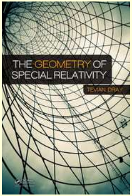
The Geometry of Relativity

Tevian Dray

Department of Mathematics
Oregon State University
http://www.math.oregonstate.edu/~tevian

Oregon State
UNIVERSITY

Books

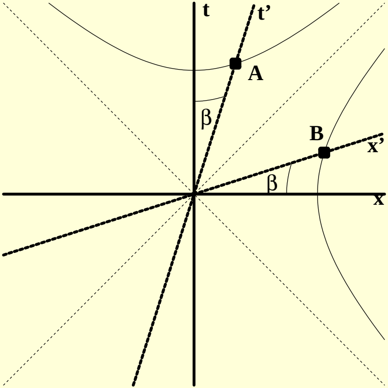


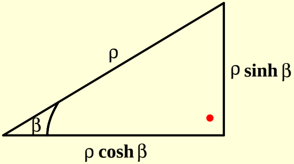
The Geometry of Special Relativity
Tevian Dray
A K Peters/CRC Press 2012
ISBN: 978-1-4665-1047-0
http://physics.oregonstate.edu/coursewikis/GSR

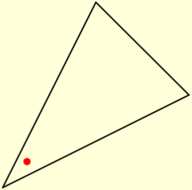
Differential Forms and the Geometry of General Relativity
Tevian Dray
A K Peters/CRC Press 2014
http://physics.oregonstate.edu/coursewikis/GDF
http://physics.oregonstate.edu/coursewikis/GGR

The Geometry of Vector Calculus
Tevian Dray & Corinne A. Manogue
online only: http://physics.oregonstate.edu/coursewikis/GVC

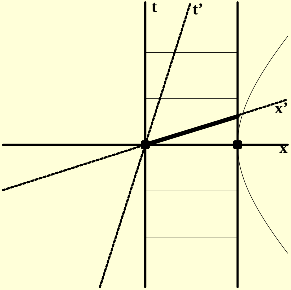
Trigonometry

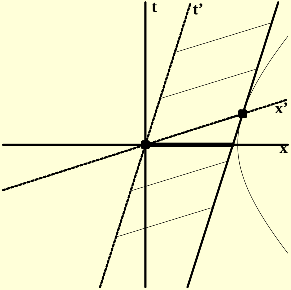




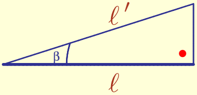


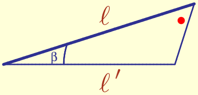
Length Contraction



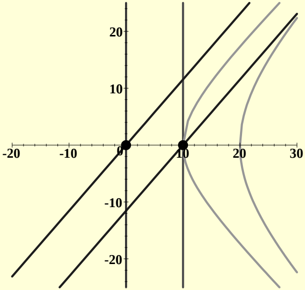


$$\ell' = \frac{\ell}{\cosh \beta}$$

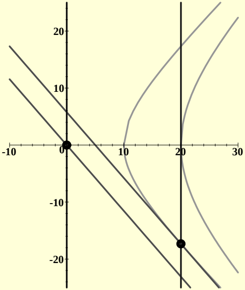




A 20 foot pole is moving towards a 10 foot barn fast enough that the pole appears to be only 10 feet long. As soon as both ends of the pole are in the barn, slam the doors. How can a 20 foot pole fit into a 10 foot barn?



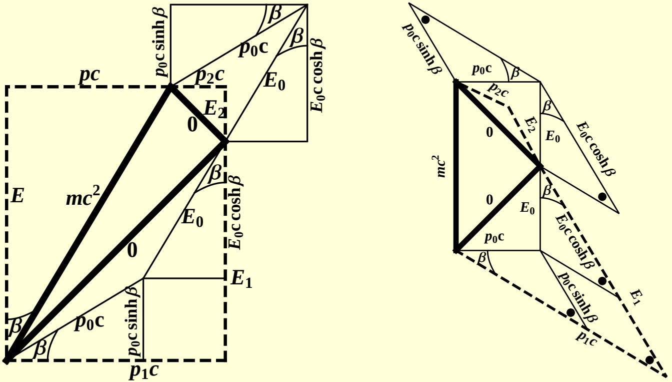
barn frame



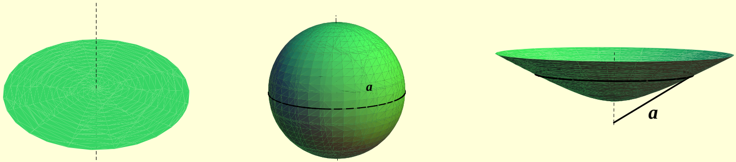
pole frame

Relativistic Mechanics

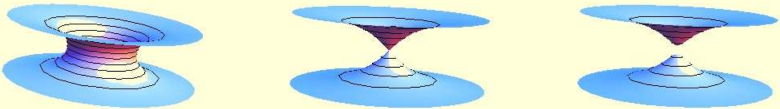
A pion of (rest) mass m and (relativistic) momentum $p = \frac{3}{4}mc$ *decays into 2 (massless) photons. One photon travels in the same direction as the original pion, and the other travels in the opposite direction. Find the energy of each photon.* $[E_1 = mc^2,\ E_2 = \frac{1}{4}mc^2]$



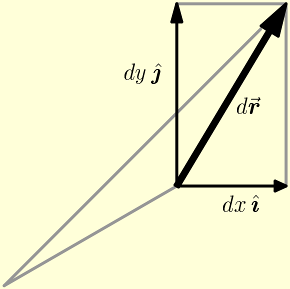
Line Elements

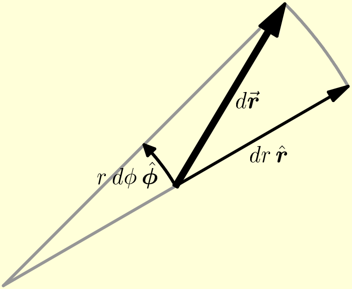


$$dr^2 + r^2\, d\phi^2 \qquad d\theta^2 + \sin^2\theta\, d\phi^2 \qquad d\beta^2 + \sinh^2\beta\, d\phi^2$$



$$ds^2 = d\vec{\mathbf{r}} \cdot d\vec{\mathbf{r}}$$





$$d\vec{\mathbf{r}} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$$

Differential Forms in a Nutshell ($\mathbb{R}^3$)

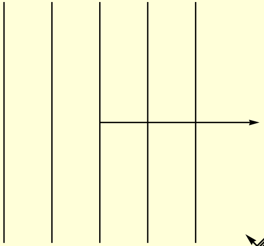
Differential forms are integrands: $(*^2 = 1)$

$$
\begin{aligned}
f &= f && \text{(0-form)} \\
F &= \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} && \text{(1-form)} \\
*F &= \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}} && \text{(2-form)} \\
*f &= f\, dV && \text{(3-form)}
\end{aligned}
$$

Exterior derivative: $(d^2 = 0)$

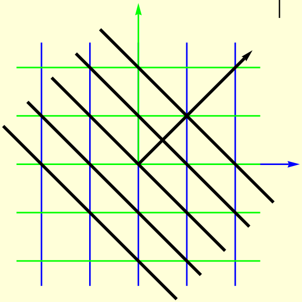
$$
\begin{aligned}
df &= \vec{\nabla} f \cdot d\vec{\mathbf{r}} \\
dF &= \vec{\nabla} \times \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}} \\
d{*F} &= \vec{\nabla} \cdot \vec{\mathbf{F}}\, dV \\
d{*f} &= 0
\end{aligned}
$$

The Geometry of Differential Forms

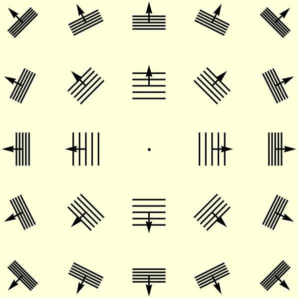


dx

$dx + dy$



$r\,dr = x\,dx + y\,dy$



Geodesic Equation

$$d\vec{\mathbf{r}} = \sigma^i \, \hat{\mathbf{e}}_i$$

Connection:

$$\omega_{ij} = \hat{\mathbf{e}}_i \cdot d\hat{\mathbf{e}}_j$$

$$d\sigma^i + \omega^i{}_j \wedge \sigma^j = 0$$

$$\omega_{ij} + \omega_{ji} = 0$$

Geodesics:

$$\vec{\mathbf{v}} \, d\lambda = d\vec{\mathbf{r}}$$

$$\dot{\vec{\mathbf{v}}} = 0$$

Symmetry:

$$d\vec{\mathbf{X}} \cdot d\vec{\mathbf{r}} = 0$$

$$\Longrightarrow \vec{\mathbf{X}} \cdot \vec{\mathbf{v}} = \text{const}$$

Einstein's Equation

Curvature:

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

Einstein tensor:

$$\gamma^i = -\frac{1}{2}\Omega_{jk} \wedge *(\sigma^i \wedge \sigma^j \wedge \sigma^k)$$

$$G^i = *\gamma^i = G^i{}_j\,\sigma^j$$

$$\vec{\mathbf{G}} = G^i\hat{\mathbf{e}}_i = G^i{}_j\,\sigma^j\hat{\mathbf{e}}_i$$

$$\Longrightarrow d*\vec{\mathbf{G}} = 0$$

Field equation:

$$\boxed{\vec{\mathbf{G}} + \Lambda\, d\vec{\mathbf{r}} = 8\pi\vec{\mathbf{T}}}$$

(vector valued 1-forms, not tensors)

SUMMARY

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http://physics.oregonstate.edu/coursewikis/GGR
http://physics.oregonstate.edu/coursewikis/GVC

- Special relativity is hyperbolic trigonometry!
- Spacetimes are described by metrics!
- General relativity can be described without tensors!
- BUT: Need vector-valued differential forms...

THE END