Reimagining Second-Year Calculus: The Vector Calculus Bridge Project

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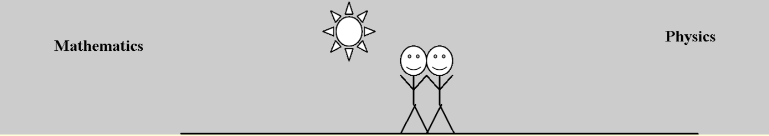
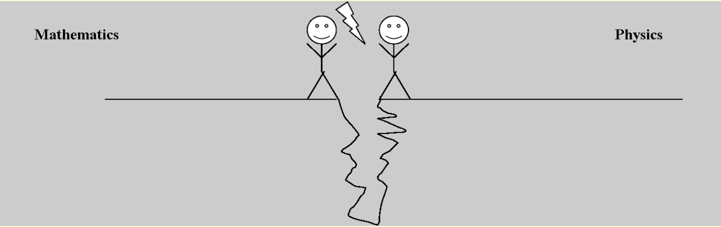
Mathematics vs. Physics



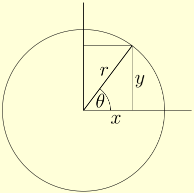


What are Functions?

Suppose the temperature on a rectangular slab of metal is given by

$$T(x,y) = k(x^2 + y^2)$$

where k is a constant. What is $T(r,\theta)$?



A: $T(r,\theta) = kr^2$

B: $T(r,\theta) = k(r^2 + \theta^2)$

What are Functions?

MATH

$$T = f(x,y) = k(x^2+y^2)$$
$$T = g(r,\theta) = kr^2$$

PHYSICS

$$T = T(x,y) = k(x^2+y^2)$$
$$T = T(r,\theta) = kr^2$$

Two disciplines separated by a common language...

physical quantities $\neq$ functions

Mathematics vs. Physics

- **Physics is about things.**
- **Physicists can't change the problem.**

- **Mathematicians do algebra.**
- **Physicists do geometry.**

The Vector Calculus Bridge Project

- **Differentials** *(Use what you know!)*
- **Multiple representations**
- **Symmetry** *(adapted bases, coordinates)*
- **Geometry** *(vectors, div, grad, curl)*

- Small group activities
- Instructor's guide
- Online text (http://www.math.oregonstate.edu/BridgeBook)

http://www.math.oregonstate.edu/bridge

DUE–0088901, DUE–0231032, DUE–0618877

The Vector Calculus Bridge Project

Bridge Project homepage hits in 2009

Mathematicians' Line Integrals

- Start with Theory

$$
\begin{aligned}
\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int \vec{\mathbf{F}} \cdot \hat{\mathbf{T}}\, ds \\
&= \int \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|} |\vec{\mathbf{r}}'(t)|\, dt \\
&= \int \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt \\
&= \ldots = \int P\, dx + Q\, dy + R\, dz
\end{aligned}
$$

- Do examples starting from next-to-last line

Need parameterization $\vec{\mathbf{r}} = \vec{\mathbf{r}}(t)$

Physicists' Line Integrals

- Theory
  - Chop up curve into little pieces $d\vec{\mathbf{r}}$.
  - Add up components of $\vec{\mathbf{F}}$ parallel to curve (times length of $d\vec{\mathbf{r}}$)
- Do examples directly from $\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$

Need $d\vec{\mathbf{r}}$ along curve

Mathematics

$$\vec{\mathbf{F}}(x,y) = \frac{-y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}}{x^2+y^2} \qquad \vec{\mathbf{r}} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$$

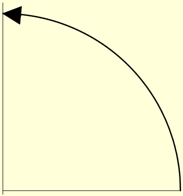
$$\begin{aligned} x &= 2\cos\theta \\ y &= 2\sin\theta \end{aligned}$$

$$\begin{aligned} \int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^{\frac{\pi}{2}} \vec{\mathbf{F}}(x(\theta),y(\theta)) \cdot \vec{\mathbf{r}}\,'(x(\theta),y(\theta))\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2}(-\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}) \cdot 2(-\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}})\,d\theta \\ &= \ldots = \frac{\pi}{2} \end{aligned}$$

Physics

$$\vec{\mathbf{F}} = \frac{\hat{\phi}}{r}$$

$$d\vec{\mathbf{r}} = r\,d\phi\,\hat{\phi}$$



I: $|\vec{\mathbf{F}}| = \text{const}; \ \vec{\mathbf{F}} \parallel d\vec{\mathbf{r}} \Longrightarrow$

$$\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \frac{1}{2}(2\ \frac{\pi}{2})$$

II: Do the dot product $\longmapsto$

$$\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^{\frac{\pi}{2}} \frac{\hat{\phi}}{2} \cdot 2\,d\phi\,\hat{\phi} = \int_0^{\frac{\pi}{2}} d\phi = \frac{\pi}{2}$$

Vector Differentials

$d\vec{\mathbf{r}} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}}$

$d\vec{\mathbf{r}} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$

$$\begin{aligned} ds &= |d\vec{\mathbf{r}}| \\ d\vec{\mathbf{A}} &= d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2 \\ dA &= |d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2| \\ dV &= (d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2) \cdot d\vec{\mathbf{r}}_3 \end{aligned}$$

The Geometry of Gradient

$$df = \vec{\nabla} f \cdot d\vec{\mathbf{r}}$$

$df \longleftrightarrow$ "small change in f"

$d\vec{\mathbf{r}} \longleftrightarrow$ "small step"

$$\begin{aligned} \text{level curve} \quad &\Longrightarrow \quad f = \text{constant} \\ &\Longrightarrow \quad df = 0 \\ &\Longrightarrow \quad \vec{\nabla} f \cdot d\vec{\mathbf{r}} = 0 \\ &\Longrightarrow \quad \vec{\nabla} f \perp \text{level curve} \end{aligned}$$

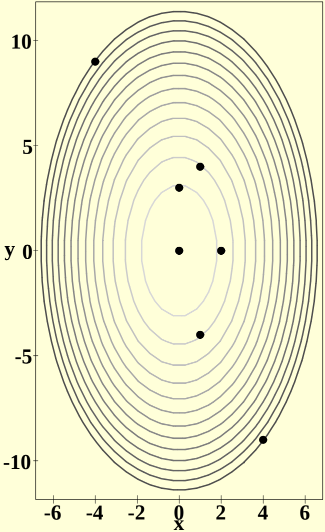
The gradient points "uphill"...

The Hill

Suppose you are standing on a hill. You have a topographic map, which uses rectangular coordinates (x, y) measured in miles. Your global positioning system says your present location is at one of the points shown. Your guidebook tells you that the height h of the hill in feet above sea level is given by

$$h = a - bx^2 - cy^2$$

where $a = 5000$ ft, $b = 30\,\frac{\text{ft}}{\text{mi}^2}$, and $c = 10\,\frac{\text{ft}}{\text{mi}^2}$.



A Radical View of Calculus

- The central idea in calculus is not the limit.
- The central idea of derivatives is not slope.
- The central idea of integrals is not area.
- The central idea of curves and surfaces is not parameterization.
- The central representation of physical quantities is not functions.

- Calculus is about infinitesimal reasoning.
- Derivatives are ratios of small quantities.
- The central idea of integrals is “chop and add”.
- The central idea of curves and surfaces is “use what you know”.
- The central representation of physical quantities is relationships (equations) between them.

Differentials

Instead of:

- chain rule
- related rates
- implicit differentiation
- derivatives of inverse functions
- difficulties of interpretation (units!)

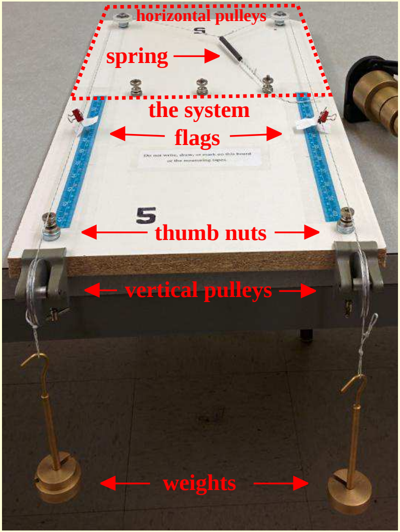
One coherent idea:

"Zap equations with *d*"

Tevian Dray & Corinne A. Manogue,
Putting Differentials Back into Calculus,
College Math. J. **41**, 90–100 (2010).

Partial Derivatives Machine

- Developed for junior-level thermodynamics course
- Two positions, x_i, two string tensions (masses), F_i.
- "Find $\frac{\partial x}{\partial F}$."
- Idea: Measure Δx, ΔF; divide.
- Mathematicians:
  "That's not a derivative!"



Paradigms in Physics Project
DUE–1023120, DUE–1323800

Limits

Is there a difference between $\frac{x^2 - 4}{x - 2}$ and $x + 2$?

What is a *numerical* representation of a derivative?

What is an *experimental* representation of a derivative?

(roundoff error, measurement error, quantum mechanics...)

“thick” derivatives

David Roundy, Tevian Dray, Corinne A. Manogue, Joseph F. Wagner, and Eric Weber, *An Extended Theoretical Framework for the Concept of Derivative*, RUME Proceedings 2015, MAA, pp. 838–843. (http://sigmaa.maa.org/rume/Site/Proceedings.html)

CONCLUSIONS

Context is everything!

- θ is an angle; x is a distance; t is time
  (use $\cos\theta$ rather than $\cos(x)$)
- units matter
  (use $\cos(\omega t)$ rather than $\cos(t)$)
- ...
- physical operators are diagonalizable
- power series for $\sin(kx)\,e^{-mx}$
- ...

Save the fine print for later!
(e.g. limits $\longmapsto$ thick derivatives & differentials)