The Geometry of Relativity

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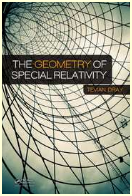
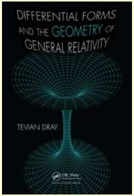
Books



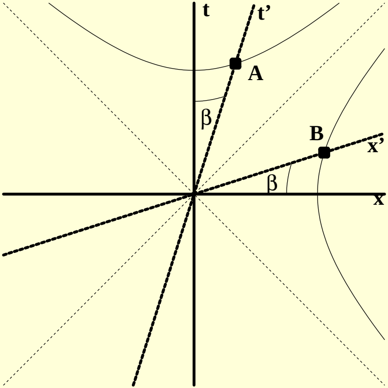
The Geometry of Special Relativity
Tevian Dray
A K Peters/CRC Press 2012
ISBN: 978-1-4665-1047-0
`http://relativity.geometryof.org/GSR`

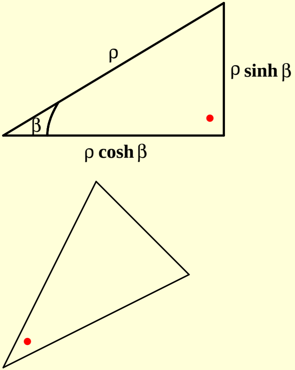


Differential Forms and the Geometry of General Relativity
Tevian Dray
A K Peters/CRC Press 2014
ISBN: 978-1-4665-1000-5
`http://relativity.geometryof.org/GDF`
`http://relativity.geometryof.org/GGR`

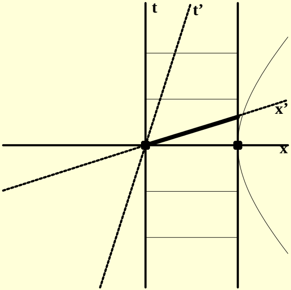
The Geometry of Relativity, AJP **85**, 683–691 (2017).

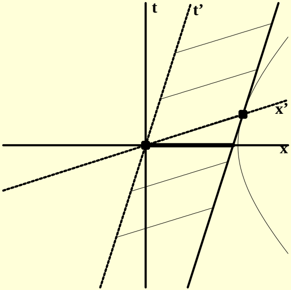
Trigonometry



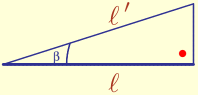


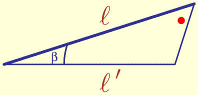
Length Contraction





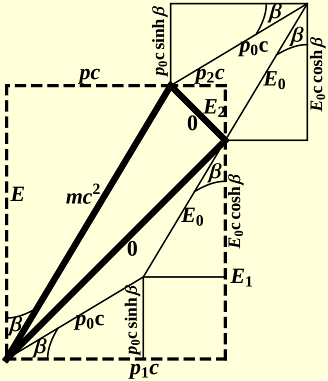
$$\ell' = \frac{\ell}{\cosh\beta}$$

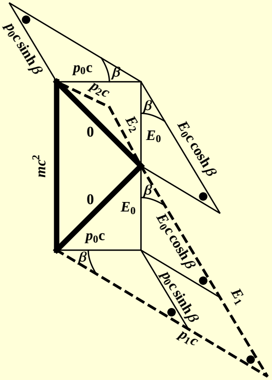




Relativistic Mechanics

A pion of (rest) mass m and (relativistic) momentum $p = \frac{3}{4}mc$ *decays into 2 (massless) photons. One photon travels in the same direction as the original pion, and the other travels in the opposite direction. Find the energy of each photon.* $[E_1 = mc^2,\ E_2 = \frac{1}{4}mc^2]$



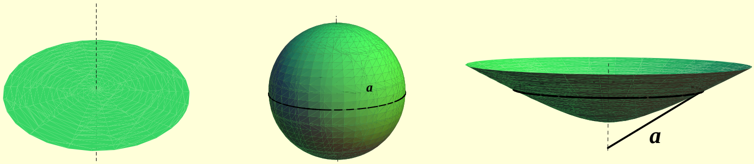


Design Principles:

- Accessible to undergraduate math (and physics) majors.
- No tensors! (Almost.)
- Use orthonormal bases!
  ($\vec{\mathbf{B}} = B_\phi \hat{\boldsymbol{\phi}} = (rB_\phi)\vec{\nabla}\phi$; $d\phi$ vs. $r\,d\phi$)

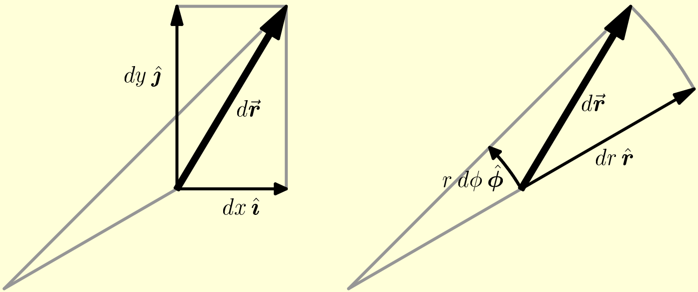
$\longmapsto$ **Build on the language of vector calculus.**

Line Elements



$dr^2 + r^2\,d\phi^2$ $\qquad$ $d\theta^2 + \sin^2\theta\,d\phi^2$ $\qquad$ $d\beta^2 + \sinh^2\beta\,d\phi^2$

$$ds^2 = d\vec{\mathbf{r}} \cdot d\vec{\mathbf{r}}$$



$$d\vec{\mathbf{r}} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$$

Differential Forms in a Nutshell ($\mathbb{R}^3$)

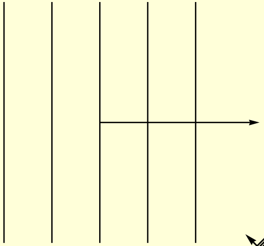
Differential forms are integrands: $(*^2 = 1)$

$$f = f \qquad \text{(0-form)}$$
$$F = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \qquad \text{(1-form)}$$
$$*F = \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}} \qquad \text{(2-form)}$$
$$*f = f\,dV \qquad \text{(3-form)}$$

Exterior derivative: $(d^2 = 0)$

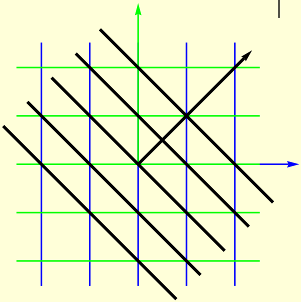
$$df = \vec{\nabla} f \cdot d\vec{\mathbf{r}}$$
$$dF = \vec{\nabla} \times \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}}$$
$$d{*}F = \vec{\nabla} \cdot \vec{\mathbf{F}}\,dV$$
$$d{*}f = 0$$

The Geometry of Differential Forms

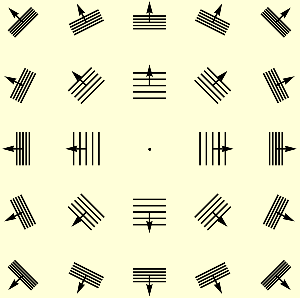


dx

$dx + dy$



$r\,dr = x\,dx + y\,dy$



Geodesic Equation

Orthonormal basis: $$d\vec{\mathbf{r}} = \sigma^i\,\hat{\mathbf{e}}_i \qquad (\mathbf{g^i{}_j = \delta^i{}_j}!)$$

Connection:

$$\omega_{ij} = \hat{\mathbf{e}}_i \cdot d\hat{\mathbf{e}}_j$$

$$d\sigma^i + \omega^i{}_j \wedge \sigma^j = 0 \qquad (d^2\vec{\mathbf{r}} = 0)$$

$$\omega_{ij} + \omega_{ji} = 0 \qquad (d\cdot = 0)$$

Geodesics:

$$\vec{\mathbf{v}}\,d\lambda = d\vec{\mathbf{r}}$$

$$\dot{\vec{\mathbf{v}}} = 0 \qquad (d\vec{\mathbf{v}} = 0)$$

Symmetry:

$$d\vec{\mathbf{X}} \cdot d\vec{\mathbf{r}} = 0$$

$$\Longrightarrow \vec{\mathbf{X}} \cdot \vec{\mathbf{v}} = \text{const}$$

Einstein's Equation

Curvature:

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

Einstein tensor:

$$\gamma^i = -\frac{1}{2}\,\Omega_{jk} \wedge *(\sigma^i \wedge \sigma^j \wedge \sigma^k)$$

$$G^i = *\gamma^i = G^i{}_j\,\sigma^j$$

$$\vec{\mathbf{G}} = G^i \hat{\mathbf{e}}_i = G^i{}_j\,\sigma^j \hat{\mathbf{e}}_i$$

$$\Longrightarrow d{*}\vec{\mathbf{G}} = 0$$

Field equation:

$$\boxed{\vec{\mathbf{G}} + \Lambda\, d\vec{\mathbf{r}} = 8\pi \vec{\mathbf{T}}}$$

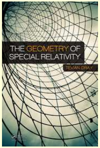
(vector valued 1-forms, not tensors)

Does it work?

- I am a mathematician...
- There is no GR course in physics department.
  (I developed the SR course.)
- Core audience is undergraduate math and physics majors.
  (Many double majors.)
- Hartle's book:
  Perfect for physics students, but tough for math majors.
- My course: 10 weeks differential forms, then 10 weeks GR.
  (Some physics students take only GR, after "crash course".)

In this context: **YES!**

SUMMARY



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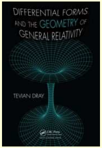

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- Special relativity is hyperbolic trigonometry!
- General relativity can be described without tensors!
- BUT: Need vector-valued differential forms...

THE END