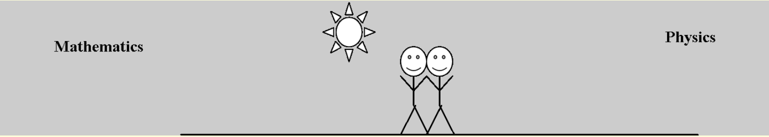
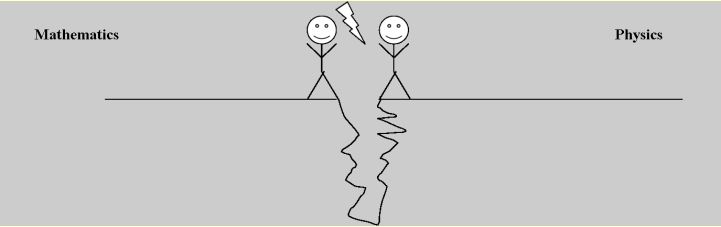
Bridging the Gap between Mathematics and Physics

Tevian Dray

Department of Mathematics
Oregon State University
http://www.math.oregonstate.edu/~tevian

Mathematics vs. Physics




Mathematics
Physics

What are Functions?

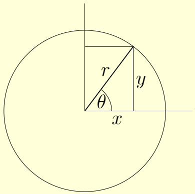
Suppose the temperature on a rectangular slab of metal is given by

$$T(x, y) = k(x^2 + y^2)$$

where k is a constant. What is $T(r, \theta)$?

A: $T(r, \theta) = kr^2$

B: $T(r, \theta) = k(r^2 + \theta^2)$



What are Functions?

MATH

$$T = f(x,y) = k(x^2+y^2)$$
$$T = g(r,\theta) = kr^2$$

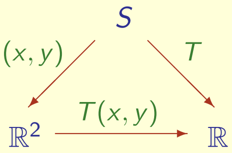
PHYSICS

$$T = T(x,y) = k(x^2+y^2)$$
$$T = T(r,\theta) = kr^2$$

Two disciplines separated by a common language...

Differential Geometry!

$$T(x,y) \longleftrightarrow T \circ (x,y)^{-1}$$
$$T(r,\theta) \longleftrightarrow T \circ (r,\theta)^{-1}$$



Two disciplines separated by a common language...

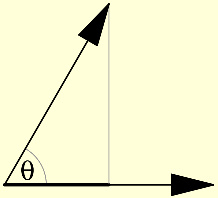
physical quantities $\neq$ functions

Mathematics vs. Physics

- **Physics is about things.**
- **Physicists can't change the problem.**

- **Mathematicians do algebra.**
- **Physicists do geometry.**

Write down something that you know about the dot product.

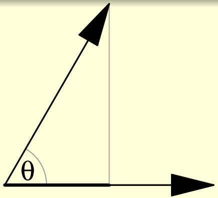


Geometry:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$

Algebra:

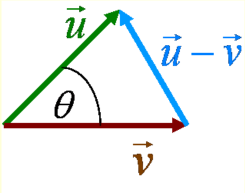
$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_x v_x + u_y v_y$$



Projection:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$

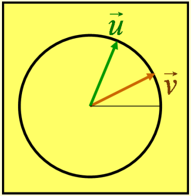
$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_x v_x + u_y v_y$$



Law of Cosines:

$$(\vec{\mathbf{u}} - \vec{\mathbf{v}}) \cdot (\vec{\mathbf{u}} - \vec{\mathbf{v}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} + \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} - 2\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$$

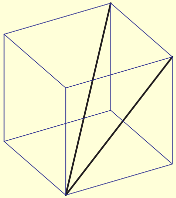
$$|\vec{\mathbf{u}} - \vec{\mathbf{v}}|^2 = |\vec{\mathbf{u}}|^2 + |\vec{\mathbf{v}}|^2 - 2|\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$



Addition Formulas:

$$\vec{\mathbf{u}} = \cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}$$
$$\vec{\mathbf{v}} = \cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath}$$
$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \cos(\alpha - \beta)$$

Find the angle between the diagonal of a cube and the diagonal of one of its faces.



Algebra:

$$\vec{\mathbf{u}} = \hat{\imath} + \hat{\jmath} + \hat{k}$$
$$\vec{\mathbf{v}} = \hat{\imath} + \hat{k}$$

$$\Longrightarrow \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 2$$

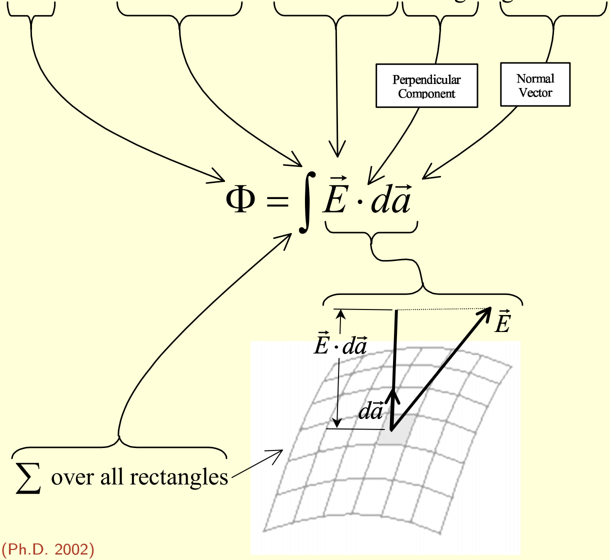
Geometry:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta = \sqrt{3}\sqrt{2}\cos\theta$$

Need both!

Flux is the total amount of electric field through a given area.

$$\Phi = \int \vec{E} \cdot d\vec{a}$$



Kerry Browne (Ph.D. 2002)

CUPM

MAA Committee on the Undergraduate Program in Mathematics

Curriculum Guide

http://www.maa.org/cupm/cupm2004.pdf

CRAFTY

Subcommittee on Curriculum Renewal Across the First Two Years

Voices of the Partner Disciplines

http://www.maa.org/cupm/crafty

The Vector Calculus Bridge Project

- **Differentials** *(Use what you know!)*
- **Multiple representations**
- **Symmetry** *(adapted bases, coordinates)*
- **Geometry** *(vectors, div, grad, curl)*

- Small group activities
- Instructor's guide
- Online text (`http://www.math.oregonstate.edu/BridgeBook`)

`http://www.math.oregonstate.edu/bridge`

The Vector Calculus Bridge Project

Bridge Project homepage hits in 2009

Mathematicians' Line Integrals

- Start with Theory

$$\begin{aligned}
\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int \vec{\mathbf{F}} \cdot \hat{\mathbf{T}}\, ds \\
&= \int \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|} |\vec{\mathbf{r}}'(t)|\, dt \\
&= \int \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt \\
&= \ldots = \int P\, dx + Q\, dy + R\, dz
\end{aligned}$$

- Do examples starting from next-to-last line

Need parameterization $\vec{\mathbf{r}} = \vec{\mathbf{r}}(t)$

Physicists' Line Integrals

- Theory
  - Chop up curve into little pieces $d\vec{\mathbf{r}}$.
  - Add up components of $\vec{\mathbf{F}}$ parallel to curve (times length of $d\vec{\mathbf{r}}$)
- Do examples directly from $\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$

Need $d\vec{\mathbf{r}}$ along curve

Mathematics

$$\vec{\mathbf{F}}(x,y) = \frac{-y\,\hat{\imath} + x\,\hat{\jmath}}{x^2+y^2} \qquad \vec{\mathbf{r}} = x\,\hat{\imath} + y\,\hat{\jmath}$$

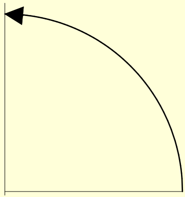
$$\begin{aligned} x &= 2\cos\theta \\ y &= 2\sin\theta \end{aligned}$$

$$\begin{aligned} \int \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} &= \int_0^{\frac{\pi}{2}} \vec{\mathbf{F}}(x(\theta),y(\theta))\cdot\vec{\mathbf{r}}\,'(x(\theta),y(\theta))\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2}(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})\cdot 2(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})\,d\theta \\ &= \ldots = \frac{\pi}{2} \end{aligned}$$

Physics

$$\vec{\mathbf{F}} = \frac{\hat{\phi}}{r}$$

$$d\vec{\mathbf{r}} = r\,d\phi\,\hat{\phi}$$



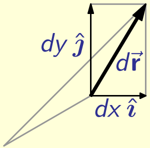
I: $|\vec{\mathbf{F}}| = \text{const};\ \vec{\mathbf{F}} \parallel d\vec{\mathbf{r}} \Longrightarrow$

$$\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \frac{1}{2}(2\ \frac{\pi}{2})$$

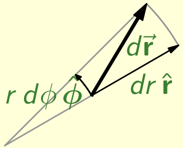
II: Do the dot product $\longmapsto$

$$\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^{\frac{\pi}{2}} \frac{\hat{\phi}}{2} \cdot 2\,d\phi\,\hat{\phi} = \int_0^{\frac{\pi}{2}} d\phi = \frac{\pi}{2}$$

Vector Differentials



$$d\vec{\mathbf{r}} = dx\,\hat{\imath} + dy\,\hat{\jmath}$$



$$d\vec{\mathbf{r}} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\phi}$$

$$\begin{aligned} ds &= |d\vec{\mathbf{r}}| \\ d\vec{\mathbf{A}} &= d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2 \\ dA &= |d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2| \\ dV &= (d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2) \cdot d\vec{\mathbf{r}}_3 \end{aligned}$$

Coherent Calculus

co-he-rent:

logically or aesthetically ordered

cal-cu-lus:

a method of computation *in a special notation*

differential calculus:

a branch of mathematics concerned chiefly with the study of the rate of change of functions with respect to their variables especially through the use of derivatives *and differentials*

Derivatives

Instead of:

- chain rule
- related rates
- implicit differentiation
- derivatives of inverse functions
- difficulties of interpretation (units!)

One coherent idea:

"Zap equations with *d*"

Tevian Dray & Corinne A. Manogue,
Putting Differentials Back into Calculus,
College Math. J. **41**, 90–100 (2010).

A Radical View of Calculus

- The central idea in calculus is not the limit.
- The central idea of derivatives is not slope.
- The central idea of integrals is not area.
- The central idea of curves and surfaces is not parameterization.
- The central representation of a function is not its graph.

- The central idea in calculus is the differential.
- The central idea of derivatives is rate of change.
- The central idea of integrals is total amount.
- The central idea of curves and surfaces is "use what you know".
- The central representation of a function is data attached to the domain.

SUMMARY

I took this class a year ago, and I still remember all of it...