Magic squares of Lie groups

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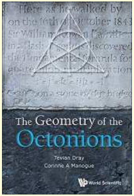
Book



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The Freudenthal–Tits Magic Square

Freudenthal (1964), Tits (1966):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{c}_3$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{a}_2$	$\mathfrak{a}_2 \oplus \mathfrak{a}_2$	$\mathfrak{a}_5$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{c}_3$	$\mathfrak{a}_5$	$\mathfrak{d}_6$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Vinberg (1966):

$$sa(3, \mathbb{A} \otimes \mathbb{B}) \oplus \mathrm{der}(\mathbb{A}) \oplus \mathrm{der}(\mathbb{B})$$

$$\mathrm{der}(\mathbb{H}) = \mathfrak{so}(3); \quad \mathrm{der}(\mathbb{O}) = \mathfrak{g}_2$$

Goal: Description as symmetry groups

[Wangberg (PhD 2007), Wangberg & Dray (JMP 2013, JAA 2014), Dray, Manogue, and Wilson (CMUC 2014)]

History

- Barton & Sudbery (2003):
  Well-understood in terms of Lie algebras.
- Satisfactory group description not yet known.
- Rosenfeld (1956/1997):
  Isometry groups of projective planes over $\mathbb{A} \otimes \mathbb{B}$.

  Cayley-Moufang plane: $F_4 \longleftrightarrow \mathbb{OP}^2$

- Baez (2002):
  OK for E_6; not for E_7, E_8.

  In short, more work must be done before we can claim to fully understand the geometrical meaning of the Lie groups E_6, E_7 and E_8.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3,\mathbb{C})$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{su}(3,\mathbb{O})$
$\mathbb{C}$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{sl}(3,\mathbb{O})$
$\mathbb{H}$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3,\mathbb{C})$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

Dray & Manogue (2010):

$F_4 \cong \mathrm{SU}(3,\mathbb{O})$, $E_6 \cong \mathrm{SL}(3,\mathbb{O})$ using $\mathrm{SL}(2,\mathbb{O}) \cong \mathrm{SO}(9,1) \subset E_6$

$$\mathcal{X} = \begin{pmatrix} X & \theta \\ \theta^\dagger & n \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

Triality!

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{su}(3,\mathbb{O})$
$\mathbb{C}$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{sl}(3,\mathbb{O})$
$\mathbb{H}$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{sp}(6,\mathbb{C})$	$\mathfrak{sp}(6,\mathbb{H})$	$\mathfrak{sp}(6,\mathbb{O})$
$\mathbb{O}$	??	??	??	??

Dray & Manogue (2010):

$F_4 \cong \mathrm{SU}(3,\mathbb{O})$, $E_6 \cong \mathrm{SL}(3,\mathbb{O})$ using $\mathrm{SL}(2,\mathbb{O}) \cong \mathrm{SO}(9,1) \subset E_6$

$$\mathcal{X} = \begin{pmatrix} X & \theta \\ \theta^\dagger & n \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

Triality!

The Subgroup Structure of E_6

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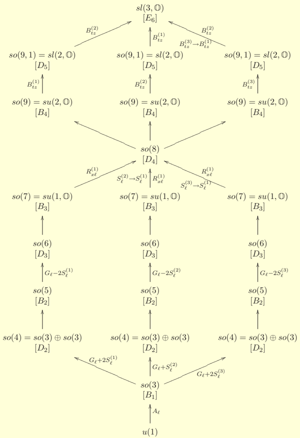

Figure 4.2: Chain of subgroups $SO(n) \subset SO(9,1) \subset SL(3,\mathbb{O})$

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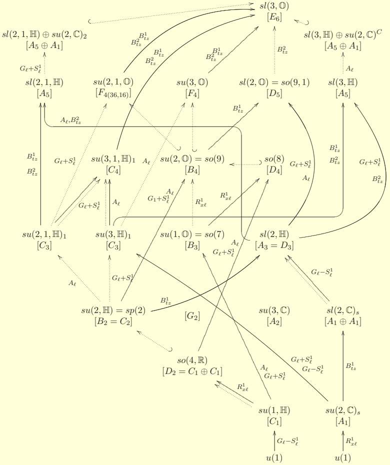


Figure 5.1: Preferred subalgebra chains of E_6 using the same basis

Wangberg (PhD 2007), Wangberg & Dray (JAA 2014)

Cartan Decompositions of E_6

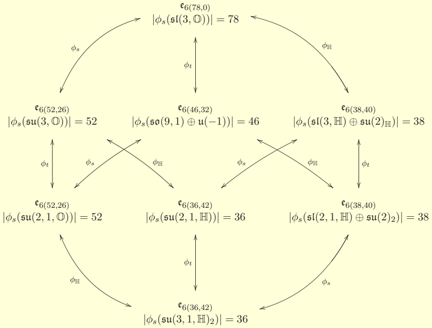


FIG. 3. Composition of associated Cartan maps of $\mathfrak{e}_6$ acting on real forms of $\mathfrak{e}_6$, showing the maximal compact subalgebra under ϕ_s.

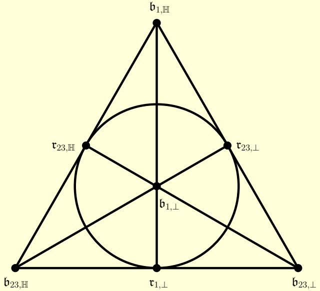


FIG. 4. Composition of associated Cartan maps of $\mathfrak{e}_6$ acting on real forms of $\mathfrak{e}_6$, showing the maximal compact subalgebra under ϕ_s.

Wangberg (PhD 2007), Wangberg & Dray (JMP 2013)

Symplectic Groups

Definition

$$\mathfrak{sp}(2k,\mathbb{R}) = \{Q \in M^{2k\times 2k}(\mathbb{R}) : Q\Omega Q^T = \Omega\}$$

$$\mathfrak{sp}(4,\mathbb{R}) = \mathfrak{so}(3,2) \qquad |\mathfrak{sp}(4,\mathbb{R})| = 10$$

$$\mathfrak{sp}(6,\mathbb{R}) = \mathfrak{c}_3 \text{ (split)} \qquad |\mathfrak{sp}(6,\mathbb{R})| = 21$$

$$\Omega = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$$

Symplectic Groups

Definition (traditional)

$$\mathfrak{sp}(2k, \mathbb{K}) = \{Q \in M^{2k \times 2k}(\mathbb{K}) : Q\Omega Q^T = \Omega\}$$

$$\mathfrak{sp}(4, \mathbb{C}) = \mathfrak{c}_2 \text{ (complex)} \qquad |\mathfrak{sp}(4, \mathbb{C})| = 20$$

$$\mathfrak{sp}(6, \mathbb{C}) = \mathfrak{c}_3 \text{ (complex)} \qquad |\mathfrak{sp}(6, \mathbb{C})| = 42$$

$$\Omega = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$$

Symplectic Groups

Definition (Sudbery)

$$\mathfrak{sp}(2k, \mathbb{K}) = \{Q \in M^{2k \times 2k}(\mathbb{K}) : Q\Omega Q^\dagger = \Omega\}$$

Fine Print: Count "phases" separately

$$\mathfrak{sp}(4, \mathbb{C}) = \mathfrak{su}(2,2) \qquad |\mathfrak{sp}(4, \mathbb{C})| = 15$$

$$\mathfrak{sp}(6, \mathbb{C}) = \mathfrak{su}(3,3) \qquad |\mathfrak{sp}(6, \mathbb{C})| = 35$$

$$\Omega = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$$

Conformalization

SO(4, 2) is the "conformalization" of SO(3, 1)

$$\text{conformalization} = \text{Lorentz} + \text{translations} + \text{conformal} + \text{dilation}$$
$$15 = 6 + 4 + 4 + 1$$

$$\text{acts on } \mathbf{P} = \begin{pmatrix} p+q & X \\ -\widetilde{X} & p-q \end{pmatrix} \longleftrightarrow \{X, p, q\}$$

$E_{7(-25)}$ is the "conformalization" of $E_{6(-26)}$

$$\text{conformalization} = \text{group} + (2 \times \text{null rotations}) + \text{phase}$$
$$133 = 78 + (2 \times 27) + 1$$

acts on Freudenthal "tower": $\{\mathcal{X}, \mathcal{Y}, p, q\}$

Freudenthal Tower

Elements of $\mathfrak{e}_7$: $(\phi \in \mathfrak{e}_6,\ \mathcal{A},\mathcal{B} \in \mathrm{H}_3(\mathbb{O}),\ \rho \in \mathbb{R})$

$$\Theta = (\phi, \mathcal{A}, \mathcal{B}, \rho)$$

Minimal representation: $(\mathcal{X},\mathcal{Y} \in \mathrm{H}_3(\mathbb{O}),\ p, q \in \mathbb{R})$

$$\mathcal{P} = (\mathcal{X}, \mathcal{Y}, p, q)$$

Action (Freudenthal):

$$\begin{aligned}
\mathcal{X} &\longmapsto \phi(\mathcal{X}) + \frac{1}{3}\rho\mathcal{X} + 2\mathcal{B} * \mathcal{Y} + \mathcal{A}q \\
\mathcal{Y} &\longmapsto 2\mathcal{A} * \mathcal{X} + \phi'(\mathcal{Y}) - \frac{1}{3}\rho\mathcal{Y} + \mathcal{B}p \\
p &\longmapsto \mathrm{tr}(\mathcal{A} \circ \mathcal{Y}) - \rho p \\
q &\longmapsto \mathrm{tr}(\mathcal{B} \circ \mathcal{X}) + \rho q
\end{aligned}$$

Tower:

$$\mathcal{A} : q = 1 \longmapsto \mathcal{X} = \mathcal{A} \longmapsto \mathcal{Y} = \mathcal{A} * \mathcal{A} \longmapsto p = \det \mathcal{A} \longmapsto 0$$

$$\mathcal{B} : 0 \longleftarrow q = \det \mathcal{B} \longleftarrow \mathcal{X} = \mathcal{B} * \mathcal{B} \longleftarrow \mathcal{Y} = \mathcal{B} \longleftarrow p = 1$$

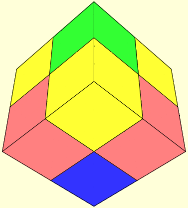
Squares and Cubes

2-forms in d = 4: $(6 = 1 + 4 + 1)$

00	01
10	11

$$00 \longleftrightarrow p$$
$$01, 10 \longleftrightarrow X$$
$$11 \longleftrightarrow q$$

3-forms in d = 6: $(20 = 1 + 9 + 9 + 1)$



$$000 \longleftrightarrow p$$
$$001, 010, 100 \longleftrightarrow \mathcal{X}$$
$$011, 110, 101 \longleftrightarrow \mathcal{Y}$$
$$111 \longleftrightarrow q$$

$\mathrm{Sp}(6,\mathbb{O})$?

Idea: $\dot{\mathbf{M}}[u \wedge v \wedge w] = \dot{\mathbf{M}}u \wedge v \wedge w + u \wedge \dot{\mathbf{M}}v \wedge w + u \wedge v \wedge \dot{\mathbf{M}}w$

- Not well defined over $\mathbb{H}$!
- No way (yet...) to fix ordering.
- BUT: works for $\mathfrak{e}_6$ and other real generators.
- Use commutators to *define* remaining elements!
- Get: $E_7 \cong \mathrm{Sp}(6,\mathbb{O})$!

[Dray, Manogue, Wilson (CMUC 2014)]

The 2 × 2 Magic Square

Barton & Sudbery (2003):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{d}_1$	$\mathfrak{a}_1$	$\mathfrak{b}_2$	$\mathfrak{b}_4$
$\mathbb{C}$	$\mathfrak{a}_1$	$\mathfrak{a}_1 \oplus \mathfrak{a}_1$	$\mathfrak{d}_3$	$\mathfrak{d}_5$
$\mathbb{H}$	$\mathfrak{b}_2$	$\mathfrak{d}_3$	$\mathfrak{d}_4$	$\mathfrak{d}_6$
$\mathbb{O}$	$\mathfrak{b}_4$	$\mathfrak{d}_5$	$\mathfrak{d}_6$	$\mathfrak{d}_8$

"Vinberg":

$$sa(2, \mathbb{A} \otimes \mathbb{B}) \oplus \mathfrak{so}(\mathrm{Im}\,\mathbb{A}) \oplus \mathfrak{so}(\mathrm{Im}\,\mathbb{B})$$

$$\mathfrak{so}(\mathrm{Im}\,\mathbb{H}) = \mathfrak{so}(3); \quad \mathfrak{so}(\mathrm{Im}\,\mathbb{O}) = \mathfrak{so}(7)$$

Unified Clifford algebra description using division algebras

[Kincaid (MS 2012), Kincaid and Dray (MPLA 2014),
Dray, Kincaid, & Huerta (LMP 2014)]

Symmetry Groups

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{su}(2,\mathbb{R})$	$\mathfrak{su}(2,\mathbb{C})$	$\mathfrak{su}(2,\mathbb{H})$	$\mathfrak{su}(2,\mathbb{O})$
$\mathbb{C}'$	$\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{sl}(2,\mathbb{H})$	$\mathfrak{sl}(2,\mathbb{O})$
$\mathbb{H}'$	$\mathfrak{sp}(4,\mathbb{R})$	$\mathfrak{sp}(4,\mathbb{C})$	$\mathfrak{sp}(4,\mathbb{H})$	$\mathfrak{sp}(4,\mathbb{O})$
$\mathbb{O}'$	??	??	??	??

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$\mathbb{C}'$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(5,1)$	$\mathfrak{so}(9,1)$
$\mathbb{H}'$	$\mathfrak{so}(3,2)$	$\mathfrak{so}(4,2)$	$\mathfrak{so}(6,2)$	$\mathfrak{so}(10,2)$
$\mathbb{O}'$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

Orthogonal Groups

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	SO(2)	SO(3)	SO(5)	SO(9)
$\mathbb{C}$	SO(2,1)	SO(3,1)	SO(5,1)	SO(9,1)
$\mathbb{H}$	SO(3,2)	SO(4,2)	SO(6,2)	SO(10,2)
$\mathbb{O}$	SO(5,4)	SO(6,4)	SO(8,4)	SO(12,4)

$$d = 3, 4, 6, 10$$

(1980s: Corrigan,Evans,Fairlie,Manogue,Sudbery)

isogeny: $\mathrm{SL}(2,\mathbb{K}) \cong \mathrm{SO}(k+1,1)$, e.g.

$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

Clifford algebras!

From Pauli to Dirac ($C\ell(3,0) \longrightarrow C\ell(3,1)$)

$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$
$$= t\,\sigma_t + x\,\sigma_x + y\,\sigma_y + z\,\sigma_z$$

$$\Gamma(X) = \begin{pmatrix} 0 & X \\ -\widetilde{X} & 0 \end{pmatrix}$$
$$= t\,\gamma_t + x\,\gamma_x + y\,\gamma_y + z\,\gamma_z$$

$$\widetilde{X} = X - \mathrm{tr}(X)\,I \Longrightarrow \Gamma(X)^2 = \det(X) = t^2 - x^2 - y^2 - z^2$$

Basic Idea

isogeny: $\mathrm{SO}(2, \mathbb{K}' \otimes \mathbb{K}) \cong \mathrm{SO}(k + \frac{1}{2}k', \frac{1}{2}k')$, e.g.

$$X = \begin{pmatrix} A & a \\ \bar{a} & -A^* \end{pmatrix}$$

with $a \in \mathbb{K}$, $A \in \mathbb{K}'$.

$$\det X = -|A|^2 - |a|^2$$

Idea: $\Gamma(X)$ generates $\mathrm{C}\ell(k + \frac{1}{2}k', \frac{1}{2}k')$!

From Clifford to Lorentz

Reflections:

$\mathbf{Q} \longmapsto \mathbf{PQP}^{-1}$ reflects $\mathbf{Q}$ about $\mathbf{P}$.

Flips:

Successive flips about $\mathbf{P}_1$, $\mathbf{P}_2$ result in a (finite) rotation in the plane spanned by $\mathbf{P}_i$.

The quadratic elements of $\mathrm{C}\ell(p,q)$ generate $\mathrm{SO}(p,q)$

Notation

Identify $\mathbf{P}$ *with* $\mathbf{P}_L \in \mathrm{End}(\mathcal{K}^{4\times4})$*:* $\mathbf{P}_L(\mathbf{Q}) = \mathbf{PQ}$.

Lemma

$$(\mathbf{P}_L)^2 = (\mathbf{P}^2)_L$$

Proof.

By alternativity:

$$\mathbf{P}(\mathbf{PQ}) = \mathbf{P}^2\mathbf{Q}$$

□

Theorem

The subalgebra of $\mathrm{End}(\mathcal{K}^{4\times 4})$ *generated by* $\Gamma_L(\mathbf{V}_2)$ *is a Clifford algebra, that is,* $\mathrm{C}\ell(\mathbf{V}_4) = \mathrm{C}\ell(k + \frac{1}{2}k', \frac{1}{2}k')$.

Proof.

$$\Gamma(X)^2 = -\det(X)\,\mathbf{I} \Longrightarrow \Gamma_L(X)^2 = |X| = -\det(X)$$

Polarize:

$$\mathbf{P}(\mathbf{QR}) + \mathbf{Q}(\mathbf{PR}) = (\mathbf{PQ} + \mathbf{QP})\,\mathbf{R} = 2g(\mathbf{P},\mathbf{Q})\,\mathbf{R}$$

That is, we have

$$\{\mathbf{P}_L, \mathbf{Q}_L\} = \{\mathbf{P}, \mathbf{Q}\}_L$$

□

Lemma

Let $\mathbf{P}, \mathbf{Q} \in \Gamma(\mathbf{V}_2)$, *with* $\det \mathbf{P} \neq 0$. *Then*

$$(\mathbf{PQ})\mathbf{P}^{-1} = \mathbf{P}(\mathbf{QP}^{-1}) \tag{1}$$

and this matrix, which we denote $\mathbf{PQP}^{-1}$, *also lies in* $\Gamma(\mathbf{V}_2)$.

Proof.

By the discussion above, $\mathbf{P}^{-1}$ is proportional to $\mathbf{P}$, so that the elements of $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{P}^{-1}$ jointly contain only two independent imaginary directions in each of $\mathbb{K}$ and $\mathbb{K}'$. Thus, there are no associativity issues when multiplying these matrices, which establishes (1). Direct computation further establishes the fact that $\mathbf{PQP}^{-1} \in \Gamma(\mathbf{V}_2)$. □

Lemma

Let $\mathbf{P}, \mathbf{Q} \in \Gamma(\mathbf{V}_2)$ *with* $|\mathbf{P}| = 1$, *so that* $\mathbf{P}_L, \mathbf{Q}_L \in \mathbf{V}_4$. *Then*

$$R_{\mathbf{P}_L}(\mathbf{Q}_L) = -\left(\mathbf{P}\mathbf{Q}\mathbf{P}^{-1}\right)_L .$$

Proof.

Given that $\mathbf{P}^2$ is a multiple of the identity, it is enough to show that

$$\mathbf{P}_L \circ \mathbf{Q}_L \circ \mathbf{P}_L = (\mathbf{P}\mathbf{Q}\mathbf{P})_L$$

in $\mathcal{K}^{4\times 4}$, that is, that for $\mathbf{R} \in \Gamma(\mathbf{V}_2)$

$$\mathbf{P}(\mathbf{Q}(\mathbf{P}(\mathbf{R}))) = (\mathbf{P}\mathbf{Q}\mathbf{P})(\mathbf{R}) \tag{2}$$

But (2) follows immediately from the Moufang identity

$$p(q(p(r))) = (pqp)r$$

□

Lemma

There is a homomorphism

$$R\colon SO(\mathbf{V}_4) \to SO(\mathbf{V}_4) \tag{3}$$

which sends a product of unit vectors $g = \mathbf{P}_1 \cdot \mathbf{P}_2$ *in* $SO(\mathbf{V}_4)$ *to the element* R_g *of* $SO(\mathbf{V}_4)$ *given by:*

$$R_g(\mathbf{Q}) = \mathbf{P}_1(\mathbf{P}_2\mathbf{Q}\mathbf{P}_2^{-1})\mathbf{P}_1^{-1}. \tag{4}$$

Theorem

The second-order homogeneous elements of $\mathrm{C}\ell(k+\frac{1}{2}k', \frac{1}{2}k')$ *generate an action of* $SO(k+\frac{1}{2}k', \frac{1}{2}k')$ *on* $\mathbf{V}_4 = \Gamma(\mathbf{V}_2)$.

Explicit Construction

Flips: $$\mathbf{P} \longmapsto e_p \mathbf{P} e_p^{-1}$$

Nested flips: $$\mathbf{P} \longmapsto \mathbf{M}_2 \left(\mathbf{M}_1 \mathbf{P} \mathbf{M}_1^{-1}\right) \mathbf{M}_2^{-1}$$

where

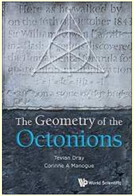
$$\begin{aligned}
\mathbf{M}_1 &= -e_p \, \mathbf{I} \\
\mathbf{M}_2 &= \left(e_p \, c(\tfrac{\theta}{2}) + e_q \, s(\tfrac{\theta}{2})\right) \, \mathbf{I} \\
&= \begin{cases} \left(e_p \cosh(\frac{\theta}{2}) + e_q \sinh(\frac{\theta}{2})\right) \mathbf{I}, & (e_p e_q)^2 = 1 \\ \left(e_p \cos(\frac{\theta}{2}) + e_q \sin(\frac{\theta}{2})\right) \mathbf{I}, & (e_p e_q)^2 = -1 \end{cases}
\end{aligned}$$

Theorem

The nested flips generate an action of $SO(k + \frac{1}{2}k', \frac{1}{2}k')$ on $\mathbf{V}_4$.

SUMMARY

- **Have:** $E_6 \cong \mathrm{SL}(3, \mathbb{O})$
  [Dray & Manogue (2010)]
- **Have:** Structure of E_6
  [Wangberg (PhD 2007), Wangberg & Dray (2013; 2014)]
- **Have:** 2×2 Magic Square
  [Kincaid (MS 2012), Kincaid and Dray (2014),
  Dray, Kincaid, & Huerta (2014)]
- **(Mostly) Have:** $E_7 \cong \mathrm{Sp}(6, \mathbb{O})$
  [Dray, Manogue, Wilson (CMUC 2014)]
- **Want:** $E_8 \cong ??$



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