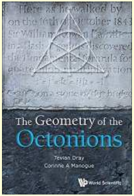
# The Octonions

Tevian Dray

Department of Mathematics
Oregon State University
http://www.math.oregonstate.edu/~tevian




**The Geometry of the Octonions**

*Tevian Dray & Corinne A. Manogue*

World Scientific 2015

ISBN: 978-981-4401-81-4

`http://physics.oregonstate.edu/coursewikis/GO`

(`http://octonions.geometryof.org/GO`)

**MTH 679: Topics in Geometry**

Spring 2016: *The Geometry of the Exceptional Lie Groups*

*Tevian Dray*

WF 4–5:20 PM

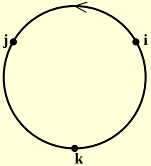
## Real Numbers

$$\mathbb{R}$$

## Quaternions

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$$

$$q = (x + yi) + (r + si)j$$



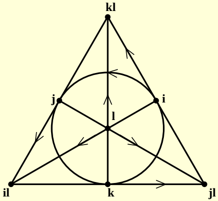

## Complex Numbers

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$$

$$z = x + yi$$

## Octonions

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$$




$$\boxed{i^2 = j^2 = \ell^2 = -1}$$

**Cayley–Dickson (1919)**

**Noncommutative:**

$$ji = -ij$$

**Nonassociative:**

$$(ij)\ell = -i(j\ell)$$

**Norm:**

$$|x|^2 = x\overline{x}$$

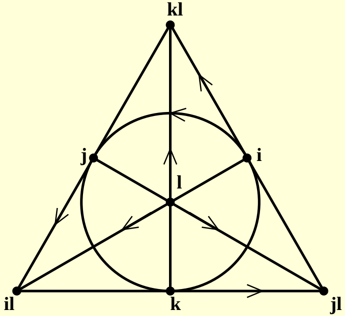
$$|x| = 0 \implies x = 0$$

**Composition:**

$$|xy| = |x||y|$$

**Inverses (Division!):**

$$x \neq 0 \implies x^{-1} = \overline{x}/|x|^2$$



**Reality: $(\mathbf{A}^\dagger = \mathbf{A} \implies \overline{\lambda} = \lambda)$**

$$Av = \lambda v \implies \overline{\lambda} v^\dagger v = (Av)^\dagger v = v^\dagger A v = v^\dagger \lambda v \neq \lambda v^\dagger v$$

$$Av = v\lambda \implies \overline{\lambda}(v^\dagger v) \neq (Av)^\dagger v \neq v^\dagger (Av) \neq (v^\dagger v)\lambda$$

**Orthogonality: $(\lambda_1 \neq \lambda_2 \implies \mathbf{v}_1^\dagger \mathbf{v}_2 = 0)$**

$$Av_m = \lambda_m v_m \implies \lambda_1 v_1^\dagger v_2 = (Av_1)^\dagger v_2 \neq v_1^\dagger (Av_2) = \lambda_2 v_1^\dagger v_2$$

**Theorem (Dray–Manogue 1998)**

*$v_m \in \mathbb{O}^3, A^\dagger = A, Av_m = \lambda_m v_m, \lambda_m \in \mathbb{R} \implies (v_1 v_1^\dagger) v_2 = 0$*
*(and $\exists$ **6** real eigenvalues...)*

**Example**

$$A = \begin{pmatrix} 0 & -\ell \\ \ell & 0 \end{pmatrix}, \quad v = \begin{pmatrix} j \\ k\ell \end{pmatrix} \implies Av = vi$$

# Simultaneous Eigenstates

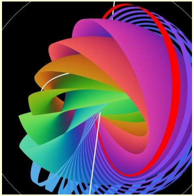
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -\ell \\ \ell & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$L_m\psi := -\frac{\hbar}{2}(\ell\sigma_m\psi)\ell$$

$$\psi = \begin{pmatrix} 1 \\ k \end{pmatrix} \Longrightarrow \quad \begin{aligned} 2\,\mathbf{L_z}\psi &= \hbar\,\psi \\ 2\,\mathbf{L_x}\psi &= -\hbar\,\psi\,\mathbf{k} \\ 2\,\mathbf{L_y}\psi &= -\hbar\,\psi\,\mathbf{k}\ell \end{aligned}$$

"spin-up" is simultaneous eigenstate of $L_x$, $L_y$, $L_z$!
(but **eigenvalues** don't commute!)

$$v = \begin{pmatrix} b \\ c \end{pmatrix} \in \mathbb{C}^2 \cong \mathbb{R}^4 \qquad \mathbf{X} = vv^\dagger = \begin{pmatrix} |b|^2 & b\bar{c} \\ c\bar{b} & |c|^2 \end{pmatrix}$$



$$\det \mathbf{X} = |b|^2|c|^2 - |b\bar{c}|^2 = 0$$

$$\operatorname{tr} \mathbf{X} = v^\dagger v = |b|^2 + |c|^2 = 2 \implies v \in \mathbb{S}^3 \subset \mathbb{R}^4$$

$$\therefore \mathbf{X} = \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

$$0 = \det \mathbf{X} = 1 - x^2 - y^2 - z^2 \implies \mathbf{X} \in \mathbb{S}^2 \subset \mathbb{R}^3$$

$\therefore$ **Hopf (1931):**

$$\mathbb{S}^3 \longrightarrow \mathbb{S}^2$$
$$v \longmapsto vv^\dagger$$

$$\mathbb{CP}^1 = \{v \in \mathbb{C}^2 : v \sim \lambda v, \lambda \in \mathbb{C}\}$$

(quantum mechanics, twistor theory, ...)

## Four division algebras $\Longrightarrow$ four Hopf maps:

$$v \longmapsto vv^\dagger$$

$$v \in \mathbb{R}^2 \Longrightarrow \mathbb{S}^1 \longrightarrow \mathbb{S}^1 \qquad (\mathbb{RP}^1)$$

$$v \in \mathbb{C}^2 \Longrightarrow \mathbb{S}^3 \longrightarrow \mathbb{S}^2 \qquad (\mathbb{CP}^1)$$

$$v \in \mathbb{H}^2 \Longrightarrow \mathbb{S}^7 \longrightarrow \mathbb{S}^4 \qquad (\mathbb{HP}^1)$$

$$v \in \mathbb{O}^2 \Longrightarrow \mathbb{S}^{15} \longrightarrow \mathbb{S}^8 \qquad (\mathbb{OP}^1)$$

Fibers in each case correspond to units in $\mathbb{K}$:

$$|u| = 1 \longleftrightarrow \mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3, \mathbb{S}^7$$

("phases": $u = e^{s\theta}$, $s^2 = -1$)

**Spacetime:**

$$\mathbf{X} = \begin{pmatrix} t+z & \bar{a} \\ a & t-z \end{pmatrix}$$

$$\det \mathbf{X} = t^2 - |a|^2 - z^2$$

**Four division algebras $\Longrightarrow$ four Lorentz groups:**

$$\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SO}(2,1)$$
$$\mathrm{SL}(2,\mathbb{C}) \cong \mathrm{SO}(3,1)$$
$$\mathrm{SL}(2,\mathbb{H}) \cong \mathrm{SO}(5,1)$$
$$\mathrm{SL}(2,\mathbb{O}) \cong \mathrm{SO}(9,1) \qquad \text{(all double covers)}$$

**Spinors:** $\mathbf{X} = vv^\dagger \Longrightarrow \det \mathbf{X} = 0 \Longrightarrow$ null vector

Clifford algebras, supersymmetry, ...

**Theorem**

*Classical supersymmetry exists only in dimensions* $3, 4, 6, 10$

## Definition

A *Lie Group* $G$ is a group that is also a smooth manifold, and on which the group operations are smooth:

$$G \times G \longrightarrow G$$
$$(X, Y) \longmapsto X^{-1}Y$$

## Example

$$\begin{aligned} \mathrm{SO}(2) &= \{A \in M^{2\times 2}(\mathbb{R}) \mid AA^t = I, \det A = 1\} \\ &= \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \, \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\} \cong \mathbb{S}^1 \end{aligned}$$

continuous symmetry groups (rotations)

$|G| = \#$ of parameters

## Definition

A *Lie algebra* is a vector space $g$ together with a binary operation

$$g \times g \longrightarrow g$$
$$(x, y) \longmapsto [x, y]$$

which is *bilinear* and satisfies

$$[x, y] = -[y, x]$$
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

## Example

$$\mathfrak{so}(3) = \{a \in M^{3\times 3}(\mathbb{R}) | a^t = -a, \mathrm{tr}(a) = 0\}$$
$$= \langle r_x, r_y, r_z \rangle$$

$$r_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{d}{d\theta}\bigg|_{\theta=0} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[r_x, r_y] = r_z$$

infinitesimal symmetries ($g = TG|_e$)
(WARNING: physicists use $-i\frac{d}{d\theta}$ to get Hermitian operators.)

$$|g| = \dim g = \dim TG = |G|$$

## Theorem (Killing 1888–1890, Cartan 1894)

*The simple Lie groups (no nontrivial normal subroups) are the classical groups*

| | |
|---|---|
| $A_n$ | $\mathrm{SU}(n+1)$ |
| $B_n$ | $\mathrm{SO}(2n+1)$ |
| $C_n$ | $\mathrm{Sp}(n)$ |
| $D_n$ | $\mathrm{SO}(2n)$ |

*together with the exceptional groups $G_2$, $F_4$, $E_6$, $E_7$, and $E_8$.*

$$\begin{aligned} \mathrm{SU}(n) &\cong \mathrm{SU}(n,\mathbb{C}) \\ \mathrm{SO}(n) &\cong \mathrm{SU}(n,\mathbb{R}) \\ \mathrm{Sp}(n) &\cong \mathrm{SU}(n,\mathbb{H}) \\ F_4 &\cong \mathrm{SU}(3,\mathbb{O}) \end{aligned}$$

$$\begin{aligned} G_2 &\cong \mathrm{Aut}(\mathbb{O}) \\ E_6 &\cong \mathrm{SL}(3,\mathbb{O}) \quad [\mathbb{OP}^2!] \\ E_7 &\cong \mathrm{Sp}(6,\mathbb{O}) \\ E_8 &\cong ?? \end{aligned}$$

# The Freudenthal-Tits Magic Square

**Freudenthal (1964), Tits (1966):**

| | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{H}$ | $\mathbb{O}$ |
|---|---|---|---|---|
| $\mathbb{R}$ | $\mathfrak{a}_1$ | $\mathfrak{a}_2$ | $\mathfrak{c}_3$ | $\mathfrak{f}_4$ |
| $\mathbb{C}$ | $\mathfrak{a}_2$ | $\mathfrak{a}_2 \oplus \mathfrak{a}_2$ | $\mathfrak{a}_5$ | $\mathfrak{e}_6$ |
| $\mathbb{H}$ | $\mathfrak{c}_3$ | $\mathfrak{a}_5$ | $\mathfrak{d}_6$ | $\mathfrak{e}_7$ |
| $\mathbb{O}$ | $\mathfrak{f}_4$ | $\mathfrak{e}_6$ | $\mathfrak{e}_7$ | $\mathfrak{e}_8$ |

**Vinberg (1966):**

$$\mathfrak{sa}(3, \mathbb{A} \otimes \mathbb{B}) \oplus \mathrm{der}(\mathbb{A}) \oplus \mathrm{der}(\mathbb{B})$$

$$\mathrm{der}(\mathbb{H}) = \mathfrak{so}(3)$$
$$\mathrm{der}(\mathbb{O}) = \mathfrak{g}_2$$

# Lie Group Description

| | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{H}$ | $\mathbb{O}$ |
|---|---|---|---|---|
| $\mathbb{R}$ | $A_1$ | $A_2$ | $C_3$ | $F_4$ |
| $\mathbb{C}$ | $A_2$ | $A_2 \oplus A_2$ | $A_5$ | $E_6$ |
| $\mathbb{H}$ | $C_3$ | $A_5$ | $D_6$ | $E_7$ |
| $\mathbb{O}$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |

**Want:**

$$\text{“}\mathrm{SU}(3, \mathbb{A} \otimes \mathbb{B})\text{”}$$

# SUMMARY

- There are four division algebras: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.
- Spacetime dimensions 3, 4, 6, 10 are special.
- Symmetry groups in those dimensions are special.
- Some of these groups are exceptional...
- **Have:** $E_6 \cong SL(3,\mathbb{O})$, $E_7 \cong Sp(6,\mathbb{O})$
  (Dray–Manogue 2010, Wangberg–Dray 2013, 2015, Dray–Manogue–Wilson 2014)
- **Should Have:** $E_8 \cong SU(3,\mathbb{O}' \otimes \mathbb{O})$.
  (Kincaid–Dray 2014, Dray–Huerta–Kincaid 2014, Wilson–Dray–Manogue in preparation)
- **Want:** physics interpretation!
  (Manogue–Dray 1999, 2000, 2010, ...)