Using non-Euclidean Geometry to teach Euclidean Geometry to K–12 teachers

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Abstract

The Oregon Mathematics Leadership Institute (OMLI) is an NSF-funded partnership aimed at increasing mathematics achievement of students in partner K–12 schools through the creation of sustainable leadership capacity. OMLI's 3-week summer institute offers content and leadership courses for in-service teachers. We report here on one of the content courses, entitled Comparing Different Geometries, which enhances teachers' understanding of the (Euclidean) geometry in the K–12 curriculum by studying two non-Euclidean geometries: taxicab geometry and spherical geometry. By confronting teachers from mixed grade levels with unfamiliar material, while modeling protocol-based pedagogy intended to emphasize a cooperative, risk-free learning environment, teachers gain both content knowledge and insight into the teaching of mathematical thinking.

Keywords: Cooperative groups, Euclidean geometry, K–12 teachers, Norms and protocols, Rich mathematical tasks, Spherical geometry, Taxicab geometry

1 Introduction

The Oregon Mathematics Leadership Institute (OMLI) is a Mathematics/Science Partnership aimed at increasing mathematics achievement of K–12 students by providing professional development opportunities for in-service teachers. Teachers participate in three 3-week intensive summer institutes, covering mathematics content as well as leadership skills. Six content courses were developed, covering Number and Operation, Geometry, Abstract Algebra, Probability and Statistics, Measurement and Change, and Discrete Mathematics; each was offered in fifteen 2-hour sessions. We report here on the course in Geometry.

2 The Geometry Course

2.1 The Geometry Team

The geometry course was originally developed by a 4-member team consisting of one faculty member from Oregon State University (OSU) with geometry expertise and interests in mathematics education, one faculty member from Western Oregon University (WOU) with experience teaching in-service and pre-service K–12 teachers, a master teacher with university experience working with pre-service teachers, and an instructor from OSU with extensive background and interest working with pre-service teachers. Another OSU instructor later joined the group. This combination of instructors from diverse backgrounds is a fundamental part of the OMLI vision, and proved remarkably effective. All members of the team also participated in classroom instruction, resulting in several instructors being present at all times.

2.2 Course Goals

The primary goal of the course was to improve the geometry content knowledge of a group of K–12 teachers. Mastering Euclidean Geometry requires not only knowledge of the geometry itself, but also familiarity with the process of mathematical reasoning. We chose to develop both sets of skills by working with non-Euclidean geometry, thus forcing our group of teachers to rethink their prior knowledge of geometry. In short, our teachers were put in the same position as their students, needing to learn both content knowledge and reasoning skills.

We chose to focus on two non-Euclidean geometries, Taxicab Geometry and Spherical Geometry, which are described in more detail below. In each case, we began with definitions or interpretations of primary objects such as points and lines, then explored such notions as parallel lines, midpoints, circles, triangles, and other geometric shapes. This approach helped to develop the structure of an axiomatic system, while allowing K–12 teachers to make connections between Euclidean and non-Euclidean geometries by examining the similarities and differences between them.

2.3 Course Structure

We began the course with a discussion of axioms, undefined terms such as points and lines, and models. We then led the class to define some basic objects, such as line segments, midpoints, and circles, and created posters with our definitions, which hung in the classroom for the remainder of the course. As we explored each new geometry, we extended and modified these definitions.

We spent one week comparing Taxicab Geometry with Euclidean Geometry, and one week comparing Spherical Geometry with Euclidean Geometry. The capstone activity in each of these weeks was the development of a comparison chart showing similarities and differences between the geometries, examples of which are shown below in Figures 2 and 3.

We made heavy use of hands-on explorations in cooperative groups. We reassigned groups 3–4 times during the course, but made a point of including teachers from different levels in each group. We assigned group roles (which at various times included Team Captain, Explainer, Facilitator, Resource Manager, Recorder) to encourage maximum accountability, and provided brief descriptions of the expectations of each one. These roles were rotated within each group on successive tasks. We also made heavy use of manipulatives, ranging from colored pencils and markers, to grid paper and Etch-a-Sketch toys for Taxicab Geometry, to Lénárt Spheres (plastic spheres which can be drawn on) for Spherical Geometry. String, rulers, compasses — and appropriate spherical analogues — were provided as needed. The tasks themselves were carefully selected, with the ideal task being open-ended, with rich mathematical content, and naturally encouraging group rather than individual effort.

The last week of the course was devoted to the completion of a group project, beginning with a choice of topic, and culminating with a poster presentation to the class as a whole. These projects were expected to explore a topic not considered in class. We provided lists of suggested topics, but there was no requirement that the topic be chosen from this list, although the choice of topic had to be defended and approved. All projects were expected to have substantial mathematical content. Some topics which led to successful projects were:

- Taxicab Geometry on hexagonal grids;
- Taxicab Geometry with one-way streets;
- Taxicab Geometry with a subway line;
- Taxicab Conic Sections;
- Spherical Triangles Area Formula;
- Spherical Tessellations.

2.4 Norms and Protocols

We developed classroom norms intended to ensure everyone's participation, and to encourage risk-taking. There were several protocols for group work, including group roles, as well as for class discussion and presentations. Most of these protocols involved private time to think, followed by specific instructions about sharing ideas first within each group, then with the class as a whole. We then directed the discussion towards the fundamental mathematical ideas of each task. Many of these protocols were based on materials developed by the Teachers Development Group [1]. This framework was designed to encourage mathematical discourse based on logical reasoning, both formal and informal. A further discussion of some of the pedagogical strategies used in this course will appear elsewhere [2].

3 The Geometries

3.1 Taxicab Geometry

Taxicab Geometry [3, 4] is the geometry used by taxicabs in an ideal city with a rectangular grid of streets. Distance is measured not as the crow flies, but as the taxicab drives, that is, the distance between two locations is the sum of the vertical and horizontal distances between them, rather than be given by the Pythagorean Theorem.

One major problem when introducing Taxicab Geometry is making clear what the model actually is. Taxicabs move only on streets, which leads many students to think that taxicab lines consist of horizontal and vertical segments. A related issue is whether Taxicab Geometry is a lattice consisting of the intersections only, or whether all Euclidean points are allowed. We encourage students to realize that these are all valid models to study, while nonetheless steering students towards the traditional model of Taxicab Geometry, which includes all Euclidean points and lines.

After watching students struggle with this, we developed a very successful activity to help clarify these choices. We had each group use an Etch-a-Sketch to measure the length of a line segment drawn on it (with washable markers), with the restriction that they could only use one knob at a time. This encouraged students to realize that the taxicab line segment was the piece of tape, and that the zig-zag path they constructed with the Etch-a-Sketch was merely a tool used to measure its length.

We used existing materials such as [3] to introduce Taxicab Geometry. A nice bonus was that this article appears in an NCTM publication, thus providing immediate reassurance that our use of non-Euclidean geometry is in fact relevant to the K–12 classroom.

Group activities in Taxicab Geometry included explorations of

- Taxicab Distance;
- Taxicab Midpoints;
- Taxicab Midsets (Points Equidistant from Two Given Points);
- Taxicab Circles and other conic sections;
- Taxicab Triangle Congruence.

Here is a sample task which investigates Taxicab Midsets:

> *Lois and Clark live in Smallville and are looking for an apartment. The street map shows each person's workplace. If they would like to live the same Taxicab distance from their jobs, where could they live?*

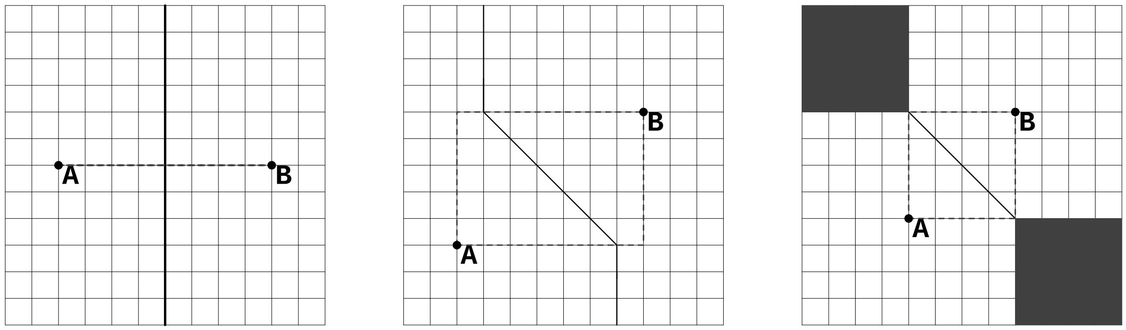

Figure 1: Three examples of Taxicab Midsets.

We provided several different maps, such as those shown in Figure 1, in which the resulting midsets are also shown. Note that the midset in the third case includes 2-dimensional regions!

3.2 Spherical Geometry

Spherical Geometry is the geometry of an ideal Earth, and is the standard example of a geometry with no parallel lines. Of course, one must first carefully define what a *line* is, making sure to discuss why lines of constant latitude aren't included (except for the equator)!

We were fortunate in having a complete classroom set of Lénárt spheres available, which were used throughout the week we spent on Spherical Geometry. A Lénárt sphere is a plastic sphere which comes with tools designed to mimic the use of straightedge and compass in Euclidean Geometry, including spherical transparencies which can be drawn on. A workbook [5] is also available, containing numerous projects, some of which we used.

Group activities in Spherical Geometry included explorations of

- Lines on the sphere;
- Perpendicular lines;
- Spherical triangles;
- Analogues of squares on the sphere;
- Other spherical polygons.

Here is a sample task which investigates lines on the sphere:

> *How do you measure the distance between two points on the plane? How do you measure the distance between two points on the sphere? In each case, explain what units of measure you can use.*

This task naturally leads to (or builds on) a discussion of what the lines are on a sphere, how many line segments connect two given points, and what sort of rulers to use to measure distance.

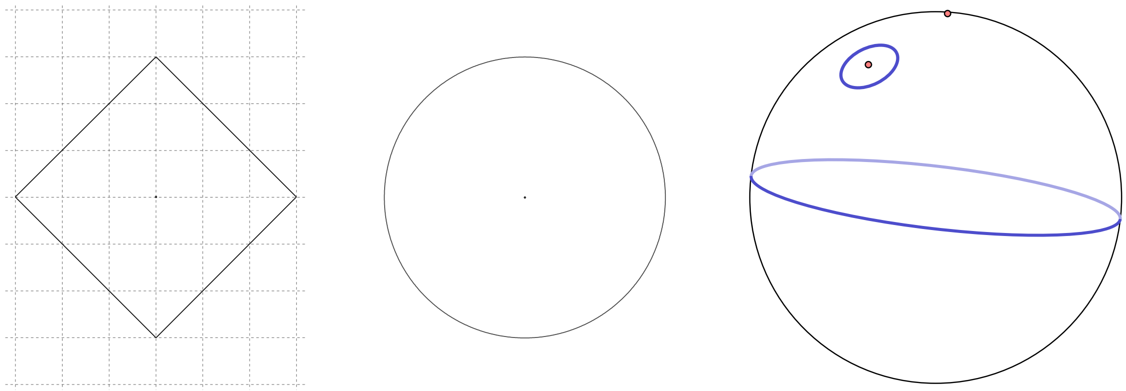

Figure 2: Circles in Taxicab, Euclidean, and Spherical Geometries, for which the ratio of circumference to diameter is $\frac{C}{d} = 2$, $\frac{C}{d} = \pi$, and $\frac{C}{d} \in [2, \pi]$, respectively.

3.3 Comparing Different Geometries

As already mentioned, the capstone experience in each of the first two weeks was the development of a comparison chart showing similarities and differences between the geometries. For example, students constructed circles in each geometry, then analyzed the ratio of circumference to diameter, thus determining the value of "π" in each geometry, defined to be the ratio of the circumference C of a circle to its diameter d. As shown in Figure 2, $C/d = 2$ in Taxicab Geometry, but $C/d \in [2, \pi]$ in Spherical Geometry, with the smallest values applying to great circles such as the equator, whereas small circles are nearly Euclidean. A similar construction of equilateral triangles in each geometry, as shown in Figure 3, leads not only to the perhaps unsurprising result that the angles of such triangles are not 60°, but also the remarkable fact that in Taxicab Geometry the angles need not be congruent.

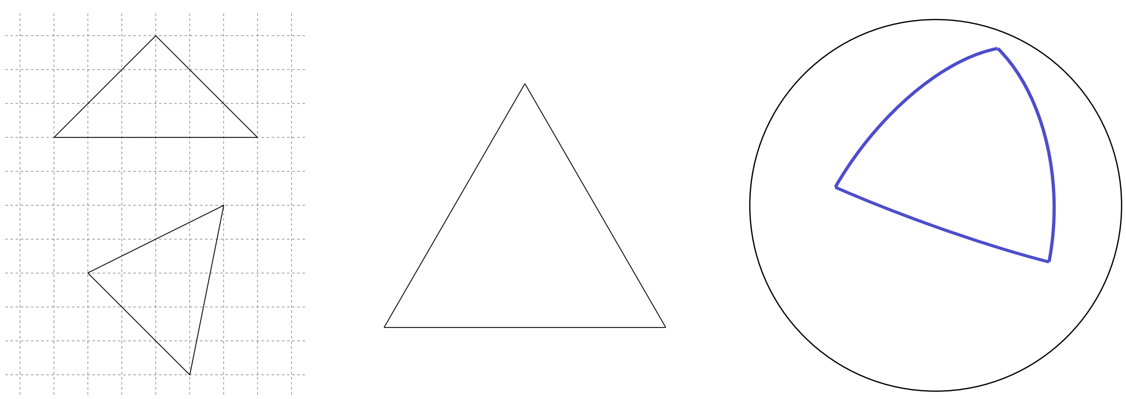

Figure 3: Equilateral triangles in Taxicab, Euclidean, and Spherical Geometries, showing that the angles are different.

4 Successes and Challenges

The quality of the projects can only be described as stunning. Watching teachers initially ignorant of non-Euclidean geometry present high quality solutions to interesting problems in this area can only be described as a richly rewarding experience for all concerned.

Teachers completing the geometry course reported an improved understanding of the role of definitions and undefined terms in geometry. This was confirmed by OMLI's evaluators using the University of Michigan's Learning Mathematics for Teaching assessment measures [6, 7], which found statistically significant gains ($p < .05$) in teachers' mathematics content knowledge for teaching. Remarkably, this was true for all content areas, and for teachers at all levels, with the greatest gain being for elementary teachers in geometry.

We did have some difficulty balancing the needs of elementary teachers with those of high school teachers, although we were able to take advantage of the latter group's expertise in mixed groups. And the high school teachers learned quite a bit from the elementary teachers about hands-on explorations. All the teachers benefited from comparing and connecting the various geometric representations and problem-solving strategies developed by their classmates.

To address this diverse audience, several tasks were successfully designed to be completed with varying sophistication. We did in fact add one such task, analyzing parallel transport on the sphere, which investigates the somewhat surprising fact that after walking around a closed polygonal path on a sphere, without turning, you wind up facing in a different direction than you started. But in retrospect we should have designed more tasks aimed at groups of high school teachers.

We also experimented with the order of the topics covered, first choosing to discuss Taxicab Geometry before Spherical Geometry, then reversing the order. Each choice has its advantages, but we felt the teachers appreciated starting with a concrete model. In particular, this allowed us to postpone discussing the subtleties of the taxicab model, referred to above, until the second week.

In either order, we believe this course could be successfully implemented on a regular basis as a geometry course for middle school teachers. A course with similar content, aimed at all math majors (including future teachers), has been offered at OSU for many years.

Finally, we worked hard to design our tasks to be as open-ended as possible, then worked hard to maintain that expectation in the classroom. This was perhaps the most important, and most successful, aspect of the course, as it modeled a classroom atmosphere of joint mathematical exploration.

Acknowledgments

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