \cdot [(\mu \boldsymbol{\nabla} \mathbf{u}_i - p \mathbf{e}_i) w_i]$. The generalized Stokes theorem implies that its integral over Ω is given by

$$\int_{\Gamma_1} \gamma_{\mathbf{n}}(\mu \boldsymbol{\nabla} u_i - p \mathbf{e}_i) \, w_i \, ds ,$$

and so from above we obtain

$$\int_{\Gamma_1} \left(\mu \frac{\partial \mathbf{u}}{\partial n} - p \, \mathbf{n} \right) \cdot \mathbf{w} \, dx = \int_{\Gamma_1} \mathbf{g} \cdot \mathbf{w} \, ds \text{ for all } \mathbf{w} \in \mathbf{V} .$$

It is in this sense that we have

$$\mu \frac{\partial \mathbf{u}}{\partial n} - p \, \mathbf{n} = \mathbf{g} \text{ on } \Gamma_1 .$$

Remark 1.2. *Because of the special properties of the space $\mathbb{B}$, the condition $\gamma_n(\mathbf{w}) = 0$ on Γ_1 is different from the condition $\gamma_n(\mathbf{w}) = 0$ for all $w \in \mathbf{V}$.*