TEACHING CALCULUS COHERENTLY

Oregon State University

Tevian Dray

I: Coherent Calculus
II: Vector Calculus Bridge Project
III: ConcepTests
IV: Calculus Concept Inventory

Coherent Calculus

co-her-ent:

logically or aesthetically ordered

cal-cu-lus:

a method of computation *in a special notation*

A Radical View of Calculus

- **The central idea in calculus is not the limit.**
- **The central idea of derivatives is not slope.**
- **The central idea of integrals is not area.**
- **The central idea of curves and surfaces is not parameterization.**
- **The central representation of a function is not its graph.**

A Radical View of Calculus

- **The central idea in calculus is the differential.**
- **The central idea of derivatives is rate of change.**
- **The central idea of integrals is total amount.**
- **The central idea of curves and surfaces is "use what you know".**
- **The central representation of a function is data attached to the domain.**

Coherent Calculus

coherent:

logically or aesthetically ordered

calculus:

a method of computation in a special notation

differential calculus:

a branch of mathematics concerned chiefly with the study of the rate of change of functions with respect to their variables especially through the use of derivatives *and differentials*

Differentials

$$df = \frac{df}{dx}\, dx$$

- Shorthand for limit argument
- Nonstandard analysis (hyperreal numbers)
- Smooth infinitesimal analysis
- Differential forms

“Differentials of *variables*”
not
“Differentials of *functions*”!

Differentials

$$
\begin{aligned}
d(u+cv) &= du + c\,dv \\
d(uv) &= u\,dv + v\,du
\end{aligned}
$$

$$
\begin{aligned}
d\left(u^n\right) &= nu^{n-1}\,du \\
d\left(e^u\right) &= e^u\,du \\
d(\sin u) &= \cos u\,du \\
d(\cos u) &= -\sin u\,du \\
d(\ln u) &= \frac{1}{u}\,du
\end{aligned}
$$

Derivatives

Derivatives:

$$\frac{d}{du}\sin u = \frac{d\sin u}{du} = \cos u$$

Chain rule:

$$\frac{d}{dx}\sin u = \frac{d\sin u}{dx} = \frac{d\sin u}{du}\,\frac{du}{dx} = \cos u\,\frac{du}{dx}$$

Inverse functions:

$$\frac{d}{du}\ln u = \frac{d}{du}\,q = \frac{dq}{du} = \frac{1}{du/dq} = \frac{1}{de^q/dq} = \frac{1}{e^q} = \frac{1}{u}$$

Derivatives

Instead of:

- chain rule
- related rates
- implicit differentiation
- derivatives of inverse functions
- difficulties of interpretation (units!)

One coherent idea:

"Zap equations with d"

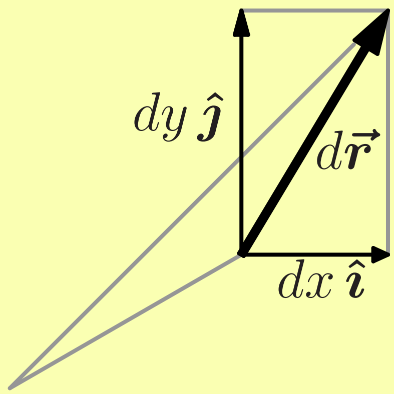
Vector Calculus Bridge Project

- **Differentials** *(Use what you know!)*
- **Multiple representations**
- **Symmetry** *(adapted bases, coordinates)*
- **Geometry** *(vectors, div, grad, curl)*

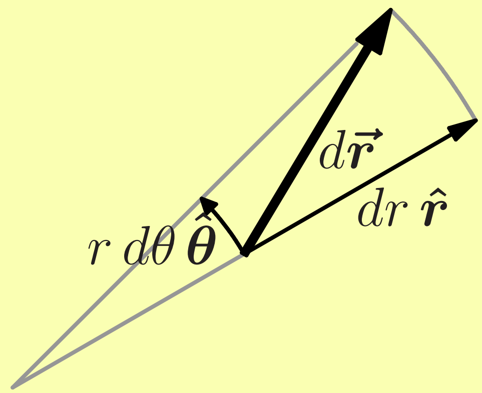
- Small group activities
- Instructor's guide (in preparation)

`http://www.math.oregonstate.edu/bridge`

Vector Differentials ($d\vec{\boldsymbol{r}}$)



$$d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}}$$



$$d\vec{\boldsymbol{r}} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}}$$

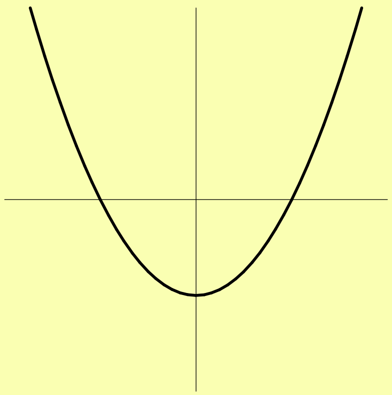
$$\begin{aligned}
ds &= |d\vec{\boldsymbol{r}}| \\
d\vec{\boldsymbol{S}} &= d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2 \\
dS &= |d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2| \\
dV &= (d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2) \cdot d\vec{\boldsymbol{r}}_3 \\
df &= \vec{\nabla} f \cdot d\vec{\boldsymbol{r}}
\end{aligned}$$

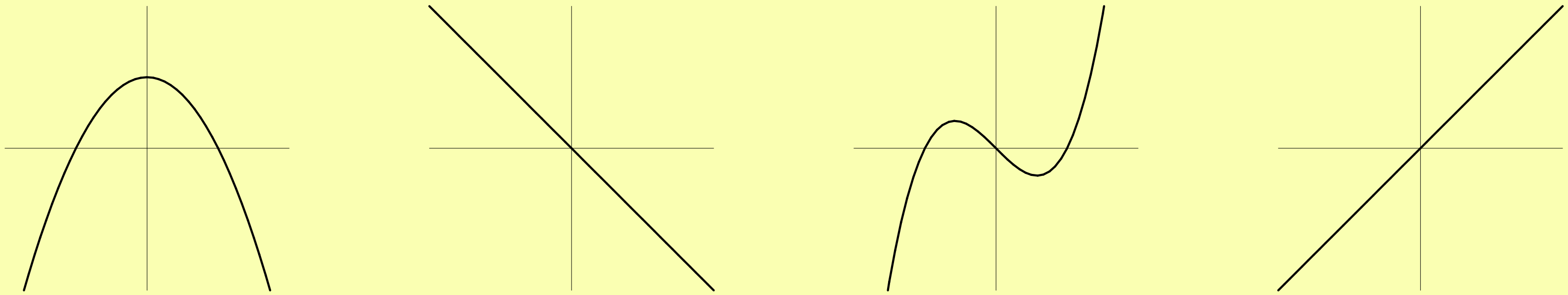
ConcepTests

- conceptual multiple-choice questions
- Eric Mazur
- `http://math.arizona.edu/~lomen/conceptests.html`
- Focus on a single concept
- Can't be solved using equations
- Have good multiple-choice answers
- Are clearly worded
- Are of intermediate difficulty

ConcepTest Example

Which of the graphs below could represent the derivative of the function graphed at the right?





A **B** **C** **D**

Calculus Concept Inventory

- pretest/posttest
- measures conceptual understanding
- Jerome Epstein
- modeled on Force Concept Inventory

Example:

If a number very close to zero is divided by another number very close (but not equal) to zero, the result

(a) must be a number very close to zero

(b) must be a number close to one

(c) could be any number

(d) might not be a number at all.

Normalized Gain

$$\text{normalized gain} = \frac{\text{gain}}{\text{possible gain}}$$

- Traditional lectures: 15–20%
- Active engagement: 30%

OSU:

- 7 sections under 20%
- 1 section @ 30%
- Made heavy use of ConcepTests
- Wasn't mine...

SUMMARY

Active engagement is essential.

Concepts matter.

Coherence is nice.