Physics vs. Mathematics Classroom Use of Differentials and Thick Derivatives

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Introduction

Is there a difference between $\frac{x^2 - 4}{x - 2}$ and $x + 2$?

What is a numerical representation of a derivative?

Math vs. Physics:

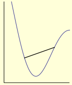
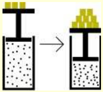
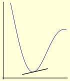
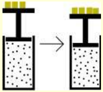
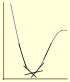
roundoff error, measurement error, quantum mechanics...

small changes...

... differentials ...

"thick" derivatives

Extended Framework for Derivative

Process-object layer	Graphical Slope	Verbal Rate of Change	Symbolic Difference Quotient	Numerical Ratio of Changes	Physical Measurement
Ratio		"avg. rate of change"	$\frac{f(x+\Delta x)-f(x)}{\Delta x}$	$\frac{y_2-y_1}{x_2-x_1}$ numerically	
Limit		"inst. rate of change"	$\lim_{\Delta x \to 0} \cdots$	...with Δx small	
Function		"...at any point/time"	$f'(x) =$ ...	... depends on x	tedious repetition

Process-object layer	· · Symbolic · ·
	Instrumental Understanding
Function	*rules to "take a derivative"*

[Roundy et al., RUME Proceedings 2015, MAA, pp. 838–843.]

Differentials

Does $\frac{df}{dx}$ mean "$f'(x)$" or "df over dx"?

$$d(u^2) = 2u\,du$$
$$d(\sin u) = \cos u\,du$$

Instead of:

- chain rule
- related rates
- implicit differentiation
- derivatives of inverse functions
- difficulties of interpretation (units!)

One coherent idea:

"Zap equations with *d*"

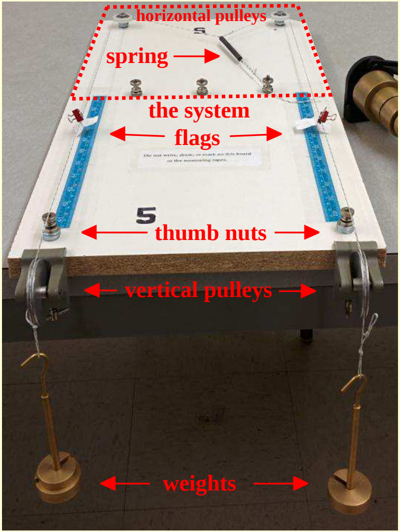
(infinitesimal reasoning)

[Dray & Manogue, CMJ **34**, 283–290 (2003); CMJ **41**, 90–100 (2010).]

Partial Derivatives Machine

- Developed for junior-level thermodynamics course
- Two positions, x_i, two string tensions (masses), F_i.
- "Find $\frac{\partial x}{\partial F}$."
- Idea: Measure Δx, ΔF; divide.

Paradigms in Physics Project
DUE–1023120, DUE–1323800



Interviews

- Partial Derivative Machine: mechanical analog of thermodynamic system with 4 state variables (x_L, F_L, x_R, F_R).
- Interviewed pairs of mathematicians, physicists, and engineers.
- Task: Find $\frac{\partial x_R}{\partial F_R}$. $\left(\frac{\partial x_R}{\partial F_R}\right)_{x_L} \neq \left(\frac{\partial x_R}{\partial F_R}\right)_{F_L}$

Physicists and Engineers:
Used measurements to find numerical approximation.
Mathematicians:
Reluctant to approximate; wanted functional form.
"That's not a derivative!"

[Roundy et al., Phys. Rev. ST Phys. Educ. Res. **11**, 020126 (2015)]

Surfaces

(Each surface is dry-erasable, as are the matching contour maps.)

Raising Calculus to the Surface (Aaron Wangberg)

Raising Physics to the Surface (+ Liz Gire, Robyn Wangberg)

http://raisingcalculus.winona.edu

Thick Derivatives

Measurement:

Ratio: Measure; compare.

Limit: Make small changes...

Measurement error: ...but not too small!

Numerical computation:

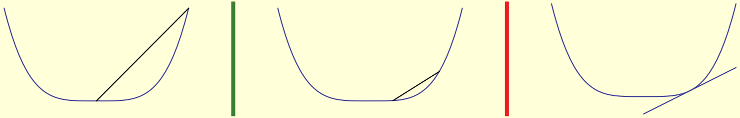
Ratio: Calculate; compare.

Limit: Make small changes...

Roundoff error: ...but not too small!

Mathematicians:
Essential difference between average and instantaneous rates of change, no matter how small the step.
Physicists and Engineers:
Very much aware of how small is "small enough".

Thick Derivatives



Math: ∃ "bright line" between *average* rate of change and *instantaneous* rate of change.

(Such averages are used to approximate derivatives.)

Physics: "Average" refers to secant lines, not (good) approximations to tangent lines.

Move the bright line!

Thick Derivatives!

(Derivatives are fundamentally ratios of small changes, not limits.)

[Dray, AMS Blog on Education, 5/31/16]