A NOTE ON THE TRANSCENDENCE OF ZEROS OF A CERTAIN FAMILY OF WEAKLY HOLOMORPHIC MODULAR FORMS

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Abstract. Recently, Duke and Jenkins have studied a certain family of modular forms $f_{k,m}$ that form a natural basis of the space of weakly holomorphic modular forms of weight $k$ on $SL_2(\mathbb{Z})$. In particular, they prove that given a simple bound on $m$, the zeros of these functions all lie on the unit circle. Using a method of Kohnen, we observe that other than $i, \rho$, all zeros of $f_{k,m}$ are transcendental.

1. Introduction

Recall that a weakly holomorphic modular form of even integer weight $k$ on the modular group $\Gamma = PSL_2(\mathbb{Z})$, is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$, where $\mathcal{H}$ is the upper half of the complex plane, such that for all $\tau \in \mathcal{H}$, and $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma,$

$$f \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k f(\tau).$$

In addition, $f$ is meromorphic at $\infty$. I.e., $f$ has a Fourier expansion of the form

$$f(\tau) = \sum_{n \geq n_0} a_n q^n,$$

Where $n_0 = \text{ord}_\infty(f) \in \mathbb{Z}$, and $q = e^{2\pi i \tau}$. We write $M_k$ to denote the infinite dimensional complex vector space of weakly holomorphic modular forms of weight $k$, and $M_k$ (resp. $S_k$) the corresponding finite dimensional space of holomorphic modular (cusp) forms of weight $k$. For the reader unfamiliar with modular forms, we recommend [12], [3], and [10] for a nice introduction.

In [4], Duke and Jenkins study a certain family of weakly holomorphic modular forms $f_{k,m} \in M_k$ of even weight $k = 12\ell + k'$, where $k' \in \{0, 4, 6, 8, 10, 14\}$. These functions have the property that for integers $m \geq -\ell$, the expansion of $f_{k,m}$ has the form

$$f_{k,m}(\tau) = q^{-m} + \sum_{n \geq \ell + 1} a_{k,m}(n)q^n.$$

Furthermore, the set $\{f_{k,m} \mid m \geq -\ell\}$ is a basis for $M_k$.

An interesting feature of these functions is that the coefficients satisfy the following duality [4]

$$a_{k,m}(n) = -a_{2-k,m}(n).$$

Zeros of certain classes of $f_{k,m}$ have been well-studied, and shown to lie on the unit circle. For example in [1], Asai, Kaneko, and Ninomiya prove this when $k = 0$. In [4], Duke and Jenkins provide a general proof that when $m \geq |\ell| - \ell$, the zeros of $f_{k,m}$ all lie on the unit circle.

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1.1. **Transcendence of zeros of modular forms.** Recall, the standard fundamental domain of the action of $\Gamma$ on $\mathcal{H}$ is given by

$$\mathcal{F} = \left\{ z \in \mathcal{H} : -\frac{1}{2} \leq \text{Re}(z) \leq 0, |z| \geq 1 \right\} \cup \left\{ z \in \mathcal{H} : 0 < \text{Re}(z) < \frac{1}{2}, |z| > 1 \right\}.$$ 

Due to the valence formula, which states that for a nonzero modular form $f$ of weight $k$ on $\Gamma$,

$$\frac{k}{12} = \text{ord}_\infty(f) + \frac{1}{2} \text{ord}_r(f) + \frac{1}{3} \text{ord}_\rho(f) + \sum_{\tau \in \mathcal{F}, \tau \neq i, \rho} \text{ord}_\tau(f),$$

the algebraic points $i, \rho = e^{\frac{2\pi i}{3}} \in \mathcal{F}$ are frequently zeros of modular forms.

For $k \geq 4$ even, let $E_k \in M_k(\Gamma)$ denote the classical Eisenstein series of weight $k$,

$$E_k(\tau) = 1 + \frac{2k}{B_k} \sum_{n \geq 1} \sigma_k(n) q^n,$$

where as usual $q = e^{2\pi i \tau}$, $B_k$ is the $k$th Bernoulli number, and $\sigma_k(n) = \sum_{d \mid n} d^{k-1}$.

In 2003, Kohnen [8] showed in an elegant argument that other than $i$ and $\rho$, the zeros of $E_k$ in $\mathcal{F}$ are transcendental. In addition, he shows that the weakly holomorphic modular forms defined by

$$J_n = (j - 744)T(n),$$

where $T(n)$ is the usual $n$th Hecke operator, have this property as well. In fact, the $J_n$ are polynomials in $j$ with integer coefficients, and $J_n(\tau) = f_{0,n}(\tau)$. Kohnen’s argument uses the beautiful result of F. Rankin and Swinnerton-Dyer [11] which states that the zeros of $E_k$ all lie on the following arc of the unit circle,

$$A = \left\{ e^{i \theta} : \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \right\}.$$

In 2006, Gun uses Kohnen’s technique, together with work of Getz [5], [6] to obtain similar results for another class of modular forms [7]. Here, we apply this method to show that in general, other than $i, \rho$, the zeros of $f_{k,m}$ in $\mathcal{F}$ are transcendental. In particular, we prove the following.

**Theorem 1.1.** Let $k$ be an even integer, and write $k = 12\ell + k'$ such that $k' \in \{0, 4, 6, 8, 10, 14\}$. If $m \geq |\ell| - \ell$, and $z_0$ is a zero of $f_{k,m}$ in the fundamental domain $\mathcal{F}$, then either $z_0 \in \{i, \rho\}$, or $z_0$ is transcendental over $\mathbb{Q}$.

By the results of Kohnen [8] stated above, we note the following corollary, which follows directly from the proof of Theorem 1.1.

**Corollary 1.2.** Consider a weakly holomorphic modular form $f(\tau)$ of the form

$$f(\tau) = \Delta(\tau)^a \prod_{i=2}^{N} E_{2i}(\tau)^{b_i} \prod_{k=1}^{M} P_k(j(\tau))^{c_k},$$

where $b_i, c_k$ are nonnegative integers, $a$ is any integer, and $P_k$ are any polynomials in $j(\tau)$ with integer coefficients, such that the zeros of $P_k$ all lie on the unit circle. Then other than $i, \rho$, all zeros of $f(\tau)$ in the fundamental domain $\mathcal{F}$ of $\Gamma$ are transcendental over $\mathbb{Q}$.

In particular, for the functions $J_n \in M_0(\Gamma)$ defined above, we have

**Corollary 1.3.** If $f(\tau) \in M_k$ has the form

$$f(\tau) = \Delta(\tau)^a \prod_{i=2}^{N} E_{2i}(\tau)^{b_i} \prod_{n=1}^{M} J_n(\tau)^{c_n},$$

where $b_i, c_n$ are nonnegative integers and $a$ is any integer, then all zeros of $f(\tau)$ in the fundamental domain $\mathcal{F}$ other than $i, \rho$ are transcendental over $\mathbb{Q}$.  

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In Section 2, we give a proper definition for the functions $f_{k,m}$, and review some results from class field theory. In Section 3 we give the proof of Theorem 1.1.

2. Preliminaries

Recall, the classical Delta function $\Delta(\tau) \in S_{12}(\Gamma)$ is defined by

$$\Delta(\tau) = \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728},$$

and has the following infinite product expansion where $q = e^{2\pi i \tau}$,

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$ 

In addition, the $j$-function $j(\tau) \in \mathcal{M}_0(\Gamma)$ defined by

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)},$$

has a simple pole at $\infty$.

For an even integer $k$, let $k = 12\ell + k'$ where $\ell$ is an integer and $k' \in \{0, 4, 6, 8, 10, 14\}$. For each integer $m \geq -\ell$, there exists a unique weakly holomorphic modular form of weight $k$, call it $f_{k,m}$, such that the $q$-expansion of $f_{k,m}$ is of the form

$$f_{k,m}(\tau) = q^{-m} + O(q^{\ell+1}).$$

In particular,

$$f_{k,m} = \Delta^\ell E_{k'} F_{k,D}(j),$$

where $F_{k,D}(x)$ is a monic polynomial in $j(\tau)$ of degree $D = \ell + m$ with integer coefficients. The polynomials $F_{k,D}(x)$ are Faber polynomials. In [4], Duke and Jenkins prove that for $m \geq |\ell| - \ell$, all zeros of $f_{k,m}$ in $\mathcal{F}$ lie on the arc

$$A = \left\{ e^{i\theta} : \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \right\},$$

and in particular that $f_{k,m}$ has $D$ simple zeros on the interior of $A$.

The following lemma of Scheider, which can be found in [9] (Corollary 3.4), is instrumental in the proof of Theorem 1.1, as well as in the theorems of Kohnen [8] and Gun [7].

**Lemma 2.1.** (Schneider, 1937) If $z \in \mathcal{H}$ and $j(z)$ is algebraic over $\mathbb{Q}$, then either $z$ is transcendental over $\mathbb{Q}$ or $z$ is imaginary quadratic (i.e., $\mathbb{Q}(z)$ is a degree 2 extension of $\mathbb{Q}$, with $z \notin \mathbb{R}$).

2.1. Some results from class field theory. We now recall some basic facts from class field theory and complex multiplication (see [2]). Consider an integer $D < 0$ so that $K = \mathbb{Q}(\sqrt{D})$ is an imaginary quadratic field. An order $\mathcal{O}$ of $K$, is a subring of $K$ containing 1 that is a free $\mathbb{Z}$-module of rank 2. In addition, a proper fractional ideal of $\mathcal{O}$ is a non-zero fractional ideal $\mathfrak{A}$ of $\mathcal{O}$ such that

$$\mathcal{O} = \left\{ \alpha \in K : \alpha \mathfrak{A} \subset \mathfrak{A} \right\}.$$ 

The set of all proper fractional ideals of $K$ forms a multiplicative group with many nice properties.

Consider a polynomial $P(x) = ax^2 + bx + c$ of negative discriminant $D = b^2 - 4ac$, with integer coefficients such that $a > 0$ and $\gcd(a,b,c) = 1$. If $z \in \mathcal{H}$ is a root of $P(x)$, then as seen in [2] (Lemma 7.5), $\mathcal{O} = [1, az]$ is an order of $K$ and $L = [1, z]$ is a proper fractional ideal of $\mathcal{O}$.

To see the structure of $\mathcal{O}$, we notice that since $z \in \mathcal{H}$ is a root of the polynomial $ax^2 + bx + c$, by the quadratic formula, $z = \frac{-b + \sqrt{D}}{2a}$. Thus,

$$[1, az] = \left[ 1, \frac{-b + \sqrt{D}}{2} \right] = \begin{cases} [1, \frac{\sqrt{D}}{2} ] & \text{if } b \text{ is even} \\ [1, \frac{1 + \sqrt{D}}{2} ] & \text{if } b \text{ is odd}. \end{cases}$$
Since $D = b^2 - 4ac$, we have that $b$ is even if and only if $D \equiv 0 \pmod{4}$. Similarly, $b$ is odd if and only if $D \equiv 1 \pmod{4}$. We have thus shown the following.

**Lemma 2.2.** Let $a, b, c \in \mathbb{Z}$ such that $a > 0$, $\gcd(a, b, c) = 1$, and $D = b^2 - 4ac < 0$. If $z \in \mathcal{H}$ is a root of the polynomial $ax^2 + bx + c$, then the lattice $[1, z]$ is a proper fractional ideal of the order $\mathcal{O} = [1, az]$ of $K = \mathbb{Q}(\sqrt{D})$. Moreover,

$$\mathcal{O} = \begin{cases} [1, \frac{\sqrt{D}}{2} ] & D \equiv 0 \pmod{4}, \\ [1, \frac{1+\sqrt{D}}{2} ] & D \equiv 1 \pmod{4}. \end{cases}$$

In light of Lemma 2.2, we see that the order $\mathcal{O}$ doesn’t depend on $z$, but instead on the discriminant $D$ of the reduced integer polynomial that has $z$ as a root.

Recall, if $L$ a lattice of $\mathbb{C}$ we define $j(L) = j(z)$, where $z \in \mathcal{H}$ and $L = [1, z]$ (the choice of $z \in \mathcal{H}$ is well defined up to $\Gamma$-equivalence). From Lemma 2.2, we see we can map a point $z \in \mathcal{H}$ to the proper fractional ideal $L = [1, z]$ of $\mathcal{O}$, where $j([1, z]) = j(z)$.

The following lemma follows from Theorem 11.1 and Proposition 13.2 in [2], and is the last result we need before the proof of Theorem 1.1.

**Lemma 2.3.** If $\mathfrak{A}$ is a proper fractional ideal of an order $\mathcal{O}$ of an imaginary quadratic field $K$, then $j(\mathfrak{A})$ is an algebraic integer over $\mathbb{Q}$. If $\mathfrak{B}$ is any other proper fractional ideal of $\mathcal{O}$, then $K(j(\mathfrak{A})) = K(j(\mathfrak{B}))$ and $j(\mathfrak{A})$ and $j(\mathfrak{B})$ are conjugate over $K$.

3. The proof of Theorem 1.1

**Proof.** Let $k$ be an even integer, and write $k = 12\ell + k'$ such that $k' \in \{0, 4, 6, 8, 10, 14\}$. Recall, 

$$f_{k,m}(\tau) = \Delta(\tau)^\ell F_{k,D}(j(\tau)),$$

where $F_{k,D}(x)$ is a degree $D = m + \ell$ polynomial with integer coefficients. By Kohnen [8], the only possible zeros of $F_{k,D}(x)$ are $i$ and $\rho$. Also, we see from the valence formula (1), that $\Delta(\tau)$ is never zero on $\mathcal{H}$. Thus, the only zeros of $f_{k,m}(\tau)$ in $\mathcal{H}$ other than $i, \rho$ are the zeros of $F_{k,D}(j(\tau))$.

Suppose $z_0 \in \mathcal{F}$ such that $F_{k,D}(j(z_0)) = 0$. Since $F_{k,D}(x)$ is a polynomial with integer coefficients, then $j(z_0)$ is algebraic over $\mathbb{Q}$. Thus by Lemma 2.1, $z_0$ is either transcendental or imaginary quadratic.

If $z_0$ is imaginary quadratic, then $z_0$ is a root of a polynomial $P(x) = ax^2 + bx + c$, where $\gcd(a, b, c) = 1$, $a > 0$, and the discriminant $D_0 = b^2 - 4ac < 0$. Let $K = \mathbb{Q}(\sqrt{D_0})$.

Consider the order $\mathcal{O} = [1, az]$ of $K$. By Lemma 2.2, the lattice $[1, z_0]$ is a proper fractional ideal of $\mathcal{O}$, and the order $\mathcal{O}$ has the form

$$\mathcal{O} = \begin{cases} [1, \frac{\sqrt{D_0}}{2} ] & D_0 \equiv 0 \pmod{4} \\ [1, \frac{1+\sqrt{D_0}}{2} ] & D_0 \equiv 1 \pmod{4}. \end{cases}$$

Thus by Lemma 2.3, if $\mathfrak{A}$ is any other proper fractional ideal of $\mathcal{O}$, $j(z_0) = j([1, z_0])$ and $j(\mathfrak{A})$ are conjugate.

Keeping this in mind, consider the point $z_1 \in \mathbb{C}$ defined by

$$z_1 = \begin{cases} \frac{i\sqrt{-D_0}}{2} & D_0 \equiv 0 \pmod{4} \\ -\frac{1+i\sqrt{-D_0}}{2} & D_0 \equiv 1 \pmod{4}. \end{cases}$$

Then $z_1 \in \mathcal{F}$, and from (4), we have $[1, z_1] = \mathcal{O}$. Thus by definition $[1, z_1]$ is a proper fractional ideal of $\mathcal{O}$, and so $j(z_0)$ and $j(z_1)$ are conjugate.
We may now take an automorphism, $\sigma$, of $K(j(\mathcal{O}))$ such that $\sigma(j(z_0)) = j(z_1)$. Since $\sigma$ acts as the identity on $\mathbb{Q}$ and $F_{k,D}$ is a polynomial with integer coefficients, we have that

$$0 = \sigma(0) = \sigma(F_{k,D}(j(z_0))) = F_{k,D}(\sigma(j(z_0))) = F_{k,D}(j(z_1)).$$

Thus $z_1$ is also a zero of $F_{k,D}$ and hence a zero of $f_{k,m}$. Since $z_1 \in \mathcal{F}$, by Duke and Jenkins [4] we have that $z_1$ must lie on the arc of the unit circle given by

$$\left\{ e^{i\theta} : \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \right\}.$$

Suppose $D_0 \equiv 0 \pmod{4}$, so that $D_0 = -4k$ for some positive integer $k$. Then $z_1 = i\sqrt{k}$, but since $z_1$ must lie on the unit circle we must have $k = 1$. Thus, $D_0 = -4$. Since $z_0 \in \mathcal{H}$, we have by the quadratic formula that

$$z_0 = \frac{-b + 2i}{2a}.$$

But $z_0 \in \mathcal{F}$, and so $\text{Im}(z_0) \geq \frac{\sqrt{3}}{2}$. Thus $a = 1$, and so

$$z_0 = -\frac{b}{2} + i.$$

But again by Duke and Jenkins [4] we have that $z_0$ must lie on the unit circle, so $b = 0$ and $z_0 = i$.

If $D_0 \equiv 1 \pmod{4}$, then $D_0 = -4k + 1$ for some positive integer $k$. Hence,

$$z_1 = \frac{-1 + i\sqrt{4k - 1}}{2},$$

and thus $|z_1|^2 = k$. Again, since $z_1$ must lie on the unit circle we must have $k = 1$. Thus, $D_0 = -3$. By the quadratic formula,

$$z_0 = \frac{-b + i\sqrt{3}}{2a}.$$

And again since $z_0 \in \mathcal{F}$ this forces $a = 1$ so that

$$z_0 = -\frac{b}{2} + i\frac{\sqrt{3}}{2}.$$

But again by Duke and Jenkins [4] we have that $z_0$ must lie on the unit circle, so $b = 1$ and $z_0 = \rho$.

Thus, a zero $z_0 \in \mathcal{F}$ of $f_{m,k}$ is either transcendental, or is one of $i$ or $\rho$. \hfill \Box

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