The Geometry of Relativity

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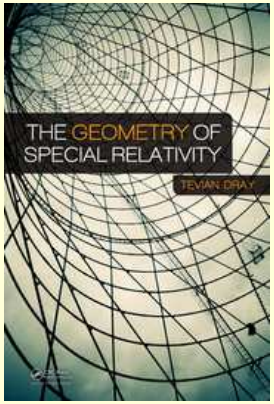
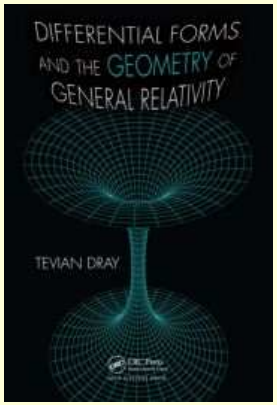
Books



The Geometry of Special Relativity
Tevian Dray
A K Peters/CRC Press 2012
ISBN: 978-1-4665-1047-0
http://physics.oregonstate.edu/coursewikis/GSR



Differential Forms and the Geometry of General Relativity
Tevian Dray
A K Peters/CRC Press 2014
ISBN: 978-1-4665-1000-5
http://physics.oregonstate.edu/coursewikis/GDF
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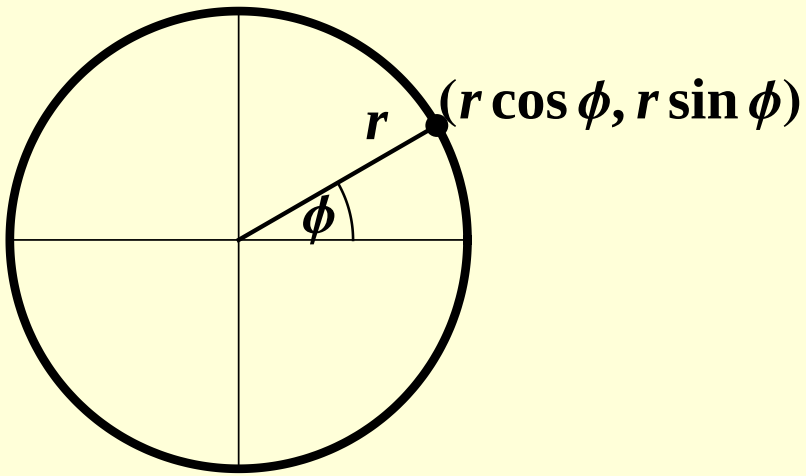
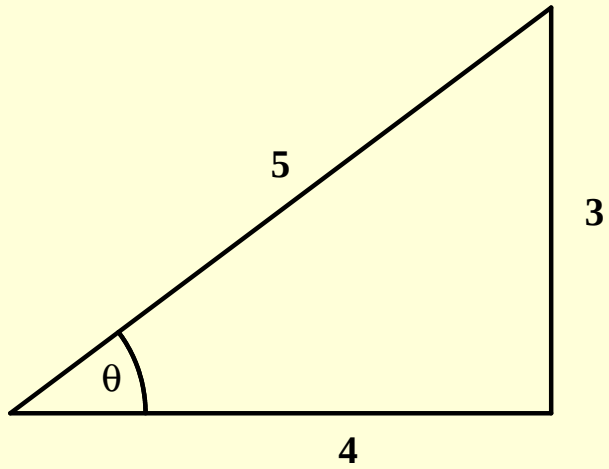
Trigonometry

$$ds^2 = dx^2 + dy^2$$

$$x^2 + y^2 = r^2$$



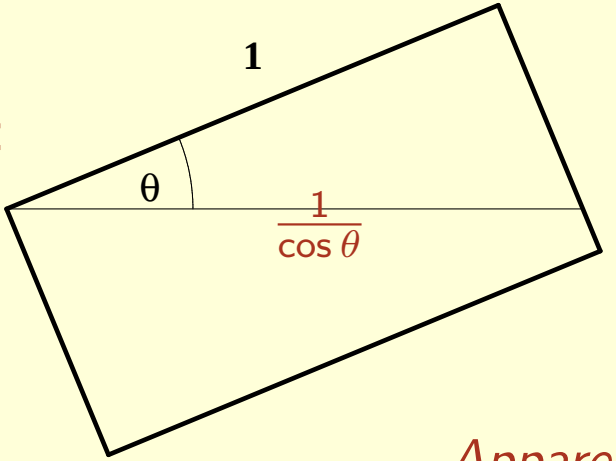


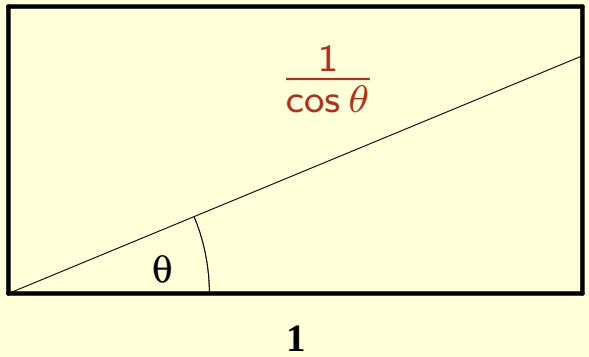
$r\phi =$ arclength

$$\tan\theta = \frac{3}{4} \Longrightarrow \cos\theta = \frac{4}{5}$$

Measurements

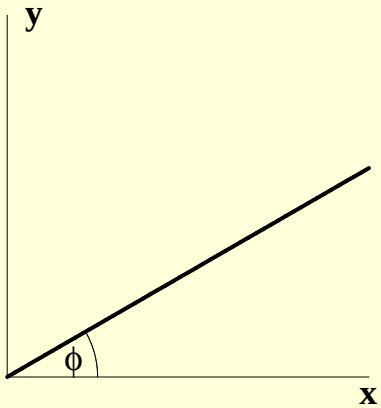
Width:

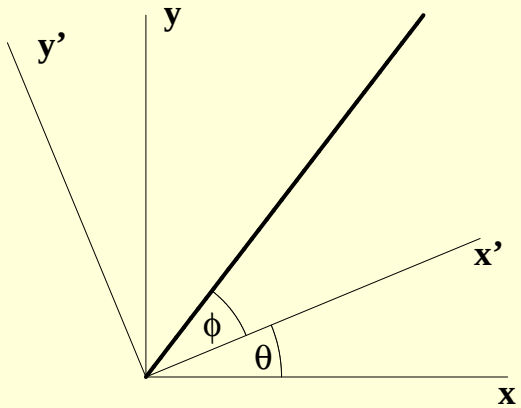




Apparent width > 1

Slope:

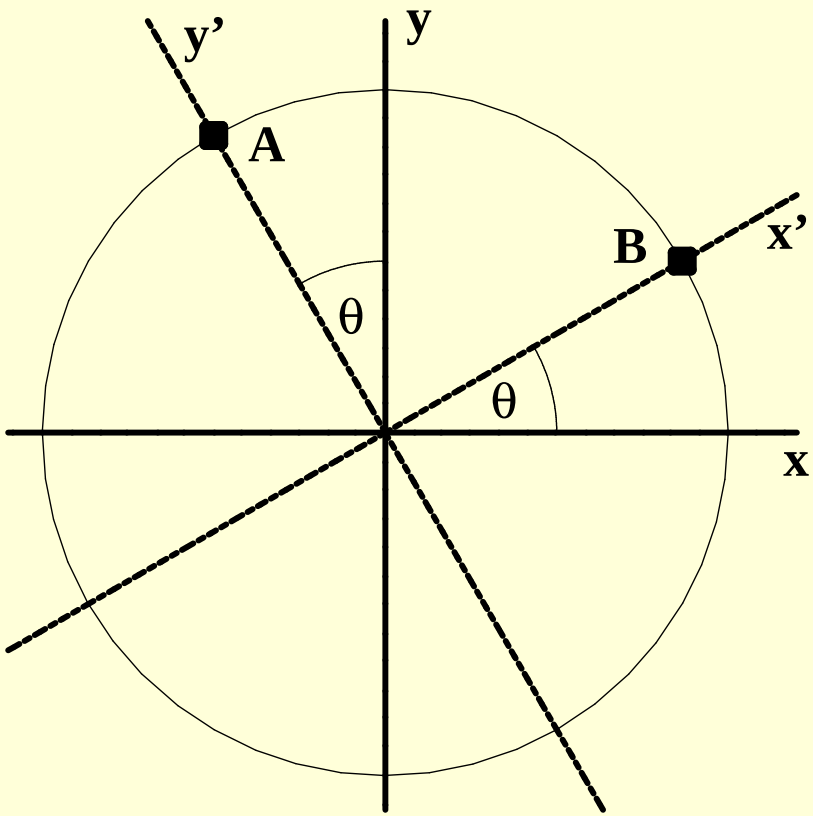




$$m \neq m_1 + m_2$$

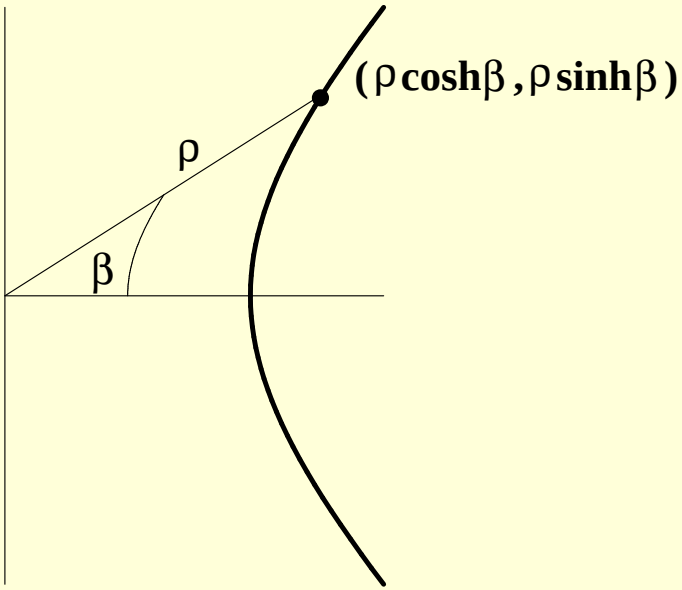
$$\tan(\theta+\phi) = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi} = \frac{m_1+m_2}{1-m_1 m_2}$$

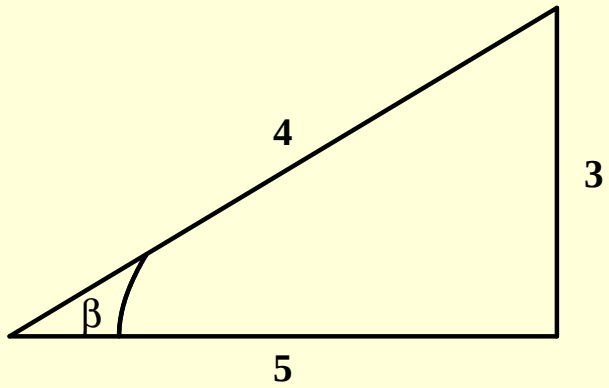
Rotations



Trigonometry

$$ds^2 = -c^2\,dt^2 + dx^2$$



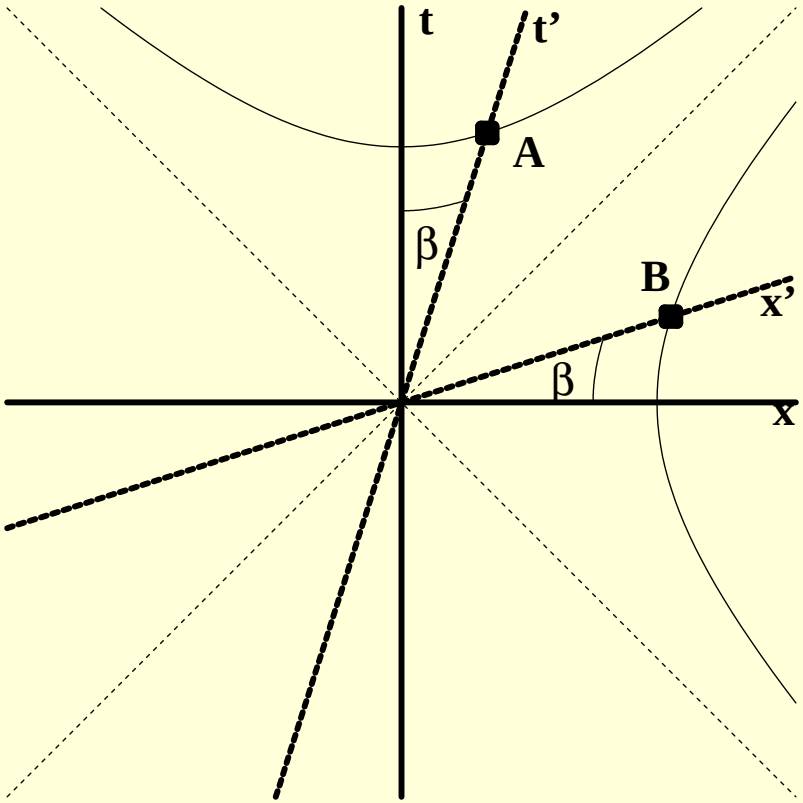


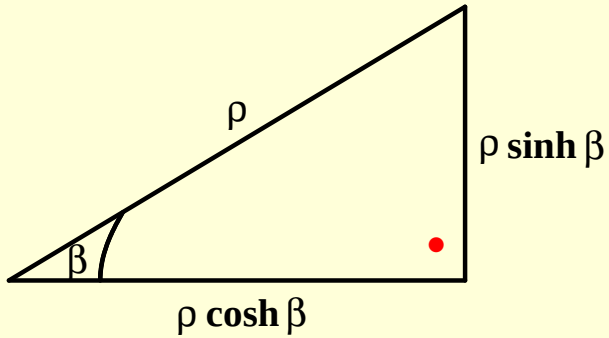
$$\tanh\beta = \frac{3}{5} \Longrightarrow \cosh\beta = \frac{5}{4}$$

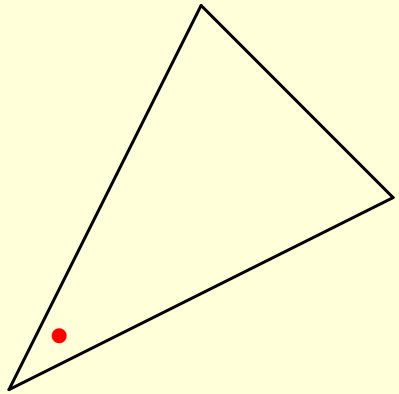
$\rho\beta =$ arclength

$(\cosh\beta \geq 1; \tanh\beta < 1)$

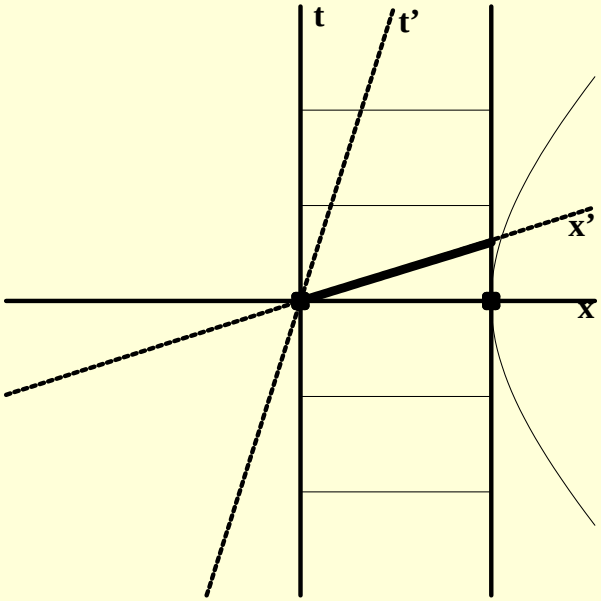
Trigonometry

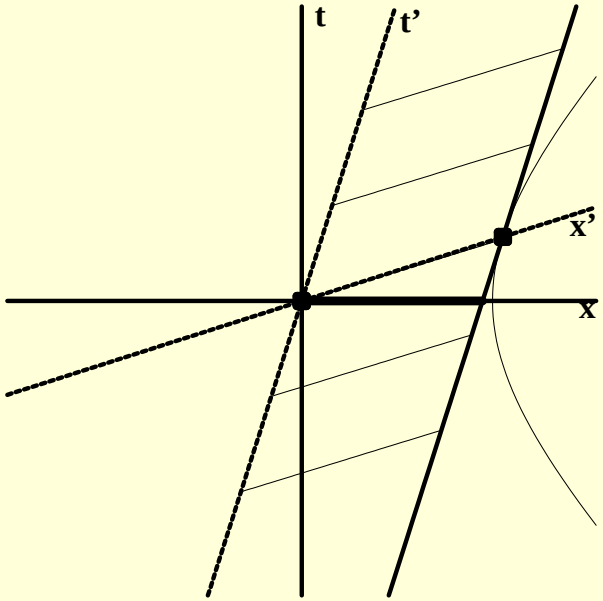




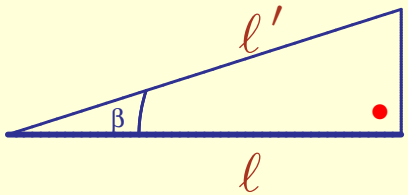


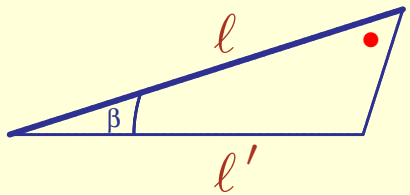
Length Contraction



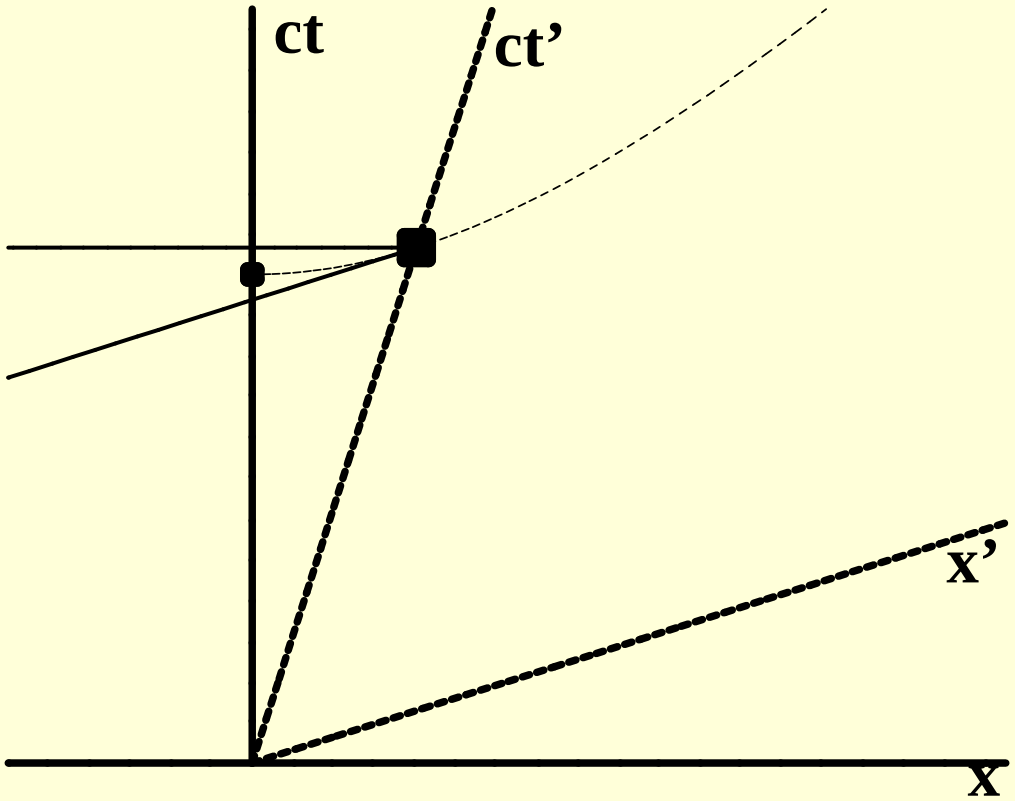


$$\ell' = \frac{\ell}{\cosh \beta}$$



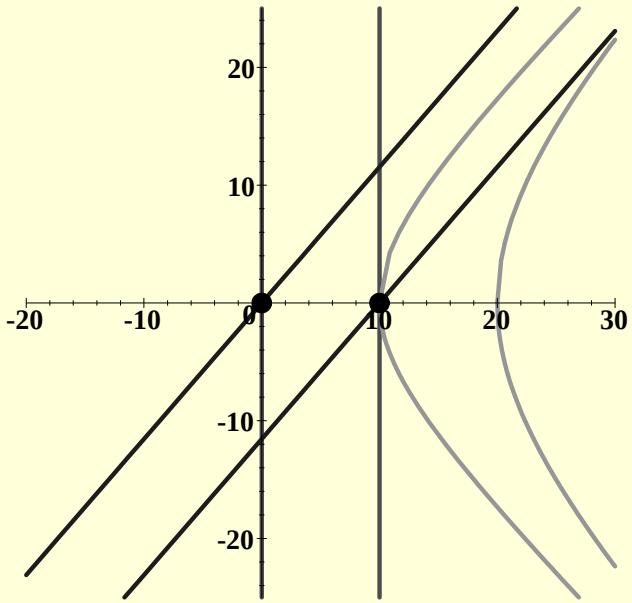


Time Dilation

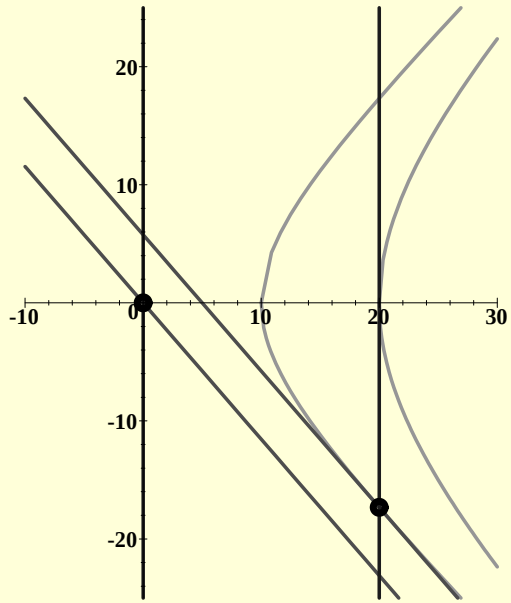


Pole & Barn

A 20 foot pole is moving towards a 10 foot barn fast enough that the pole appears to be only 10 feet long. As soon as both ends of the pole are in the barn, slam the doors. How can a 20 foot pole fit into a 10 foot barn?



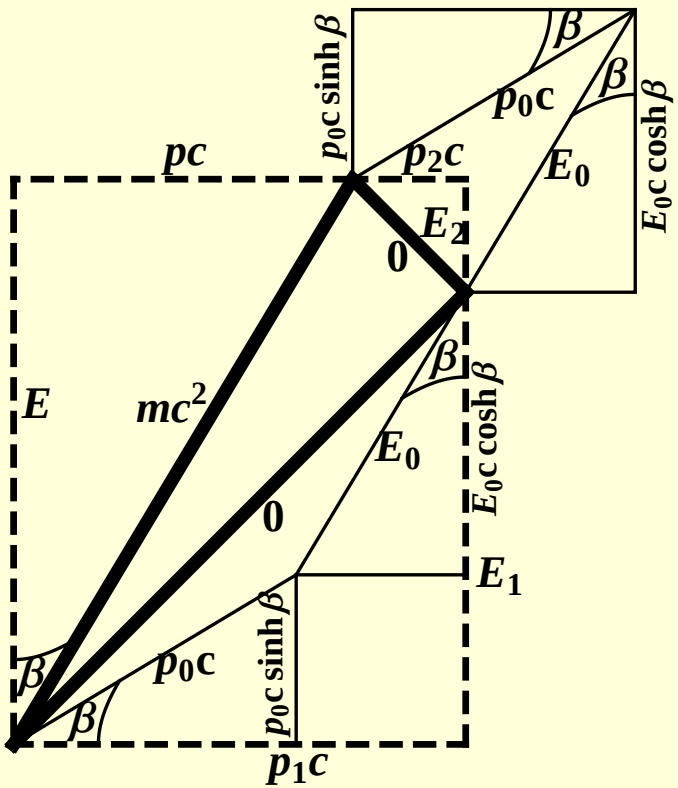
barn frame

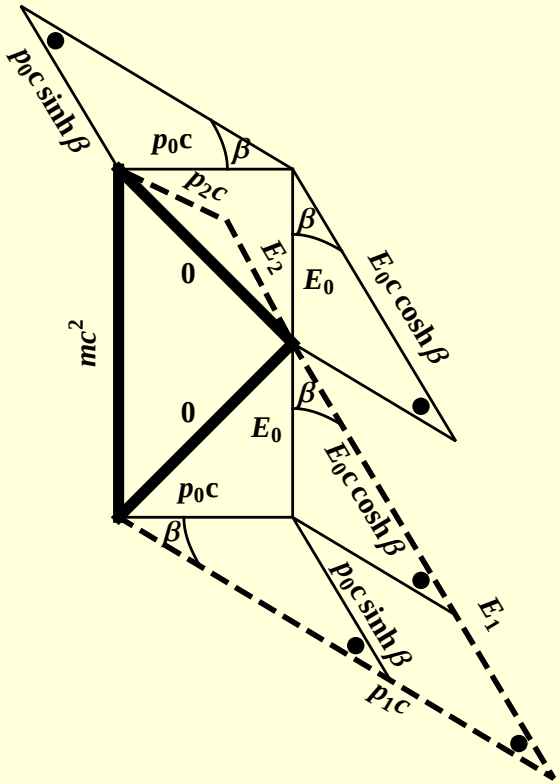


pole frame

Relativistic Mechanics

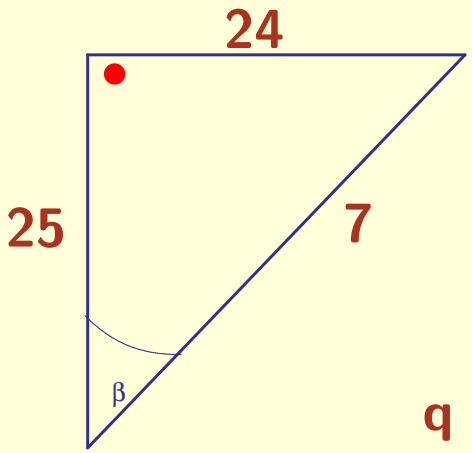
A pion of (rest) mass m and (relativistic) momentum $p = \frac{3}{4}mc$ decays into 2 (massless) photons. One photon travels in the same direction as the original pion, and the other travels in the opposite direction. Find the energy of each photon. $[E_1 = mc^2, E_2 = \frac{1}{4}mc^2]$

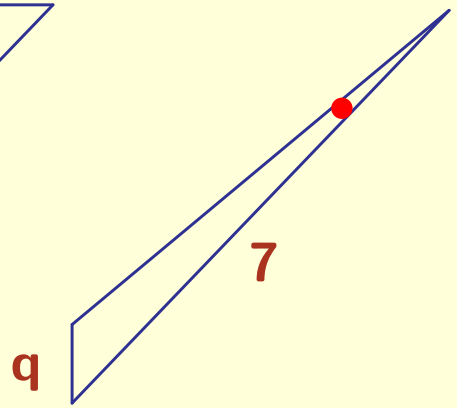


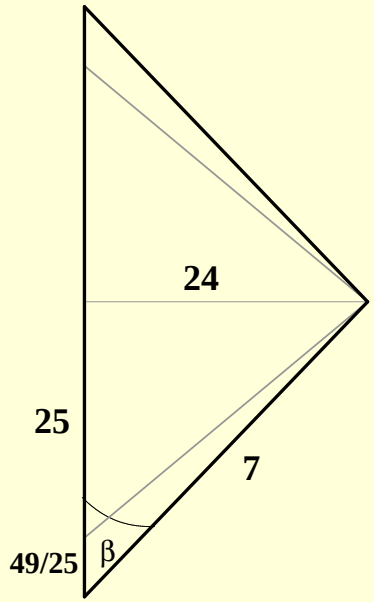


Twin Paradox

One twin travels 24 light-years to star X at speed $\frac{24}{25}c$; her twin brother stays home. When the traveling twin gets to star X, she immediately turns around, and returns at the same speed. How long does each twin think the trip took?



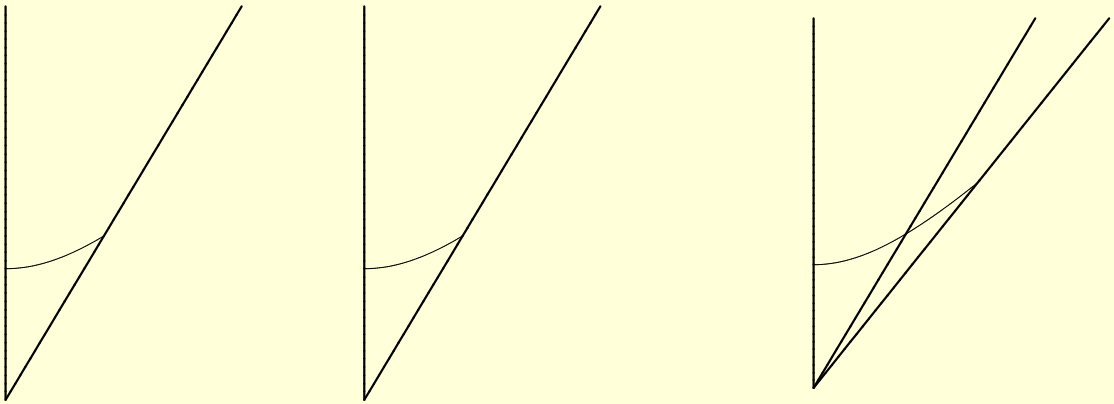




$$\cosh\beta = \frac{25}{7} \qquad q = \frac{7}{\cosh\beta} = \frac{49}{25}$$

Straight path takes longest!

Addition of Velocities

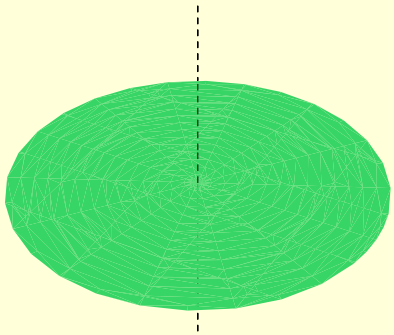


$$\frac{v}{c} = \tanh\beta$$

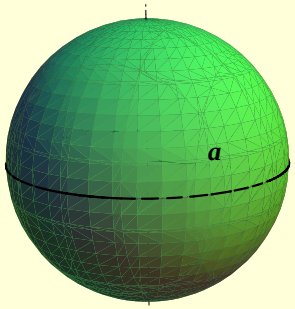
$$\tanh(\alpha+\beta) = \frac{\tanh\alpha + \tanh\beta}{1+\tanh\alpha\tanh\beta} = \frac{\frac{u}{c}+\frac{v}{c}}{1+\frac{uv}{c^2}}$$

Einstein addition formula!

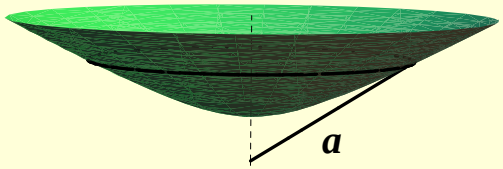
Line Elements



$dr^2 + r^2\, d\phi^2$



$d\theta^2 + \sin^2\theta\, d\phi^2$



$d\beta^2 + \sinh^2\beta\, d\phi^2$

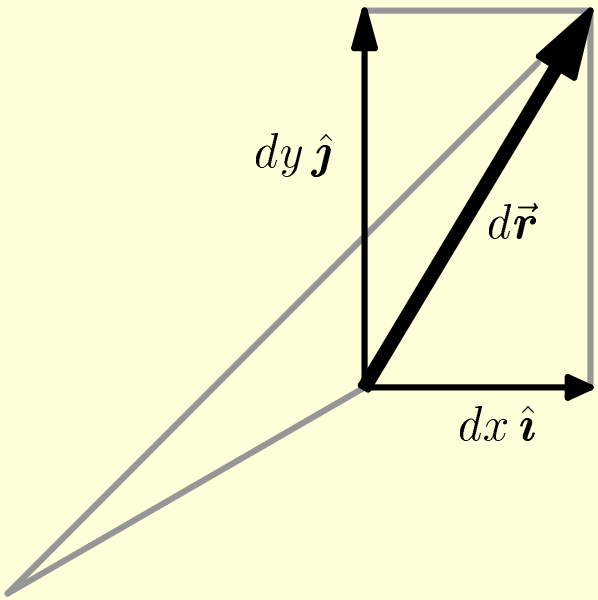
Black Hole: $ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2$

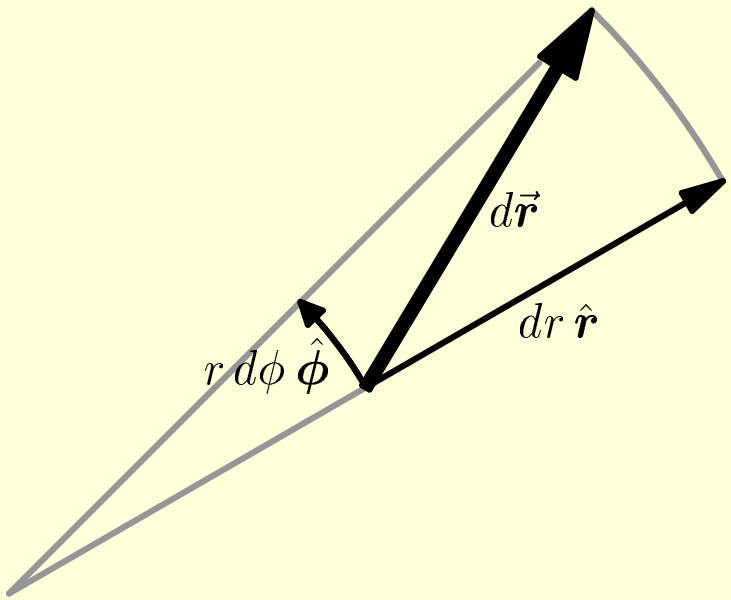
Cosmology: $ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right)$

	s = 0	**s = 1**
flat	Euclidean	Minkowskian **(SR)**
curved	Riemannian	Lorentzian **(GR)**

Vector Calculus

$$ds^2 = d\vec{\mathbf{r}} \cdot d\vec{\mathbf{r}}$$





$$d\vec{\mathbf{r}} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$$

Differential Forms in a Nutshell ($\mathbb{R}^3$)

Differential forms are integrands: $(*^2 = 1)$

$$f = f \qquad \text{(0-form)}$$

$$F = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \qquad \text{(1-form)}$$

$$*F = \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}} \qquad \text{(2-form)}$$

$$*f = f\, dV \qquad \text{(3-form)}$$

Exterior derivative: $(d^2 = 0)$

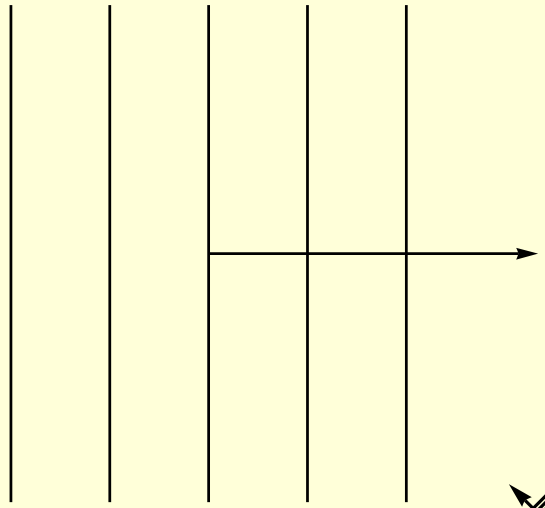
$$df = \vec{\nabla} f \cdot d\vec{\mathbf{r}}$$

$$dF = \vec{\nabla} \times \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}}$$

$$d*F = \vec{\nabla} \cdot \vec{\mathbf{F}}\, dV$$

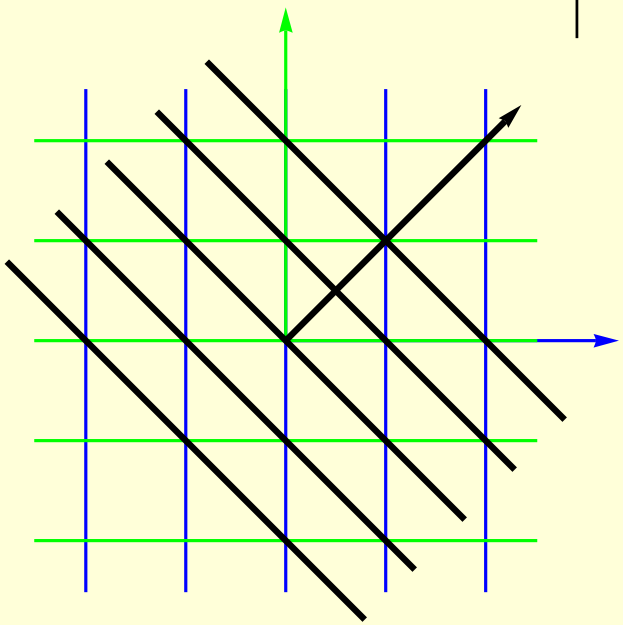
$$d*f = 0$$

The Geometry of Differential Forms

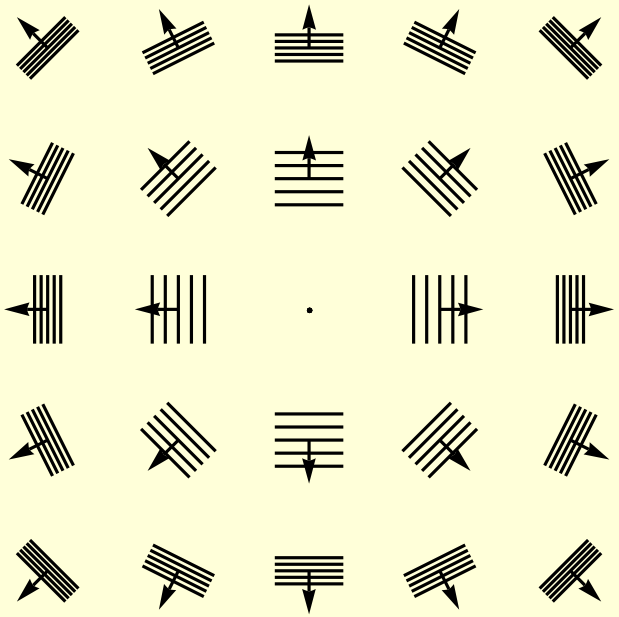


dx

$dx + dy$



$r\,dr = x\,dx + y\,dy$



Geodesic Equation

Orthonormal basis:

$$\boxed{d\vec{\mathbf{r}} = \sigma^i\, \hat{\mathbf{e}}_i}$$

Connection:

$$\omega_{ij} = \hat{\mathbf{e}}_i \cdot d\hat{\mathbf{e}}_j$$

$$d\sigma^i + \omega^i{}_j \wedge \sigma^j = 0$$

$$\omega_{ij} + \omega_{ji} = 0$$

Geodesics:

$$\vec{\mathbf{v}}\, d\lambda = d\vec{\mathbf{r}}$$

$$\dot{\vec{\mathbf{v}}} = 0$$

Symmetry:

$$d\vec{\mathbf{X}} \cdot d\vec{\mathbf{r}} = 0$$

$$\Longrightarrow \vec{\mathbf{X}} \cdot \vec{\mathbf{v}} = \text{const}$$

Einstein's Equation

Curvature:

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

Einstein tensor:

$$\gamma^i = -\frac{1}{2}\,\Omega_{jk} \wedge *(\sigma^i \wedge \sigma^j \wedge \sigma^k)$$

$$G^i = *\gamma^i = G^i{}_j\,\sigma^j$$

$$\vec{\mathbf{G}} = G^i \hat{\mathbf{e}}_i = G^i{}_j\,\sigma^j \hat{\mathbf{e}}_i$$

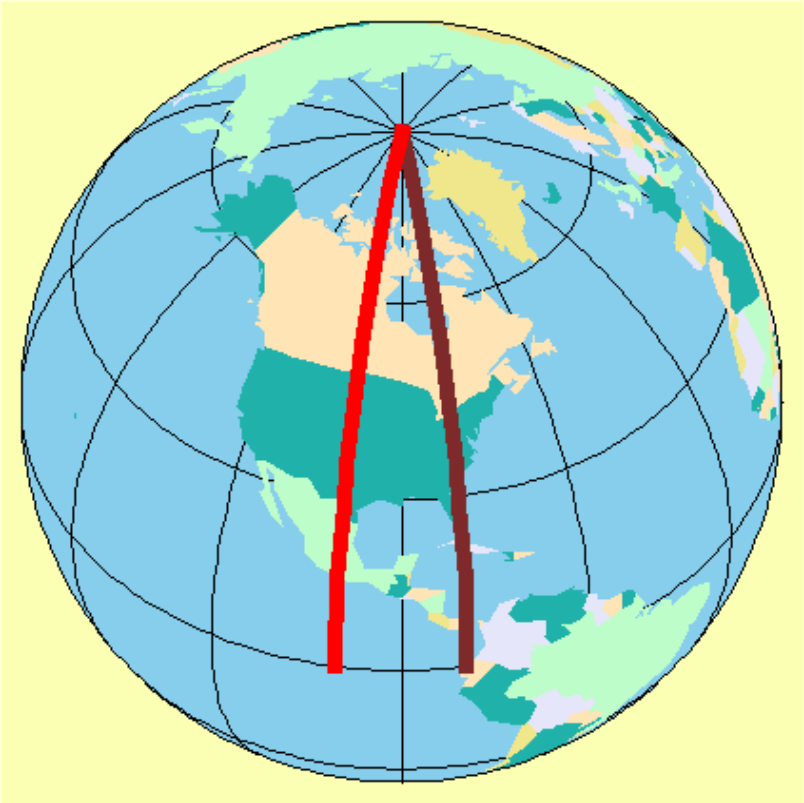
$$\Longrightarrow d*\vec{\mathbf{G}} = 0$$

Field equation:

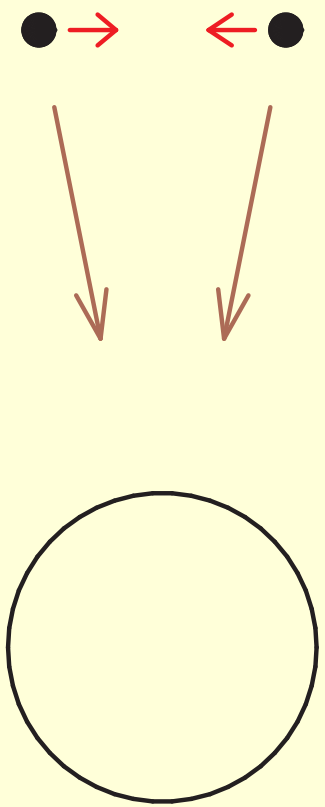
$$\boxed{\vec{\mathbf{G}} + \Lambda\, d\vec{\mathbf{r}} = 8\pi \vec{\mathbf{T}}}$$

(curvature = matter)

Curvature



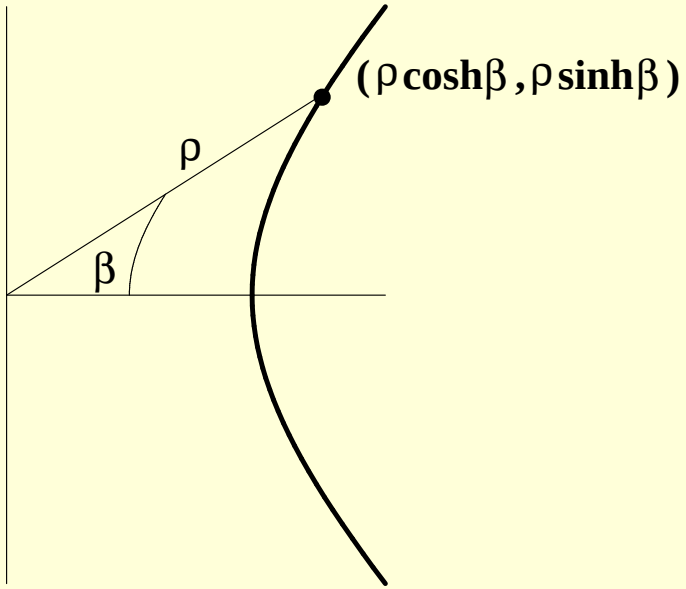
$$ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

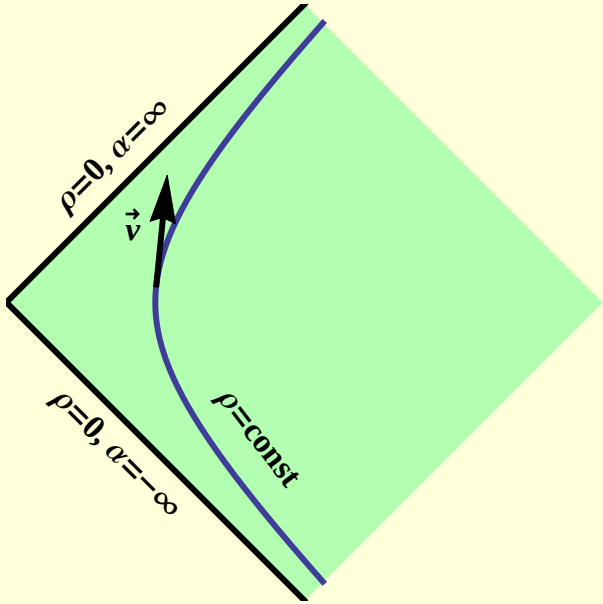


Tidal forces!

Acceleration

constant curvature = constant acceleration

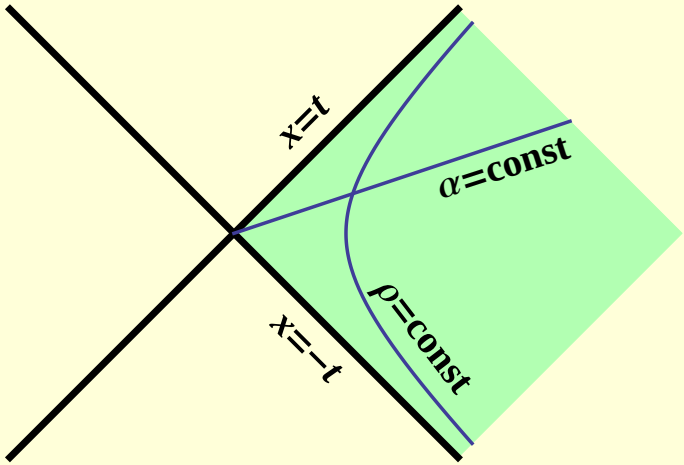


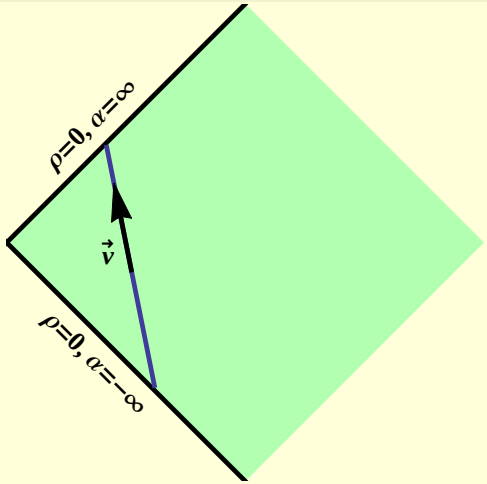


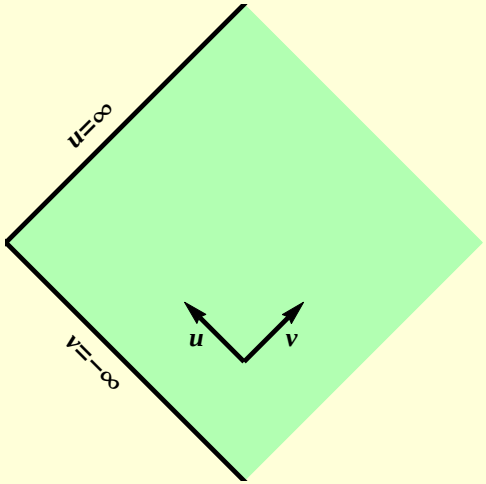
Can outrun lightbeam!

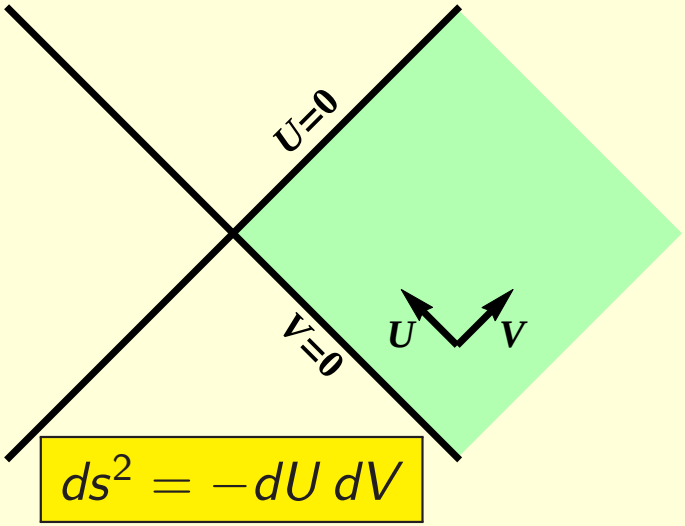
$$x = \rho \cosh \alpha$$
$$t = \rho \sinh \alpha$$
$$\Longrightarrow$$
$$ds^2 = d\rho^2 - \rho^2 \, d\alpha^2$$

From Rindler to Minkowski







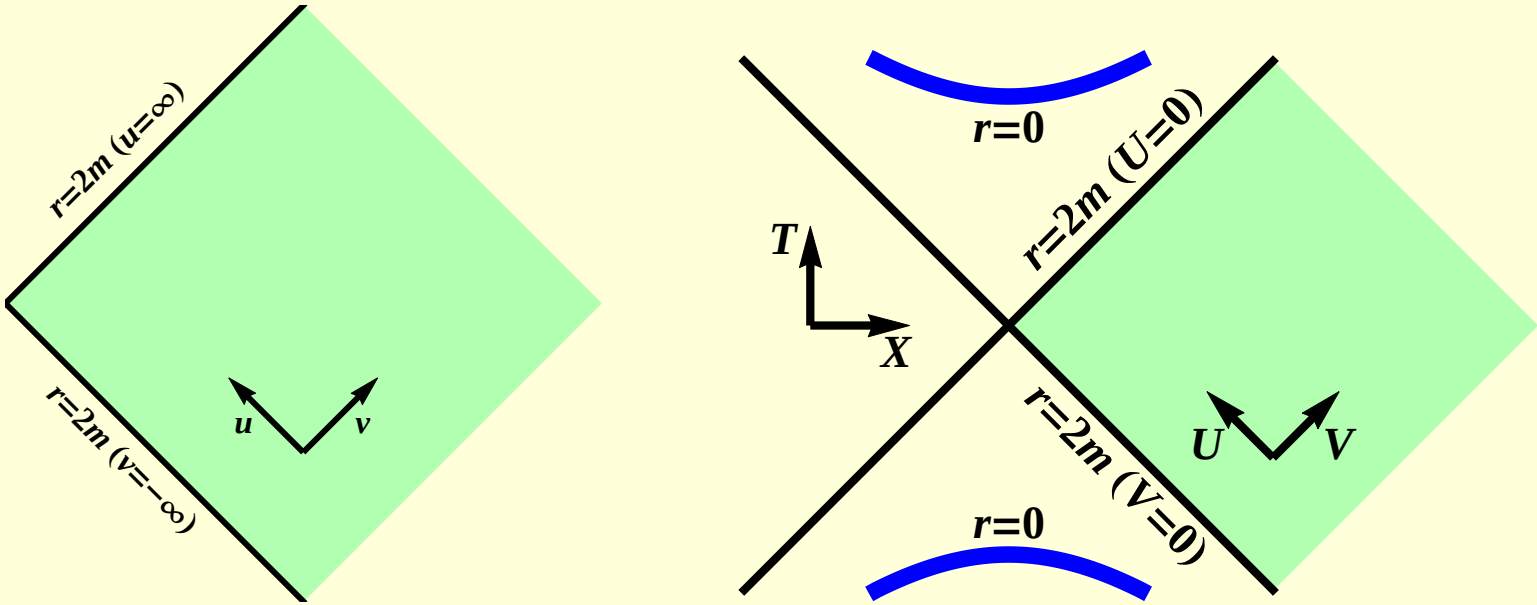


$ds^2 = -dU\,dV$

$u = \alpha - \ln\rho,\ v = \alpha + \ln\rho$ $\qquad$ $U = -e^{-u} = -\rho\, e^{-\alpha},\ V = e^{v} = \rho\, e^{\alpha}$

From Schwarzschild to Kruskal

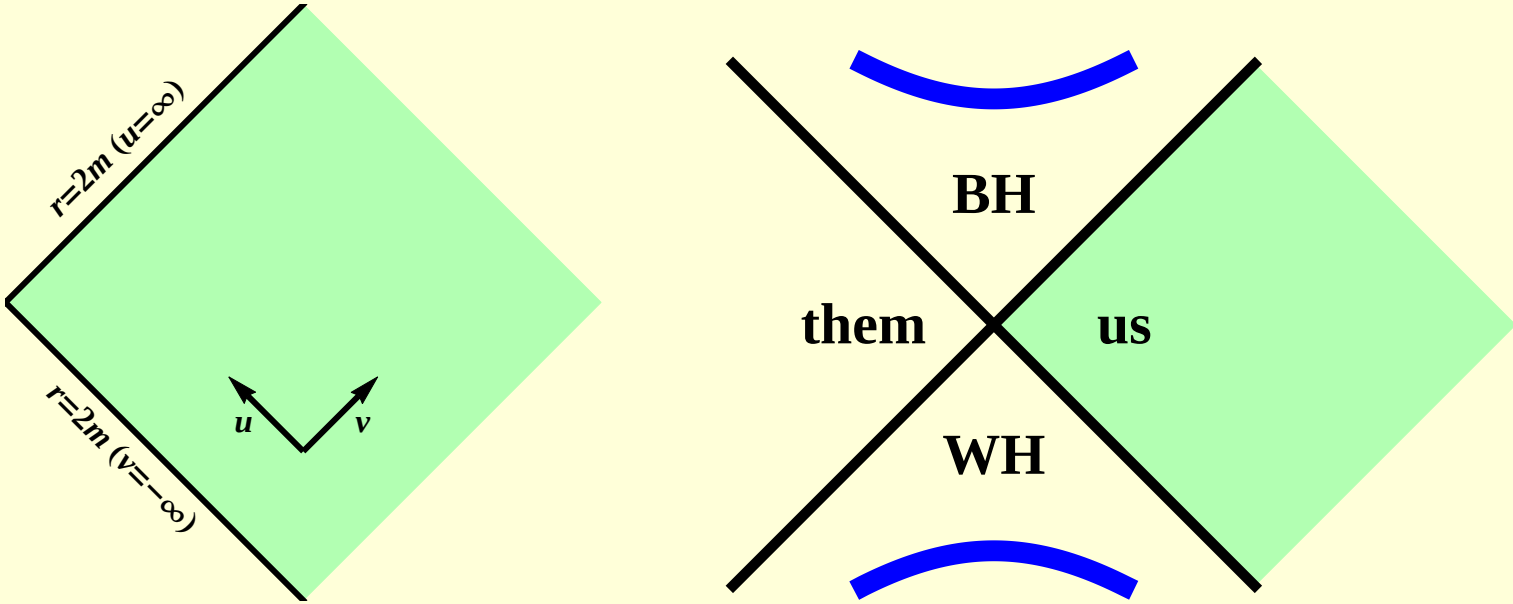
$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$



$$ds^2 = -\frac{32m^3}{r} e^{-r/2m}\, dU\, dV$$

From Schwarzschild to Kruskal

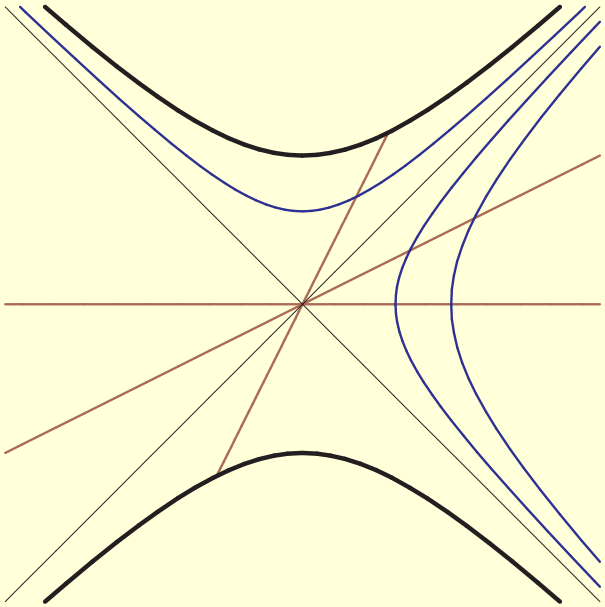
$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$



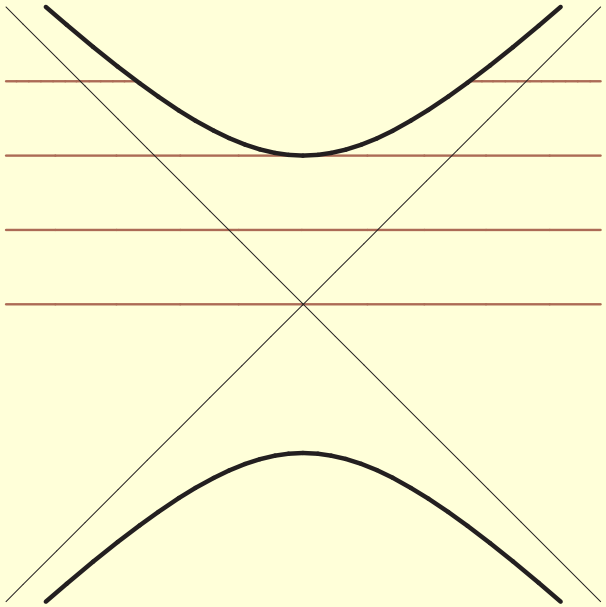
$$ds^2 = -\frac{32m^3}{r} e^{-r/2m}\, dU\, dV$$

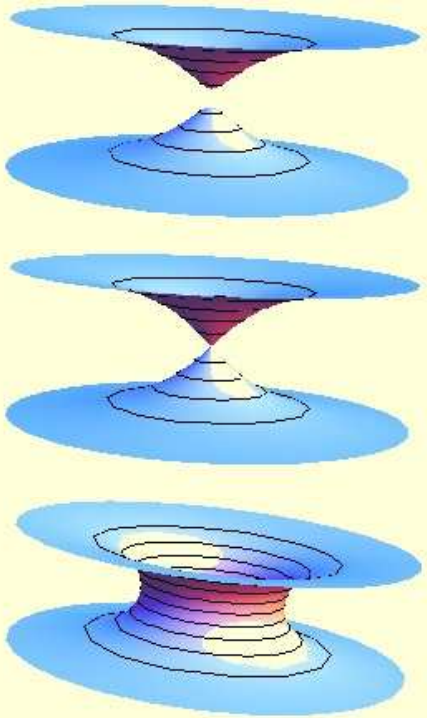
Wormholes

Constant radius = constant acceleration!

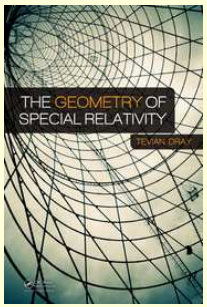


Wormholes

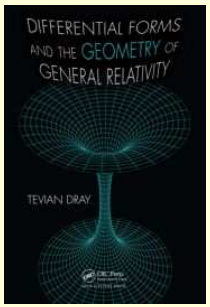




SUMMARY



http://relativity.geometryof.org/GSR
http://relativity.geometryof.org/GDF
http://relativity.geometryof.org/GGR



- Special relativity is hyperbolic trigonometry!
- Spacetimes are described by line elements!
- Curvature = gravity!
- Geometry = physics!

THE END