Calculus Meets Electromagnetism and Thermodynamics: *A Tale of Two Disciplines*

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Introduction

Is there a difference between $\dfrac{x^2-4}{x-2}$ and $x+2$?

Mathematics and Physics are two disciplines separated by a common language!

Physicists are bilingual (but don't know it)

What are Functions?

Suppose you know that:

$$T(x, y) = k(x^2 + y^2)$$

where k is a constant. What is $T(r, \theta)$?

Write your answer on your small whiteboard.
Share your answer with your neighbor(s).

A: $T(r, \theta) = kr^2$
B: $T(r, \theta) = k(r^2 + \theta^2)$

Are mathematicians bilingual?

My Background

- Math major (only). (No physics lab...)
- Ph.D. in mathematics. (Relativity!)
- Postdocs in both math and physics.
- My wife is a physicist. (She was a double major.)
- Our daughter is a math educator. (Also a double major.)
- Member of AMS/MAA/RUME and APS/AAPT/PERTG.

My department thinks I'm a physicist.
(The physics department knows better.)

The Paradigms in Physics Project

- Complete redesign of physics major – 20 new courses
- Junior-year “paradigms” designed around common themes.
  (*Central Forces*: CM of solar system + QM of hydrogen atom.)
- 2×3-credit in parallel $\longmapsto 3 \times 2$-credit in series.
- Senior-year “capstones” finish traditional disciplinary content.
  (Electromagnetism, Quantum Mechanics, etc.)
- 24 years of continuous NSF funding.
- Living curriculum: monthly curriculum meetings for 24 years!
- Paradigms 2.0 implemented in 2017:
  3×3-week $\longmapsto 2 \times (4 + 1)$-week courses (“Math Bits”)

Tell me something you know about derivatives.

Write your answer on your small whiteboard.

Share your answer with your neighbor(s).

Theoretical background

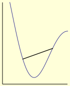
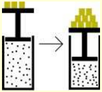
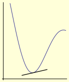
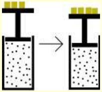
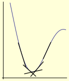
- **Vinner (1983):** A *concept image* is the set of properties associated with a concept together with the mental pictures of the concept.
- **Sfard (1991):** The *process-object* framework describes mathematics as proceeding through processes acting on objects, with those processes then becoming reified into objects.
- **Zandieh (2000):** Student understanding of the concept of derivative can be described by associating *process-object layers* with *representations* or *contexts*.

Zandieh (2000)

Process-object layer	Graphical	Verbal	Physical	Symbolic	Other
	Slope	Rate	Velocity	Difference Quotient	
Ratio					
Limit					
Function					

Michelle Zandieh, *A theoretical framework for analyzing student understanding of the concept of derivative*, CBMS Issues in Mathematics Education **8**, 103–122, 2000.

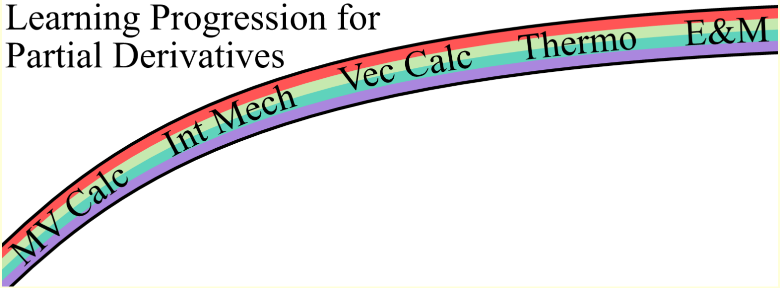
Extended Theoretical Framework for Concept of Derivative

Process-object layer	Graphical	Verbal	Symbolic	Numerical	Physical
	Slope	Rate of Change	Difference Quotient	Ratio of Changes	Measurement
Ratio		"avg. rate of change"	$\frac{f(x+\Delta x)-f(x)}{\Delta x}$	$\frac{y_2-y_1}{x_2-x_1}$ numerically	
Limit		"inst. rate of change"	$\lim_{\Delta x \to 0} \cdots$	...with Δx small	
Function		"...at any point/time"	$f'(x) =$ ...	... depends on x	tedious repeti-tion

No entry for symbolic differentiation!!

Roundy, Dray, Manogue, Wagner, & Weber, CRUME 18 Proceedings, MAA, 2015. `http://sigmaa.maa.org/rume/Site/Proceedings.html`

Learning Progression



- Successively more sophisticated ways of thinking about a topic.
- Sequences supported by research on learner's ideas and skills.
- *Lower anchor* grounded in students' prior ideas and skills.
- *Upper anchor* grounded in knowledge and practices of experts.

Duschle et al., NRC, 2007; Plummer, 2012; Sikorski et al., 2009, 2010
Manogue, Dray, Emigh, Gire, & Roundy, PERC 2017

Differentials

Does $\frac{df}{dx}$ mean "$f'(x)$" or "df over dx"?

$$d(u^2) = 2u\,du$$
$$d(\sin u) = \cos u\,du$$

Instead of:

- chain rule
- related rates
- implicit differentiation
- derivatives of inverse functions
- difficulties of interpretation (units!)

One coherent idea:

"Zap equations with *d*"

(infinitesimal reasoning)

Dray & Manogue, CMJ **34**, 283–290 (2003); CMJ **41**, 90–100 (2010).

Vector calculus is about one coherent concept:

Infinitesimal Displacement (à la Griffiths!)

$d\vec{\mathbf{r}} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}}$

$d\vec{\mathbf{r}} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\phi}$

$$\begin{aligned} ds &= |d\vec{\mathbf{r}}| \\ d\vec{\mathbf{A}} &= d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2 \\ dA &= |d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2| \\ dV &= (d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2) \cdot d\vec{\mathbf{r}}_3 \end{aligned}$$

Gradient

Tell me something you know about the gradient.

Write your answer on your small whiteboard.

Share your answer with your neighbor(s).

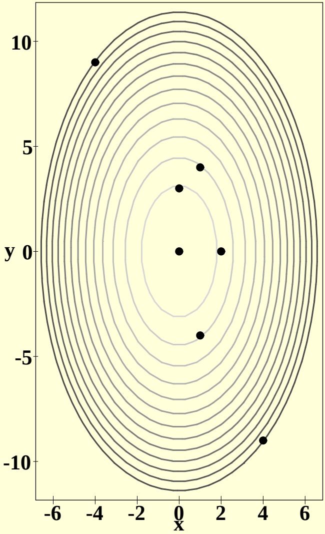
- $\vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{y}} + \ldots$
- The gradient points in the steepest direction.
- The magnitude of the gradient tells you how steep.
- The gradient is perpendicular to the level curves.

The Hill

Suppose you are standing on a hill. You have a topographic map, which uses rectangular coordinates (x, y) measured in miles. Your global positioning system says your present location is at one of the points shown. Your guidebook tells you that the height h of the hill in feet above sea level is given by

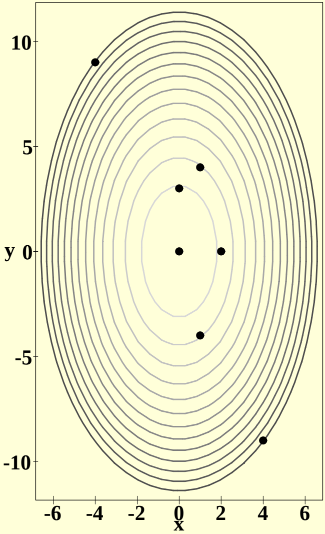
$$h = a - bx^2 - cy^2$$

where $a = 5000$ ft, $b = 30\,\frac{\text{ft}}{\text{mi}^2}$, and $c = 10\,\frac{\text{ft}}{\text{mi}^2}$.



Stand up and close your eyes.

Hold out your right arm in the direction of the gradient where you are standing.



Partial Derivatives

State Variables:

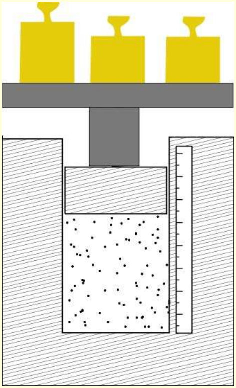
$$\begin{aligned} T &= \text{temperature} \\ S &= \text{entropy} \\ p &= \text{pressure} \\ V &= \text{volume} \end{aligned}$$

First Law:
(U = internal energy)

$$\boxed{dU = T\,dS - p\,dV}$$

- Compressibility $= -\frac{1}{V}\frac{\partial V}{\partial p}$
- Design an experiment to measure compressibility.
- What are the independent variables??

Name the Experiment



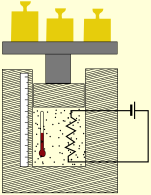
$-\frac{1}{V}\frac{\partial V}{\partial p}$

$\longleftarrow$ What is this material??

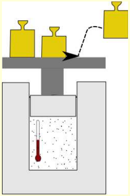
Isothermal: $-\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$

Isoentropic: $-\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_S$

Name the Experiment



$$\left(\frac{\partial p}{\partial T}\right)_V$$



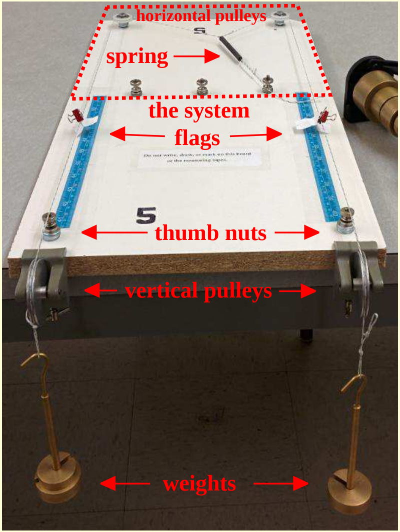
$$\left(\frac{\partial p}{\partial T}\right)_S$$

David Roundy, Mary Bridget Kustusch, and Corinne Manogue, *Name the experiment! Interpreting thermodynamic derivatives as thought experiments*, Am. J. Phys. **82**, 39–46, 2014.

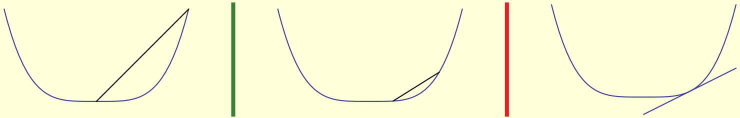
Partial Derivative Machine

- Developed for junior-level thermodynamics course
- Two positions, x_i, two string tensions (masses), F_i.
- "Find $\frac{\partial x}{\partial F}$."
- Idea: Measure Δx, ΔF; divide.
- Mathematicians:
  "That's not a derivative!"

Roundy et al., *Experts' Understanding of Partial Derivatives Using the Partial Derivative Machine*, PERC 2014



Thick Derivatives



Math: $\exists$ "bright line" between *average* rate of change and *instantaneous* rate of change.

(Such averages are used to approximate derivatives.)

Physics: "Average" refers to secant lines, not (good) approximations to tangent lines.

Move the bright line!

Thick Derivatives!

(Derivatives are fundamentally ratios of small changes, not limits.)

[Dray, AMS Blog on Education, 5/31/16]

Surfaces

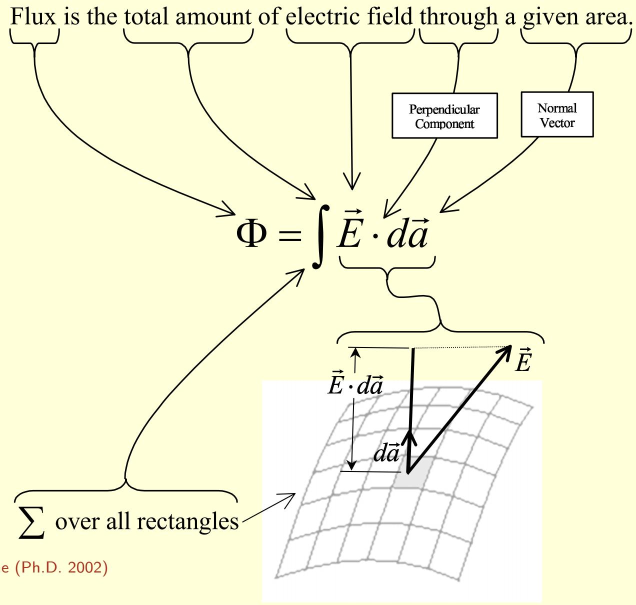
(Each surface is dry-erasable, as are the matching contour maps.)

Raising Calculus to the Surface (Aaron Wangberg)

Raising Physics to the Surface (+ Liz Gire, Robyn Wangberg)

http://raisingcalculus.winona.edu

Multiple Representations

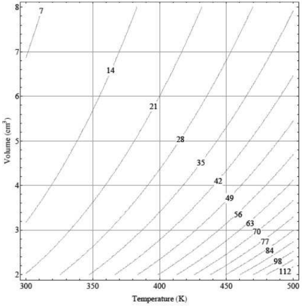


Kerry Browne (Ph.D. 2002)

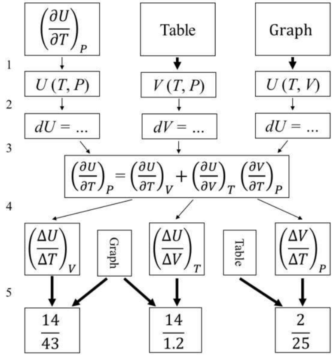
Evaluate $\left(\frac{\partial U}{\partial T}\right)_P$ at $P = 10$ atm., $T = 410$K using the information below.

P(atm.)	T(K)	V(cm^3)
10	300	1.32
10	310	1.44
10	320	1.57
10	330	1.71
10	340	1.85
10	350	2.00
10	360	2.15
10	370	2.32
10	380	2.49
10	390	2.67
10	400	2.86
10	410	3.05
10	420	3.25
10	430	3.47
10	440	3.69
10	450	3.91
10	460	4.15
10	470	4.40

Pressure P, Temperature T, and Volume



Internal Energy U (T, V).



Rabindra R. Bajracharya, Paul J. Emigh, and Corinne A. Manogue, *Students' stategies for solving a multi-representational partial derivative problem in thermodyanmics*, in preparation.

Vector Calculus Bridge Project:

http://math.oregonstate.edu/bridge

- Differentials *(Use what you know!)*
- Multiple representations
- Symmetry *(adapted bases, coordinates)*
- Geometry *(vectors, div, grad, curl)*
- Online text (http://math.oregonstate.edu/BridgeBook)

Paradigms in Physics Project:

http://physics.oregonstate.edu/portfolioswiki

- Redesign of undergraduate physics major *(18 new courses!)*
- Active engagement *(300+ documented activities!)*

Things to consider:

- Lecture is fast; use it when it works.
- What is the focus of attention? (You, the slides, their notes...)
- How busy are the slides?
- Do the figures have distracting elements?

Classroom implementation:

- Have a way to show students where you are on the slide.

Things to consider:

- Asks students to make a commitment.
- Asks students to defend an answer.
- Good questions: conceptual, focus on common mistakes.

Classroom implementation:

- Many "response" systems:
  clickers, ABCD cards, whiteboards, fingers.
- Two stages.
- Simultaneous and anonymous.
- *Convince your neighbor.*

Small Whiteboard Questions (SWBQs)

Things to consider:

- Allows open-ended responses.
- Can be used in sequence with increasing sophistication.

Classroom implementation:

- Gather responses and discuss.
- Use wrap-up as an opportunity for reflection.
- SWBQs can be spontaneous!

Small Group Activity

Things to consider:

- Can emphasize more complex problems/reasoning.
- Students practice problem solving themselves.
- Equity: moves office hours into the classroom.

Classroom implementation:

- *You have 10 minutes; GO!*
- Who needs help?
- Do you need more time?
- Pause.

Kinesthetic Activity

Things to consider:

- Everyone is awake!
- Teacher can see what everyone is thinking.
- Highlights geometric reasoning.
- Students get geometric cues from others.
- Students must make a decision.
- Student can be asked to translate representations.

Classroom implementation:

- *Please stand up.*
- *Show me...*
- *Thank you, you can sit down.*

SUMMARY

- Physics $\neq$ Mathematics ("Spherical coordinates")
- Syllabus $\neq$ Content ("Divergence Theorem")
- Hidden curriculum matters ("Think like a ______")
- Curriculum is dynamic! (Keep talking!)

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