The Asymptotic Structure of a Family of Einstein−Maxwell Solutions

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ABSTRACT

A 3-parameter family of solutions of the Einstein-Maxwell field equations (the (C-metrics) are shown to be asymptotically flat at null and spatial infinity in the sense of Geroch [17]. The Bondi news (and the electromagnetic radiation field in the charged case) is shown to be nonzero, thus resolving the issue of existence of exact radiating solutions to the Einstein and Einstein-Maxwell equations. The ADM and Bondi masses are computed and discussed; in particular the Bondi mass is shown to be negative in a neighborhood of spatial infinity, and the ADM mass is zero. This analysis supports, at least for some parameter values (including some which may be interpreted as black holes), the physical interpretation of this solution as the gravitational and electromagnetic field of two uniformly accelerating charged masses. However, evidence is presented which suggests that, for other parameter values, the C-metrics describe negative mass solutions. The compatibility with positive mass theorems is discussed.

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1: Introduction

We will analyze the asymptotic structure of a 3−parameter family of solutions to the source-free Einstein-Maxwell equations. These solutions will be called the C−metrics, following the terminology of Ehlers and Kundt [1]. The vacuum C−metrics were first discovered by Levi-Civita in 1918 [2], then rediscovered by Newman and Tamburino in 1961 [3], and again by Ehlers and Kundt in 1963 [1]. Kinnersley and Walker [4,5] were the first to suggest a physical interpretation, namely the combined gravitational and electromagnetic fields of a uniformly accelerating charged mass. Several generalizations of the C−metrics exist [6,7,8,9,10], but we will not discuss them here.

We will show that the C−metrics are asymptotically flat at both null and spacelike infinity (although not quite AEFANSI; these terms will be precisely defined in the next Chapter), thus strengthening the above physical interpretation, and verify that they have gravitational as well as electromagnetic radiation. The C−metrics are the first spacetimes to be shown to admit a conformal completion in which null infinity (I) is topologically $\mathbf{S^2} \times \mathbf{R}$ [1] and which is regular in a neighborhood of spatial infinity (i^o), and on which the Bondi-news is nonzero; the analysis thus shows that there exist exact solutions of the Einstein and Einstein-Maxwell equations which admit radiation in the sense of Bondi, Sachs, and Penrose.[2] However, since the C−metrics are time-symmetric, the presence of outgoing radiation at I^+ implies the existence of incoming radiation from I^-; the C−metrics thus do **not** represent isolated systems.

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[1] However, the generators of I are **not** complete.

[2] Schmidt [11] has recently shown that certain Einstein-Rosen waves are asymptotically empty and flat at null infinity. See also the work of Bičák [12].

The material is organized as follows: Chapter 2 is used to establish notation and conventions, and definitions of asymptotic flatness are given. In Chapter 3, we review the basic properties of the C−metrics. It turns out that, except for two 2−parameter families of electrovac solutions, the C−metrics each possess a 2−dimensional sheet of nodal singularities which considerably complicates the global analysis. In Chapter 4, we show that essentially all of the C−metrics admit a conformal completion in which I is topologically $\mathbf{S^2} \times \mathbf{R}$. In order to do so, we perform an analytic extension; the resulting spacetimes represent the field of **two** uniformly accelerating particles. We also show in this chapter that the Bondi news is nonzero. In Chapter 5 we show that the C−metrics admit a conformal completion which is smooth at spacelike infinity (i^o). We thus conclude that the total mass (energy) of the system, as measured by the ADM mass, is zero. We discuss the global structure of the C−metrics in Chapter 6, and include a discussion of the sense in which the vacuum C−metrics may be regarded as black holes. In Chapter 7, we discuss the apparent contradiction between the physical interpretation and the fact that the ADM mass is zero. We do this first by considering the various positive energy conjectures, and then by considering test particle orbits. For some parameter values we are able to find bound test particle orbits around either one of the accelerating masses, strongly supporting the physical interpretation. However, for other parameter values we provide evidence which suggests that the physical interpretation may be incorrect, and that the accelerating masses may have negative mass. An explicit calculation of the Bondi mass (for vacuum C−metrics) is presented in the Appendix; to the best of our knowledge this is the first such calculation to be done explicitly.

2. Definitions and Notation

We begin with certain definitions. By a **spacetime** (M, g_{ab}) we shall mean a manifold M, possibly with boundary, equipped with a metric g_{ab} of Lorentzian signature $(-+++)$. (All manifolds are assumed to be connected, Hausdorff, and paracompact.) A spacetime (M, g_{ab}) will be called C^∞ if both M and g_{ab} are C^∞. The Levi-Civita connection on (M, g_{ab}) will be denoted ∇_a. Our conventions for the Riemann tensor, Ricci tensor, and scalar curvature are: $\nabla_{[a}\nabla_{b]}k_c = \frac{1}{2}R_{abc}{}^d k_d$ for and k_c; $R_{ab} = R_{amb}{}^m$; $R = R^m{}_m$.[3] Square brackets denote antisymmetrization, round brackets denote symmetrization; both carry a factor of $\frac{1}{n!}$.

Definition 1: A spacetime $(\hat{M}, \hat{g}_{ab})$ will be said to be **asymptotically empty and flat at null infinity** [13] if there exists a C^∞ spacetime (M, g_{ab}) equipped with a C^∞ function Ω and an embedding of $\hat{M}$ into M (by which we shall identify $\hat{M}$ with its image in M) such that:

i) $g_{ab} = \Omega^2 \hat{g}_{ab}$ on $\hat{M}$;

ii) $\Omega = 0$ on $\partial\hat{M}$, $\nabla_a\Omega \neq 0$ on $\partial\hat{M}$;

iii) the manifold of orbits of $n^a := \nabla^a\Omega|_{\partial\hat{M}}$ is diffeomorphic to $\mathbf{S^2}$; and

iv) $\Omega^{-2}\hat{R}_{ab}$ admits a C^∞ extension to $\partial\hat{M}$.

The boundary $\partial\hat{M}$ of $\hat{M}$ in M represents null infinity and will be denoted by I ("Scri"; script I);[4] $\hat{R}_{ab}$ is the Ricci tensor of $\hat{g}_{ab}$. Note that i) of course implies that $\Omega \neq 0$ on $\hat{M}$; ii) ensures that $\Omega \approx r^{-1}$ near I; while i), ii), and iv)

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[3]We use abstract indices throughout. This is to be contrasted with the component version of e.g. the first of these equations: $(\nabla_{[X}\nabla_{Y]} - \nabla_{[X,Y]})k_Z = \frac{1}{2}R_{XYZ}{}^M k_M$.

[4]We are being somewhat sloppy here with the notation I; This construction will only yield (part of) a connected component of the boundary at null infinity (e.g. I^+).

imply that I is a null 3-surface, and that one can always perform a conformal rescaling such that the vector field n^a is (locally) divergence free (i.e. $\nabla_a \nabla^a \Omega|_I = 0$, or, equivalently, $L_n q_{ab} = 0$, where L_n denotes the Lie derivative on I along n^a and q_{ab} is the pullback of g_{ab} to I). Together with iii) they imply that the Weyl tensor of g_{ab} must vanish on I, enabling one to define e.g. the Bondi 4-momentum.[4½] The orbits (integral curves) of n^a will be called **generators** of I.

If, in addition to the above conditions, n^a is a complete vector field (in the conformal frames in which it is divergence free) then we say that $(\hat{M}, \hat{g}_{ab})$ is **asymptotically Minkowskian**.

Definition 2: A spacetime $(\hat{M}, \hat{g}_{ab})$ will be said to be **asymptotically flat at spatial infinity** [14,15] if there exists a spacetime (N, g_{ab}) which is C^∞ everywhere except possibly at a point i^o where N is $C^{>1}$ and g_{ab} is $C^{>0}$,[5] equipped with a function Ω, which is C^2 at i^o and C^∞ elsewhere, and an embedding of $\hat{M}$ into N (by which we shall identify $\hat{M}$ with its image in N) such that:

v) $g_{ab} = \Omega^2 \hat{g}_{ab}$ on $\hat{M}$;

vi) $\Omega = 0$, $\nabla_a \Omega = 0$, and $\nabla_a \nabla_b \Omega = 2 g_{ab}$ are all satisfied at i^o;

vii) $\bar{J}(i^o) = N - \hat{M}$.

$\bar{J}(i^o)$ denotes the set of all points in N which can be reached from i^o by a non-spacelike curve in N, together with the point i^o [16]. Condition vi) ensures that $\Omega \approx r^{-2}$ near i^o, while vii) tells us that the set of points in N which are spacelike related to i^o is precisely $\hat{M}$. Furthermore, if we can find conformal

[4½] Note that one **cannot** introduce the news tensor of the Bondi 4-momentum unless iii) is satisfied.

[5] The awkward differential structure $C^{>n}$ is defined in [14]; the spacetimes we will consider will turn out to be C^∞ at i^o, which is a (very) special case of the above conditions.

completions of $(\hat{M}, \hat{g}_{ab})$ such that i) thru vii) are satisfied **for the same Ω** (note that M is thus naturally related to N), then vii) ensures that I is just (a connected component of) the lightcone at i^o (in $M \cup N$).

A spacetime will be called **AEFANSI** (Asymptotically Empty and Flat at Null and Spatial Infinity) if i) thru vii) are satisfied (for the same Ω) **and** it is asymptotically Minkowskian.

We now collect from [16] some definitions about global structure. A spacetime $(\hat{M}, \hat{g}_{ab})$ will be said to satisfy the **weak energy condition** if $\hat{T}_{ab} w^a w^b \geq 0$ for any timelike vector field w^a, where $\hat{T}_{ab}$ is the stress-energy tensor constructed out of $\hat{g}_{ab}$ using the field equations. If, in addition, $\hat{T}^{ab} w_b$ is non-spacelike for all timelike w_b, then we saw that $(\hat{M}, \hat{g}_{ab})$ satisfies the **dominant energy condition.** Using the field equations one can relate these conditions to various convergence conditions; see [16].

A (global) **Cauchy surface** is a spacelike hypersurface which every non-spacelike curve intersects precisely once.

The (future) **event horizon** of a spacetime $(\hat{M}, \hat{g}_{ab})$ which is asymptotically empty and flat at null infinity is the boundary of the region from which particles or photons can escape to infinity (I^+) in the future direction. In the notation of [16] we write this as $\dot{J}^-(I^+, \hat{M})$.

3. The Metric[6]

The C−metrics are usually given in the form

$$\hat{g}_{ab} = r^2(f_{ab} + h_{ab})$$

with

$$\begin{aligned} f_{ab}dx^a dx^b &:= -F(y)dt^2 + \frac{dy^2}{F(y)} \\ h_{ab}dx^a dx^b &:= \frac{dx^2}{G(x)} + G(x)dz^2 \\ G(x) &:= 1 - x^2 - 2mAx^3 - e^2A^2x^4 \\ F(y) &:= -G(-y) = -1 + y^2 - 2mAy^3 + e^2A^2y^4 \\ r &:= A^{-1}(x+y)^{-1} \end{aligned} \tag{3.1}$$

and where $A > 0$, $m \geq 0$, and e are constants; A^{-1}, m, and e have the dimensions of length, whereas the coordinates (t, y, x, z) are dimensionless. The manifold $\hat{M}$ is determined by the coordinate ranges

$$\begin{aligned} &t \in \mathbf{R} \\ &x \in [x_1, x_2] \\ &y \in (-x, \infty) \\ &z \in [0, 2\pi\kappa] \end{aligned} \tag{3.2}$$

where $x_{i+1} > x_i$ are the (real) roots of $G(x)$, and κ is a constant. (We assume that $G(x)$ has at least two real roots, i.e. $e \neq 0$ or $mA < 3^{-3/2}$, and that all of the roots are distinct.) These coordinate ranges ensure $G(x) \geq 0$,[7] so that the signature is Lorentzian, i.e. $(- + + +)$, and that $0 < r < \infty$. There is a curvature singularity at

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[6]The description of the C−metrics presented here is essentially the same as in [4,5].

[7]Note that if $G(x)$ has four distinct real roots, one could have chosen $x \in [x_4, x_3]$ instead of the above choice. We will not consider this possibility here, although these C−metrics are essentially the same as those with only two distinct real roots.

$r = 0$ unless $e = m = 0$. The Maxwell field is given by[8]

$$\hat{F}_{ab} = 2e\hat{\nabla}_{[a}y\hat{\nabla}_{b]}t\ , \tag{3.3}$$

where $\hat{\nabla}_a$ is of course the Levi-Civita connection on $(\hat{M}, \hat{g}_{ab})$. For $e = 0$, $(\hat{M}, \hat{g}_{ab})$ satisfies the vacuum Einstein equations (and is flat as well if $(m = 0)$, while for $e \neq 0$ $(\hat{M}, \hat{g}_{ab}, \hat{F}_{ab})$ satisfies the source-free Einstein-Maxwell equations. The parameters m, e, and A are interpreted as the mass, charge, and acceleration of a particle located at $r = 0$ whose gravitational and electromagnetic fields are described by the solution. There are two independent Killing vector fields, ∂_t and ∂_z, which are both hypersurface orthogonal, and which commute. However, the C−metrics are **not** static because ∂_t is not **everywhere** timelike. In fact, the labels "t" and "z" are misleading; ∂_t will be interpreted as a boost and ∂_z as a rotation,[9] in analogy with their behavior in the flat limit $e = m = 0$. The surfaces in $\hat{M}$ on which ∂_t is null ($F(y) = 0$; $y = y_i = -x_i$ with $i \neq 1$) will be called Killing horizons. There is also a conformal Killing tensor $\hat{K}^{ab} = r^4 h^{ab}$ (which is a Killing tensor of $r^{-2}\hat{g}_{ab}$). These symmetries make it possible to integrate all null geodesics [4].

Eq. (3.2) allows us to show that the $\{(y, t)$=constant$\}$ 2−surfaces have the topology of $\mathbf{S^2}$, which will be essential in the construction of I in Chapter 4. Introduce new coordinates (ϑ, φ) via

$$\vartheta := \int_{x_2}^{x} G(x)^{-1/2} dx \tag{3.4a}$$

$$\varphi := \kappa^{-1} z.$$

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[8] $\hat{F}_{ab}$ is of course only determined up to a duality rotation. We have chosen the expression with zero magnetic charge.

[9] We will show in Chapter 5 that this interpretation is correct at i^o.

We now have

$$\vartheta \in [0, \vartheta_o] \tag{3.4b}$$

$$\varphi \in [0, 2\pi]$$

and the metric on the $\{(y, t)$=constant$\}$ 2−surfaces becomes

$$r^2 h_{ab} = r^2\left(d\vartheta^2 + \kappa^2 \rho(\vartheta)^2 d\varphi^2\right) \tag{3.5}$$

where $\vartheta_o = \vartheta(x_1)$ and $\rho(\vartheta) := G(x(\vartheta))^{1/2}$. $\rho(\vartheta)$ is thus positive and bounded in $(0, \vartheta_o)$, and vanishes at $\vartheta = 0$ and $\vartheta = \vartheta_o$. h_{ab} is clearly regular except possibly at the roots of $\rho(\vartheta)$. A necessary and sufficient condition that the metric be regular at these roots is [4] $\kappa^{-1} = |\rho'(\vartheta)|$ there. We set $\kappa^{-1} := |\rho'(0)|$, thus ensuring regularity at $\vartheta = 0$. However, we cannot in general avoid the presence of nodal (i.e. conical) singularities at $\vartheta = \vartheta_o$ unless $|\rho'(0)| = |\rho'(\vartheta_o)|$ which occurs if and only if $m = 0$ or $|e| = m > \dfrac{1}{4A}$. Note that, for these parameter values, $G(x)$ (and hence also $F(y)$) has exactly two real roots. In the general case, the points $\vartheta = \vartheta_o$ must be deleted from $\hat{M}$ in order to preserve the C^∞ differentiable structure.

We now consider the $\{(x, z)$=constant$\}$ submanifolds. These are clearly regular except possibly at the roots of $F(y)$. We introduce a retarded coordinate u via

$$Au := t + \int^y F(y)^{-1} dy \tag{3.6}$$

We thus obtain $f_{ab} dx^a dx^b = -A^2 F(y) du^2 + 2A du dy$, and the full metric becomes

$$\hat{g}_{ab} dx^a dx^b = r^2\left[-A^2 F(y) du^2 + 2A du dy + d\vartheta^2 + \kappa^2 \rho(\vartheta)^2 d\varphi^2\right] \tag{3.7}$$

Thus, in the coordinate ranges determined by Eq. (3.2), namely

$$
\begin{aligned}
&u \in \mathbf{R} \\
&\varphi \in [0, 2\pi] \\
&\vartheta \in [0, \vartheta_o] \\
&y \in (-x_2 - \int_0^{\vartheta} \rho(\vartheta) d\vartheta, \infty),
\end{aligned}
\tag{3.8}
$$

$\hat{g}_{ab}$ is thus regular everywhere except for a timelike 2−dimensional sheet of nodal singularities at $\vartheta = \vartheta_o$ (unless $m = 0$ or $|e| = m > \frac{1}{4A}$).

Using $(u, y, \vartheta, \varphi)$ as coordinates one can establish the following results: [4]

i) The Weyl tensor is of Petrov type D (2−2) [27]. The $(u, r, \vartheta, \varphi)$ coordinate system is adapted to one of the principal null vectors, namely $l^a = (\partial_r)^a$. Furthermore, l^a generates a family of null hypersurfaces on which u =constant, and r is an affine parameter along l^a. Thus, the C−metrics are Robinson-Trautman solutions [28]. In these coordinates, the other principal null vector is $k^a = (\partial_u)^a - (½)A^2 r^2 F(y)(\partial_4)^a$. Note that as r goes to ∞ the directions determined by k^a and l^a coincide; the C−metrics are thus Petrov type N [27] in this limit.

ii) For e and m fixed (and A in some neighborhood of 0), $\lim_{A\to 0} (\hat{M}, \hat{g}_{ab})$ is the Reissner-Nordström spacetime with charge e and mass m. (The condition on A is necessary because otherwise the limit may not be defined; the coordinate ranges could otherwise change in a noncontinuous manner, e.g. because the number of roots of $G(x)$ changes.)

However, the $(u, y, \vartheta, \varphi)$ coordinate system is useful for calculations.

We now give expressions for the curvature tensor of $\hat{g}_{ab}$. The Ricci tensor is

$$\hat{R}_{ab}dx^a dx^b = \frac{e^2}{r^2}(-f_{ab} + h_{ab})\, dx^a dx^b \tag{3.9}$$

$$= \frac{e^2}{r^2}\left[A^2 F(y)du^2 - 2Adudy + d\vartheta^2 + \kappa^2\rho(\vartheta)^2 d\varphi^2\right]$$

Thus, the curvature scalar $\hat{R} = \hat{R}^m{}_m$ is zero, and the Einstein tensor is the same as the Ricci tensor ($\hat{G}_{ab} = \hat{R}_{ab}$).

The Weyl tensor is

$$\begin{aligned}\hat{C}_{abcd} = 4rP(y,\vartheta)\Big[&-2A^2\hat{\nabla}_{[a}u\hat{\nabla}_{b]}y\hat{\nabla}_{[c}u\hat{\nabla}_{d]}y + A^2F(y)\hat{\nabla}_{[a}u\hat{\nabla}_{b]}\vartheta\hat{\nabla}_{[c}u\hat{\nabla}_{d]}\vartheta \\ &+A^2\kappa^2\rho(\vartheta)^2F(y)\hat{\nabla}_{[a}u\hat{\nabla}_{b]}\varphi\hat{\nabla}_{[c}u\hat{\nabla}_{d]}\varphi + 2\kappa^2\rho(\vartheta)^2\hat{\nabla}_{[a}\vartheta\hat{\nabla}_{b]}\varphi\hat{\nabla}_{[c}\vartheta\hat{\nabla}_{d]}\varphi \\ &-A\hat{\nabla}_{[a}u\hat{\nabla}_{b]}\vartheta\hat{\nabla}_{[c}y\hat{\nabla}_{d]}\vartheta - A\hat{\nabla}_{[a}y\hat{\nabla}_{b]}\vartheta\hat{\nabla}_{[c}u\hat{\nabla}_{d]}\vartheta \\ &-A\kappa^2\rho(\vartheta)^2\hat{\nabla}_{[a}u\hat{\nabla}_{b]}\varphi\hat{\nabla}_{[c}y\hat{\nabla}_{d]}\varphi - A\kappa^2\rho(\vartheta)^2\hat{\nabla}_{[a}y\hat{\nabla}_{b]}\varphi\hat{\nabla}_{[c}u\hat{\nabla}_{d]}\varphi\Big]\end{aligned} \tag{3.10}$$

where $P(y,\vartheta) = m - e^2A(y - x(\vartheta))$.

We now show that the C−metrics satisfy the dominant energy condition. We have

$$\hat{T}_{ab} = \hat{G}_{ab} = \hat{R}_{ab} = \frac{e^2}{r^2}(-f_{ab} + h_{ab}) \tag{3.11}$$

where h_{ab} is positive definite. Let w^a be a timelike vector field on $\hat{M}$. Then

$$0 > \hat{g}_{ab}w^a w^b = r^2\left(f_{ab}w^a w^b + h_{ab}w^a w^b\right) \tag{3.12}$$

which implies that

$$f_{ab}w^a w^b < 0 \tag{3.13}$$

and hence

$$\hat{T}_{ab}w^a w^b = \frac{e^2}{r^2}\left(-f_{ab}w^a w^b + h_{ab}w^a w^b\right) \geq 0 \tag{3.14}$$

with equality holding if and only if $e = 0$; this is the weak energy condition. Furthermore,

$$\hat{g}_{ab}\hat{T}^{ac}w_c\hat{T}^{bd}w_d = \frac{e^4}{r^8}\,\hat{g}_{ab}w^a w^b \leq 0 \tag{3.15}$$

where we have used the properties of f_{ab} and h_{ab}, in particular $f_{ab}h^{bc} = 0$. Again, equality holds if and only if $e = 0$. Thus, the dominant energy condition is satisfied.

We will classify the C−metrics using $n := \deg G(x) \equiv \deg F(y)$ as follows (see Figure 1):

i) $n = 2$: **flat** C−metrics ($e = m = 0$);

ii) $n = 3$, all roots real and distinct: **vacuum** C−metrics ($e = 0, mA < 3^{-3/2}$);

iii) $n = 4$, no nodes: **regular** C−metrics ((a) $m = 0$, $e \neq 0$ or (b) $|e| = m > \frac{1}{4A}$; note that in both cases there are exactly two real roots, which are distinct);

iv) $n = 4$, two distinct real roots, with nodes;

v) $n = 4$, four distinct real roots.

We will restrict ourselves to these C−metrics in the remainder of this work.[10] A complete discussion of the parameter ranges for types iv) and v) appears in [4]. Types iii, iv, and v will be referred to collectively as **charged** C−metrics.

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[10]For C−metrics which do not have two distinct real roots we cannot give the $\{(y,t)$=constant$\}$ 2−surfaces the topology of $\mathbf{S}^2$ required in the construction of I. The only remaining case is C−metrics with multiple roots; these will not be considered here.

4. Null Infinity[11]

We first establish (Section 4.1) that the regular C−metrics are asymptotically empty and flat at null infinity, but not asymptotically Minkowskian, and show that they admit gravitational (and, in the charged cases, electromagnetic) radiation. We then consider (Section 4.2) the flat C−metrics, and show how to analytically extend these spacetimes in order to obtain "all" of I (and, in particular, a neighborhood of i^o). We then use a similar procedure to show (Section 4.3) that the remaining C−metrics are also asymptotically empty and flat at null infinity. Note that one must extend all C−metrics in order to obtain a neighborhood of i^o in I, while for nodal C−metrics it is **essential** to extend the spacetimes just to obtain a I which is topologically $\mathbf{S^2} \times \mathbf{R}$.

4.1 Regular C−Metrics

We restrict ourselves in this section to the cases $m = 0$ and $|e| = m > \frac{1}{4A}$, i.e. C−metrics of types i and iii. Let M be the manifold obtained from $\hat{M}$ by extending the coordinate ranges in Eq. (3.8) to include the points $\{y = -x\}$, i.e. $\{r = \infty\}$.

[11] The material in Sections 4.1 and 4.2 is essentially the same as the presentation in [13].

Define, on M,

$$\Omega := r^{-1} \equiv A\left(y + x_2 + \int_0^{\vartheta} \rho(\vartheta) d\vartheta \right) \tag{4.1}$$

and

$$g_{ab} dx^a dx^b := \Omega^2 \hat{g}_{ab} dx^a dx^b \equiv - A^2 F(y) du^2 + 2A du dy + d\vartheta^2 + \kappa^2 \rho(\vartheta)^2 d\varphi^2.$$

g_{ab} is thus C^∞ everywhere on M, with signature $(-+++)$. We claim that (M, g_{ab}), together with Ω, satisfy the conditions of Definition 1: Let I_u denote the boundary of $\hat{M}$ in M, i.e. $\{\Omega = 0\}$. Since $d\Omega = A(dx + \rho(\vartheta) d\vartheta)$, $d\Omega$ vanishes nowhere on I_u. Furthermore, from Eq. (3.9) we have

$$\Omega^{-2} \hat{R}_{ab} dx^a dx^b = e^2 \left[A^2 F(y) du^2 - 2A du dy + d\vartheta^2 + \kappa^2 \rho(\vartheta)^2 d\varphi^2 \right], \tag{4.2}$$

which admits a smooth limit to I_u. Thus it only remains to check that the manifold S of orbits of $n^a = \nabla^a \Omega|_{I_u}$ is topologically $\mathbf{S^2}$. In the $(u, y, \vartheta, \varphi)$ chart we have

$$g^{ab} \nabla_b \Omega = (\partial_u)^a + AF(y)(\partial_y)^a + A\rho(\vartheta)(\partial_\vartheta)^a. \tag{4.3}$$

Thus $L_n \vartheta \neq 0$, and so it is not clear that the $\mathbf{S^2}$ topology of the $\{u = \text{constant}\}$ 2−surfaces (in I_u) projects down to S. However, since $L_n u = 1$, u is an affine parameter along the generators n^a. Thus, the 2−surface $\{u = 0\}$ is a crosssection of I_u, and therefore S has the same topology as this surface, namely $\mathbf{S^2}$. We can exhibit this diffeomorphism explicitly by defining a function $\psi(\vartheta, u)$ on I_u via

$$\psi(\vartheta, 0) = \vartheta \quad \text{and} \quad L_n \psi = 0. \tag{4.4}$$

We can now use (ψ, φ) as coordinates on S. We conclude that the regular C−metrics are asymptotically empty and flat at null infinity.

We now analyze the structure intrinsic to I_u, where we regard I_u as a 3−manifold coordinatized by (u, ϑ, φ). Let q_{ab} denote the pullback of g_{ab} to I_u. Then q_{ab} is the (degenerate) metric on I_u, and is given by

$$q_{ab}dx^a dx^b = A^2\rho(\vartheta)^2 du^2 - 2A\rho(\vartheta)du d\vartheta + d\vartheta^2 + \kappa^2\rho(\vartheta)^2 d\varphi^2, \tag{4.5}$$

where we have used the fact that the pullback of $dy + \rho(\vartheta)d\vartheta$ to I_u vanishes and that $F(y) = -\rho(\vartheta)^2$ on I_u. From $q_{ab}n^b = 0$ and $L_n u = 1$, we have

$$n^a = (\bar{\partial}_u)^a + A\rho(\vartheta)(\bar{\partial}_\vartheta)^a \tag{4.6}$$

where $\bar{\partial}$ denotes partial differentiation **within** I_u. Eq. (4.4) implies

$$\frac{d\psi}{\rho(\psi)} = -Adu + \frac{d\vartheta}{\rho(\vartheta)}. \tag{4.7}$$

Substituting this in Eq. (4.5) we obtain

$$q_{ab}dx^a dx^b = \frac{\rho(\vartheta)^2}{\rho(\psi)^2}\left(d\psi^2 + \kappa^2\rho(\psi)^2 d\varphi^2\right) \tag{4.8}$$

where ϑ is now a function of u and ψ. Note that the factor $\frac{\rho(\vartheta)}{\rho(\psi)}$ is in fact regular and nonzero: From $L_n\vartheta = A\rho(\vartheta)$ and the definition of ψ (Eq. (4.4)), we have

$$\vartheta = 0 \text{ iff } \psi = 0; \;\; \vartheta = \vartheta_o \text{ iff } \psi = \vartheta_o. \tag{4.9}$$

(Note that each of these represents a **single** generator of I_u.) Thus, $L_n\left[\frac{\rho(\vartheta)}{\rho(\psi)}\right] = A\rho'(\vartheta)\left[\frac{\rho(\vartheta)}{\rho(\psi)}\right]$, which implies $\left[\frac{\rho(\vartheta)}{\rho(\psi)}\right] = e^{\pm A\kappa^{-1}u}$ along the two generators $\psi = 0$ and $\psi = \vartheta_o$.[12]

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[12]Regularity implies $\rho'(\vartheta_o) = -\rho'(0) =: -\kappa^{-1}$. Note also that $n^a = (\bar{\partial}_u)^a$ in the (u, ψ, φ) chart.

We can now investigate the completeness of the generators of I_u. However, since $L_n\vartheta \neq 0$, we also have $L_n q_{ab} \neq 0$; the conformal frame (q_{ab}, n^a) is not divergence-free. Using the gauge freedom available on I_u, we transform to the conformal frame $(q'_{ab} := \omega^2 q_{ab},\ n'^a := \omega^{-1} n^a)$ with $\omega := \dfrac{\rho(\psi)}{\rho(\vartheta)}$; $L_{n'} q'_{ab}$ is then clearly zero. On each generator we wish to determine the range of u', the affine parameter along n'^a satisfying $L_{n'} u' = 1$. First of all, we have $du' = \omega du$ along any generator. On any generator with $\psi \neq 0, \psi \neq \vartheta_o$, we obtain $u' = \rho(\psi) \int \rho(\vartheta)^{-1} du$. But $\rho(\vartheta)$ is bounded above, and hence, as u ranges over $(-\infty, \infty)$, so must u'. We conclude that all generators with $\psi \neq 0, \psi \neq \vartheta_o$ are complete.

Consider next the generator $\psi = 0$. Using our previous result, we have $du' = e^{-A\kappa^{-1}u} du$ along this generator, i.e. $u' = -\dfrac{\kappa}{A}(e^{-A\kappa^{-1}u} - 1)$, where we have set $u'|_{u=0} = 0$. As u ranges over $(-\infty, \infty)$, u' ranges over $(-\infty, \dfrac{\kappa}{A})$. Thus, the generator $\psi = 0$ is incomplete in the future. Similarly, along the generator $\psi = \vartheta_o$ we have $du' = e^{+A\kappa^{-1}u} du$, and hence $u' = +\dfrac{\kappa}{A}(e^{A\kappa^{-1}u} - 1)$. As u ranges over $(-\infty, \infty)$, u' now ranges over $(-\dfrac{\kappa}{A}, \infty)$; this generator is incomplete in the past. See Figure 2. We have thus shown that these spacetimes are **not** asymptotically Minkowskian.

We now show that there is radiation. As expected from a general theorem (Theorem 11 of [17]), the Weyl tensor $C_{abcd} = \Omega^2 \hat{C}_{abcd}$ of g_{ab} vanishes on I_u, and $\Omega^{-1} C_{abcd}$ admits a smooth limit to I_u. Denote the pullback of $\Omega^{-1} C_{abcd}$ to I_u by K_{abcd} and set $K_{ac} := K_{abcd} n^b n^d$. Then

$$K_{ab} dx^a dx^b = 3A^2 \rho(\vartheta)^2 P|_I \big(-A\rho(\vartheta)^2 du^2 + 2A\rho(\vartheta) du d\vartheta \qquad (4.10)$$

$$-d\vartheta^2 + \kappa^2 \rho(\vartheta)^2 d\varphi^2\big)$$

where $P|_I = m + 2e^2 Ax(\vartheta)$. In the Newman-Penrose notation [18], the fact that K_{ab} is not identically zero on I_u (unless $e = m = 0$) means that ${\psi_4}^0$ is not identically zero on I_u, which in turn means that the news function cannot vanish identically on I_u. Alternatively, in the Geroch notation [17], K_{ab} can be expressed in terms of the derivative of the news tensor, which therefore cannot vanish identically on I_u.[13]

Finally, we turn to the electromagnetic field. We have

$$F_{ab} \equiv \hat{F}_{ab} = 2eA\hat{\nabla}_{[a}y\hat{\nabla}_{b]}u \tag{4.11}$$

so that

$$F_{ab}\nabla^b\Omega dx^a = eAdy - eA^2F(y)du \tag{4.12}$$

and thus the pullback of this expression to I_u is given by $-eA\rho(\vartheta)d\vartheta - eA^2F(y)du$, which is not identically zero on I_u (unless $e = 0$). In the Newman-Penrose notation, this means that ${\phi_2}^0$ is not identically zero on I_u. We conclude that the regular C−metrics contain both electromagnetic and gravitational radiation. Furthermore, as we shall see in Section 4.3, the above derivation is quite general, and in fact establishes the presence of gravitational radiation for **all** C−metrics except for the flat C−metrics, and the presence of electromagnetic radiation if $e \neq 0$.

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[13]Both of these results are stated in conformal frames which are divergence free. However, since $K'_{ab} = \omega^{-1}K_{ab} \neq 0$, and $\omega \neq 0$, our conformal frame in fact suffices.

4.2 Flat C−metrics

The flat C−metrics ($e = m = 0$) are a subclass of the C−metrics considered in the preceding section. Yet, our construction yielded a I whose generators are not all complete! How is this possible? It turns out that our charts only cover half of Minkowski space. More generally, all of the C−metrics admit analytic extensions, and it is these extended manifolds which will be asymptotically flat and empty at null infinity in a neighborhood of i^o. We restrict ourselves in this section to the flat C−metrics, and carry out the construction explicitly; this will then serve as a model for the general construction in the next section.

Let $(\hat{M}, \hat{g}_{ab})$ be the spacetime of Eqs. (3.7) and (3.8) with $e = m = 0$. We then have $F(y) = y^2 - 1$, $\rho(\vartheta) = \sin\vartheta$, and $\kappa = 1$. Using the method of [19], we set

$$U := e^{Au}; \;\; UV := \frac{y-1}{y+1} \tag{4.13}$$

which leads to

$$\hat{g}_{ab}dx^a dx^b = r^2\left[(y+1)^2 dUdv + d\vartheta^2 + \sin^2\vartheta d\varphi^2\right] \tag{4.14}$$

where

$$y(U,V) = \frac{1+UV}{1-UV} \tag{4.15}$$

$$Ar(U,V) = \frac{1-UV}{1-\cos\vartheta + UV(1+\cos\vartheta)}$$

The manifold in $\hat{M}$ is given by

$$\begin{aligned} \vartheta &\in [0, \pi] \\ \varphi &\in [0, 2\pi] \\ U &> 0 \\ \frac{\cos\vartheta - 1}{\cos\vartheta + 1} &< UV < 1 \end{aligned} \tag{4.16}$$

We can extend $\hat{M}$ to a new manifold $\hat{N}$ by simply dropping the condition $U > 0$; see Figure 3. Note that $\hat{g}_{ab}$ is C^∞ on $\hat{N}$; $\{y = -1\}$ is not part of $\hat{N}$.

Introducing new coordinates via

$$\begin{aligned} \bar{x} &:= r \sin\vartheta \cos\varphi \\ \bar{y} &:= r \sin\vartheta \sin\varphi \\ \bar{z} &:= r \frac{U+V}{1-UV} \\ \bar{t} &:= r \frac{U-V}{1-UV} \end{aligned} \tag{4.17}$$

we discover that

$$\hat{g}_{ab} dx^a dx^b = d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 - d\bar{t}^2; \tag{4.18}$$

$(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ are the usual coordinates on Minkowski space. It is easy to check that $\hat{N}$ corresponds to all of Minkowski space, with the "singularities" at $r = 0$ (i.e. $\bar{x} = \bar{y} = 0$, $\bar{z}^2 - \bar{t}^2 = A^{-2}$)[14] removed, while $\hat{M}$ corresponds to the submanifold of

[14]It is the behavior of the $r = 0$ "singularity" in the flat C−metrics which leads to the interpretation of (a) constantly accelerating particle(s). We make no effort to remove these coordinate singularities here, because, except for the flat C−metrics, they are curvature singularities and cannot be removed.

$\hat{N}$ with $\bar{z}+\bar{t}>0$. Thus, the I of $\hat{M}$ should, indeed, have two half generators missing, as shown in the last section. We now show that $\hat{N}$ admits a conformal completion whose generators are complete except for the two "bullet holes" corresponding to the intersection of the $r=0$ "singularities" with I.

We will repeat the construction of the previous section, again using r^{-1} as our conformal factor. Unfortunately, since $r=\infty$, $\vartheta=\pi$ implies that y=-1, the rescaled metric $g_{ab}:=\Omega^2\hat{g}_{ab}$ is not well-behaved in the (U,V,ϑ,φ) chart. We avoid this problem by using two coordinate patches to cover I. One of these will be the (u,y,ϑ,φ) coordinates in which we made detailed calculations in Section 4.1. We introduce a second, "dual", chart $(\tilde{u},y,\vartheta,\varphi)$ by defining, for $V<0$, $\dfrac{\cos\vartheta-1}{\cos\vartheta+1}<UV<1$:[15]

$$A\tilde{u}:=\ln(-V) \tag{4.19}$$

(Note that $r=\infty$ corresponds to $UV=\dfrac{\cos\vartheta-1}{\cos\vartheta+1}\le 0$.) We now have, in the $(\tilde{u},y,\vartheta,\varphi)$ chart:

$$g_{ab}dx^a dx^b=-A^2(y^2-1)d\tilde{u}^2+2Ad\tilde{u}\,dy+d\vartheta^2+\sin^2\vartheta\,d\varphi^2 \tag{4.20a}$$

while in the (u,y,ϑ,φ) chart:

$$g_{ab}dx^a dx^b=-A^2(y^2-1)du^2+2Adudy+d\vartheta^2+\sin^2\vartheta\,d\varphi^2 \tag{4.20b}$$

These charts intersect in the region $U>0$, $V<0$; the intersection is C^∞ since $u+\tilde{u}=A^{-1}\ln\left|\dfrac{y-1}{y+1}\right|$. Each of these charts can clearly be extended to include a surface at $r=\infty$ (i.e. $y=\cos\vartheta$); denote these surfaces I_u and $I_{\tilde{u}}$, respectively.

---

[15]We use "$\tilde{u}$" instead of "v" because **both** u and $\tilde{u}$ are retarded coordinates; this corresponds to the usage in [13], with w, $\tilde{w}$ replaced here by Au, $A\tilde{u}$.

Using the procedure of Section 4.1, we see that each of these is topologically $\mathbf{S^2} \times \mathbf{R}$ (with incomplete generators) and can be coordinatized by (u, ψ, φ) and $(\tilde{u}, \tilde{\psi}, \varphi)$, respectively, with

$$q_{ab}\, dx^a dx^b = \frac{\sin^2 \vartheta}{\sin^2 \psi}\left(d\psi^2 + \sin^2 \psi\, d\varphi^2\right)$$

$$L_n \psi = 0 \tag{4.21a}$$

$$n^a = (\bar{\partial}_u)^a$$

and

$$q_{ab}\, dx^a dx^b = \frac{\sin^2 \vartheta}{\sin^2 \tilde{\psi}}\left(d\tilde{\psi}^2 + \sin^2 \tilde{\psi}\, d\varphi^2\right)$$

$$L_n \tilde{\psi} = 0 \tag{4.21b}$$

$$n^a = (\bar{\partial}_{\tilde{u}})^a$$

Since $u + \tilde{u} = A^{-1} \ln \left| \frac{\cos \vartheta - 1}{\cos \vartheta + 1} \right|$ on $I_u \cap I_{\tilde{u}}$, it follows that this intersection is all of $H := I_u \cup I_{\tilde{u}}$ **except** $\{\vartheta = 0$ or $\vartheta = \pi\}$. On the intersection, the metrics of course agree, and

$$n^a = \tilde{n}^a$$

$$du + d\tilde{u} = \frac{2A^{-1} d\vartheta}{\sin \vartheta} \tag{4.22}$$

$$\frac{d\psi}{\sin \psi} = -\frac{d\tilde{\psi}}{\sin \tilde{\psi}},$$

the last of which yields $\psi + \tilde{\psi} = \pi$, since at the points $(u = 0, \vartheta = \frac{\pi}{2}, \varphi)$ we have $\tilde{u} = u = 0$ and $\psi = \tilde{\psi} = \frac{\pi}{2}$. We can thus extend ψ (and $\tilde{\psi}$) to all of H by requiring

$\psi + \tilde{\psi} = \pi$ everywhere. Introducing an affine parameter $\tilde{u}'$ on $I_{\tilde{u}}$ analogously to the introduction of u' on I_u, we see that $\tilde{u}'$ (and u') ranges over $(-\infty, \infty)$ on all generators except $\psi = 0$ and $\psi = \pi$, and that $\tilde{u}'$ ranges over $(-\infty, \frac{1}{A})$ on the generator $\psi = \pi$ ($\tilde{\psi} = 0$), and over $(-\frac{1}{A}, \infty)$ on the generator $\psi = 0$ ($\tilde{\psi} = \pi$). As shown in Section 4.1, u' ranges over $(-\frac{1}{A}, \infty)$ on $\psi = \pi$, and over $(-\infty, \frac{1}{A})$ on $\psi = 0$. It remains to determine the relationship between u' and $\tilde{u}'$ on these generators. But since both are affine parameters on $I_u \cap I_{\tilde{u}}$, we must have $\tilde{u}' = u' + \alpha(\psi, \varphi)$ for some smooth function α; one can show that

$$\lim_{\psi \to 0} \alpha(\psi, \varphi) = -\frac{2}{A} \tag{4.23}$$

$$\lim_{\psi \to \pi} \alpha(\psi, \varphi) = \frac{2}{A} .$$

We can use this to extend, say, u' to an affine parameter **everywhere** on H, and find that it takes on all values between $-\infty$ and ∞ on **all** generators, except for the values $u' = \frac{1}{A}$ on $\psi = 0$ and $u' = -\frac{1}{A}$ on $\psi = \pi$, corresponding to the two bullet holes (see Figure 4).

Note that H does **not** satisfy Definition 1: Because the two generators labeled by $\psi = 0$ and $\psi = \pi$ are disconnected, the "manifold" of orbits of n^a fails to be Hausdorff! This appears to be due to a technical flaw in Definition 1. (Is there a simple way to fix this?) We can avoid this problem by deleting e.g. the two half generators with $\vartheta = \pi$; the resulting manifold I **is** Scri in the sense of Definition 1. Finally, we note that we have in fact constructed I^+, and that I^- would have been obtained by considering the region $U > 0$, $V < 0$.

4.3 The General Case[16]

We are now ready to discuss the general nodal C−metric $(\hat{M}, \hat{g}_{ab})$. For these spacetimes, the construction of Section 4.1 yields a 3-surface I_u which is **topologically** $\mathbf{S^2} \times \mathbf{R}$, but which inherits a nodal singularity from $\hat{M}$. Since these points must be removed from I_u, the resulting **manifold** is topologically $\mathbf{R^3}$, and this $(\hat{M}, \hat{g}_{ab})$ has not been shown to be asymptotically empty and flat at null infinity. However, the results of the previous section suggest (see Figure 4) that, if we extend $\hat{M}$ prior to the conformal completion, the resulting I might be regular on a full "$\mathbf{S^2}$-worth" of generators, since the metric is regular at $\vartheta = 0$, which corresponds to the "bottom halves" of **both** the generator $\psi = 0$ **and** the generator $\psi = \pi$. We will show that this is correct, and that the **extended** spacetimes are asymptotically empty and flat at null infinity. Furthermore, the same procedure is necessary even for regular (and flat) C−metrics in order to obtain a neighborhood of i^o in I^+.

For the remainder of this section let $(\hat{M}, \hat{g}_{ab})$ be the region of the spacetime of Eqs. (3.7, 3.8) given by $y < y_3$, where $y_3 = \infty$ by convention for those C−metrics with only one Killing horizon (i.e. types i, iii, iv).[17] Introduce new coordinates $(U, V, \vartheta, \varphi)$ via [19]: (Note that $\kappa^{-1} \equiv F'(y_2)$.)

$$U := e^{\kappa^{-1} A u} = e^{\kappa^{-1}(t+y^*)}$$

$$UV := \mathrm{sgn}(y - y_2) e^{2\kappa^{-1} y^*} \tag{4.24}$$

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[16]The restriction of the procedure in this section to the vacuum C−metrics is treated explicitly in [13].

[17]Recall that $y = y_1$ does **not** determine a Killing horizon since y can only take this value at $r = \infty$.

where

$$y^* = \int \frac{dy}{F(y)} \, .$$

Note that e^{y^*} is regular on the region under consideration although y^* is not. The metric now takes the form:

$$\hat{g}_{ab} dx^a dx^b = r^2 \left(\frac{\kappa^2 F(y) dU dV}{UV} + d\vartheta^2 + \kappa^2 \rho(\vartheta)^2 d\varphi^2 \right) \tag{4.25}$$

where y is to be thought of as a function of U and V determined implicitly by Eq. (4.24). $\hat{M}$ is now given by

$$\begin{aligned} &\vartheta \in [0, \vartheta_o) \\ &\varphi \in [0, 2\pi] \\ &U > 0 \\ -e^{2\kappa^{-1}x^*} &< UV < B \end{aligned} \tag{4.26}$$

where $B = 1$ if $y_3 = \infty$, and $B = \infty$ otherwise, and $x^* = \int \frac{dx}{G(x)}$; again, note that e^{x^*} is well-behaved although x^* is not. (Note that we have removed the points $\vartheta = \vartheta_o$ even for regular C−metrics.) We extend $\hat{m}$ to a new manifold $\hat{N}$ by dropping the restriction $U > 0$ from Eq (4.26); see Figure 5. Note that g_{UV} is C^∞ at $UV = 0$ (i.e. $y = y_2$), and is thus regular everywhere on $\hat{N}$. We wish to consider the surface defined by $UV = -e^{2\kappa^{-1}x^*}$. [18] Since y can take on the value y_1 (but not y_3) on this surface, and since g_{UV} is not defined there, we see that the $(U, V, \vartheta, \varphi)$ chart is not suited to the construction of I. As in the last section, we

[18] It is straightforward to check that this condition is equivalent to $y = -x$, i.e. $r = \infty$.

introduce a new coordinate $\tilde{u}$ on the region $V < 0$ of $\hat{N}$ by

$$A\tilde{u} := \kappa \ln(-V) \tag{4.27}$$

and work in the region $U > 0$, $V < 0$ of $\hat{N}$, which is the intersection of the $(u, y, \vartheta, \varphi)$ and $(\tilde{u}, y, \vartheta, \varphi)$ charts. On the intersection we have

$$Au + A\tilde{u} = 2y^* \tag{4.28}$$

and the two charts thus have a C^∞ overlap; the conformal metric $g_{ab} := r^{-2}\hat{g}_{ab}$ is given by

$$\begin{aligned} \hat{g}_{ab}dx^a dx^b &= -A^2F(y)du^2 + 2Adudy + d\vartheta^2 + \kappa^2\rho(\vartheta)^2 d\varphi^2 \\ &= -A^2F(y)d\tilde{u}^2 + 2Ad\tilde{u}\,dy + d\vartheta^2 + \kappa^2\rho(\vartheta)^2 d\varphi^2 \end{aligned} \tag{4.29}$$

We now extend each chart separately to include 3−surfaces at $r = \infty$, defined now by $x_2 + \int_0^\vartheta \rho(\vartheta)d\vartheta + y = 0$; denote these surfaces I_u and $I_{\tilde{u}}$, respectively. Again using the procedure of Section 4.1, we see that each of these is topologically $\mathbf{S^2} \times \mathbf{R}$, and can be coordinatized by (u, ψ, φ) and $(\tilde{u}, \tilde{\psi}, \varphi)$, respectively, with:

$$\begin{aligned} \frac{d\psi}{\rho(\psi)} &= -Adu + \frac{d\vartheta}{\rho(\vartheta)} \\ \frac{d\tilde{\psi}}{\rho(\tilde{\psi})} &= -Ad\tilde{u} + \frac{d\vartheta}{\rho(\vartheta)} \\ L_n\psi &= 0 = L_n\tilde{\psi} \end{aligned} \tag{4.30}$$

The intersection $I_u \cap I_{\tilde{u}}$ is again all of $I := I_u \cup I_{\tilde{u}}$ except those generators labeled by $\psi = 0$ in I_u and $\tilde{\psi} = 0$ in $I_{\tilde{u}}$. But since, on $I_u \cup I_{\tilde{u}}$,

$$Adu + Ad\tilde{u} = \frac{2d\vartheta}{\rho(\vartheta)} \tag{4.31}$$

we have

$$-\frac{d\psi}{\rho(\psi)} = \frac{d\tilde{\psi}}{\rho(\tilde{\psi})} \tag{4.32}$$

there. Thus, ψ and $\tilde{\psi}$ are smoothly related, and satisfy

$$\lim_{\psi\to 0} \tilde{\psi} = \vartheta_o \tag{4.33}$$

$$\lim_{\psi\to \vartheta_o} \tilde{\psi} = 0 \, .$$

Unlike the flat case, it appears here that we can only make statements about limits. However, this suffices to allow us to extend ψ (and $\tilde{\psi}$) to all of I. But we have thus shown that I is topologically $\mathbf{S^2} \times \mathbf{R}$; there are generators for **all** values of ψ in $[0, \vartheta_o]$. Furthermore, since $\vartheta \neq \vartheta_o$ on I, the pullback q_{ab} of g_{ab} to I is regular everywhere on I. From Eq. (4.2) we again see that the condition on $\hat{R}_{ab}$ is satisfied. We conclude that **all** the C−metrics considered here[7,10] are asymptotically empty and flat at null infinity. Finally, we can still conclude from Eqs. (3.9) and (3.10) that the C−metrics contain gravitational radiation (unless $e = m = 0$) and electromagnetic radiation (if $e \neq 0$), since it suffices to show that K_{ab} and the pullback of $F_{ab}\nabla^b\Omega$ to I fail to vanish identically on some open region of I, e.g. the $(u, y, \vartheta, \varphi)$ chart.

In complete analogy to the results of Section 4.2, one finds that all of the generators for nodal C−metrics are complete except for $\psi = 0$ and $\psi = \vartheta_o$, which are **both** incomplete in the future and complete in the past (see Figure 6). (This situation would be reversed if we had placed the nodes at $\vartheta = 0$ instead of at $\vartheta = \vartheta_o$.) In all cases, $(\hat{N}, \hat{g}_{ab})$ is **not** asymptotically Minkowskian.

5. Spatial Infinity[19]

We are now in a position to show that the C−metrics are also asymptotically flat at spatial infinity, and to analyze the structure there. In particular, we show that the ADM mass is zero, and we justify the interpretation of the Killing vectors ∂_t and ∂_φ as a boost and a rotation, respectively.

Note that the results of Chapter 4 are unchanged if we replace the conformal factor r^{-1} by

$$\Omega' := \frac{A}{2\kappa r} \equiv \frac{A}{2\kappa}\,\Omega\,; \tag{5.1}$$

i.e. we can show that the spacetime $(\hat{N}, \hat{g}_{ab})$ of Chapter 4 is asymptotically empty and flat at null infinity using the conformal factor Ω'. Let N be the extension of $\hat{N}$ obtained by adding the surface $\Omega' = 0$ **in the** $(U, V, \vartheta, \varphi)$ **chart** (i.e. those points with $UV = -e^{2\kappa^{-1}x^*}$), and set

$$g_{ab} := \Omega'^2 \hat{g}_{ab} = \frac{A^2}{4}\left[\frac{F(y)dUdV}{UV} + \kappa^{-2}d\vartheta^2 + \rho(\vartheta)^2 d\varphi^2\right] \tag{5.2}$$

We claim that the conformal completion (N, g_{ab}) of $(\hat{N}, \hat{g}_{ab})$ already suffices to establish asymptotic flatness at spatial infinity, and that the point Q:=$(U = 0 = V,\ \vartheta = 0)$ is i^o: (N, g_{ab}) is C^∞ **everywhere,** including at Q, while $\{\Omega' = 0\} \equiv Q \cup I$, where $I := I_{\cup} I_{\tilde{u}}$ as in Chapter 4. Thus, in order for Definition 2 to be satisfied, it only remains to show that, at Q:

$$\nabla_a \Omega' = 0 \tag{5.3}$$

$$\nabla_a \nabla_b \Omega' = 2 g_{ab}.$$

---

[19]The results in this section appear in [20].

Since $\vartheta = 0$ at Q,[20] the 2−sphere coordinates (ϑ, φ) of our chart are badly behaved there. We therefore replace them in a neighborhood of Q with

$$\xi := \kappa\rho(\vartheta)\cos\varphi \tag{5.4}$$

$$\eta := \kappa\rho(\vartheta)\sin\varphi$$

and do our calculations in the (U, V, ξ, η) chart. It is now straightforward (using l'Hôpital's rule) to verify that Eq. (5.3) is satisfied. We conclude that the C−metrics are asymptotically flat at spatial infinity.

We classify the Killing vector fields of the C−metrics by their behavior near i^o (compare [21]). We must, of course, work in coordinates which are well-behaved in a neighborhood of i^o. In the (U, V, ξ, η) chart we have

$$\partial_t = \kappa^{-1}U\partial_U - \kappa^{-1}V\partial_V \tag{5.5}$$

and

$$\partial_\varphi = \xi\partial_\eta - \eta\partial_\xi.$$

This is precisely the behavior we expect for a boost and a rotation, respectively.

We now turn to the ADM mass. We have seen that the C−metrics admit a C^∞ differential structure at i^o. But C^1 suffices in order to conclude that the ADM 4−momentum P_a vanishes [14]! This result is, however, no surprise.[21] There are two independent arguments which show that P_a must vanish **if** it is well-defined, i.e. **if** i^o exists. The first of these [22] is that the ADM 4−momentum must be invariant under the action of any Killing vector fields on the physical spacetime $(\hat{M}, \hat{g}_{ab})$. We have just seen that the Killing vector fields of the

[20] Note that Q is **not** the origin in the $(U, V, \vartheta, \varphi)$ chart.

[21] However, see the discussion in Chapter 7.

C−metrics may be interpreted as a boost (∂_t) and a rotation (∂_φ); furthermore, these vector fields commute with each other. But the only vector left invariant under the action of **both** ∂_t and ∂_φ is the zero vector.

The second argument is based on the following statement: Assuming that the Bondi news tensor has the appropriate falloff at i^o, the limit of the Bondi 4−momentum at i^o will be precisely P_a [23]. It is shown in the Appendix that, at least for vacuum C−metrics, the condition on the news tensor is satisfied, and that the Bondi 4−momentum goes to zero at i^o. Michael Streubel has also shown this for regular C−metrics with $|e| = m$ [24].

Finally, note that although r was a "good" radial coordinate near I (i.e. $r^{-1} \approx \Omega'$), it is a **bad** radial coordinate near i^o (where one desires $r^{-2} \approx \Omega'$). In particular, note that for the flat C−metrics of Section 4.2 one has $\bar{r} \neq r$, where $\bar{r}$ is the radial coordinate of Minkowski space. However, this behavior seems to be a natural consequence of the accelerated motion.

6. Global Structure

We have shown that, although the C−metrics are **not** AEFANSI spacetimes, the only requirement of an AEFANSI spacetime which they fail to meet is that the generators be complete. Although this complicates the analysis of the global structure considerably,[22] a more important problem is the lack of a good angular coordinate to replace ϑ, e.g. when trying to draw Penrose diagrams. The problem is that the intersection of $\{(\vartheta, \varphi)$=constant$\}$ hypersurfaces with I is **not,** in general, null. However, as was shown in [13], the "Penrose diagrams" of [4] **are** essentially correct in that they display the basic global structure. We now summarize the results of [4,13].

Now that we have shown that the underlying **extended** spacetime $(\hat{N}, \hat{g}_{ab})$ is asymptotically flat, we may use the procedure of [19] to determine the maximal extension of the conformal spacetime (N, g_{ab}) by successively extending across the Killing horizons $y = y_i$ with $i \neq 1$. This results in the "Penrose diagrams" of [4][23], which are partially reproduced in Figure 7 for 2−root C−metrics (types i, iii, iv) and in Figure 8 for vacuum C−metrics. It is clear from these that in the former case, which includes the regular C−metrics, the singularities at $r = 0$ are naked singularities and that all partial Cauchy surfaces are contained in the causal past of I^+. These results still hold for type v) C−metrics, although the topology of the resulting extended spacetime is quite complicated, and there is more than one asymptotic region. However, for the vacuum C−metrics, the $r = 0$ singularities are spacelike, and lie behind the event horizons at $y = y_3$. (The surfaces $\{y = y_3\}$ are event horizons despite the fact that we have only

[22]In particular, many of the theorems in [16] no longer apply; the C−metrics fail to be future asymptotically predictable.

[23]Note that in [4] asymptotic flatness is **assumed.**

considered 2−surfaces with (ϑ, φ) constant.) Thus we may say that the vacuum C−metrics represent black holes; the partial Cauchy surface S in Figure 8 is such that $S - S \cap J^{-}(I^{+})$ has two disconnected components. However, as pointed out in [13], the standard definition of a black hole requires complete generators on I^{+} and is simply not designed for "constantly accelerating black holes".

7. Positive Energy Conjectures

If we accept the physical interpretation of [4], namely that the C−metrics describe two particles of (positive) mass m and charge e undergoing constant acceleration A, then we intuitively expect that the total mass (energy) of the system should be positive. Yet we have just seen that the ADM mass, which one interprets as the total mass, is zero for **all** C−metrics. Furthermore, since the Bondi news is nonzero in any neighborhood of i^o (except for the flat C−metrics), this implies that the Bondi mass, whose interpretation at any instant of retarded time is the total mass minus the energy radiated away prior to that time, is in fact negative in a neighborhood of i^o on I^+ [23].[24]

We attempt to clarify these points in this chapter. In Section 7.1 we first briefly discuss the existing positive mass theorems and conjectures, and show that **none** of the C−metrics violate any of the existing formulations. Also in Section 7.1 we present the **intuitive** resolution for nodal C−metrics of the apparent conflict between zero ADM mass and failure to violate positive mass conjectures. In Section 7.2 we examine regular C−metrics, in particular the case $|e| = m > \frac{1}{4A}$. (The case $m = 0$ is not very interesting.) We show that for these parameter values there are probably no bound test particle orbits, and argue that these C−metrics represent the fields of particles with **negative** mass, contrary to the usual physical interpretation. It is also shown that most of the C−metrics with a Reissner-Nordström limit[25] in fact **do** have bound test particle orbits, thus reinforcing the standard interpretation of these C−metrics.

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[24]This is explicitly verified for the vacuum C−metrics in the Appendix. Michael Streubel [24] has also checked this for $|e| = m > \frac{1}{4A}$. In both cases, the required condition on the falloff of the news tensor in [23] is satisfied.

[25]More precisely, for all such C−metrics except $|e| > m$ and $m = e = 0$.

7.1 Statement of Conjectures, and Relevance to C−Metrics[26]

We first state several conjectures about the positivity of the Bondi mass $M_{\rm B}$ on I^+. Let $(\hat{M}, \hat{g}_{ab})$ be an AEFANSI[27] spacetime satisfying the dominant energy condition.

Conjecture 1: If there exists a global, regular Cauchy surface, then $M_{\rm B} \geq 0$ everywhere on I^+.

Conjecture 2: If, in the (maximal) conformally completed manifold, there exists an acausal hypersurface S with (disconnected) boundary at I^+ and at an apparent horizon, which intersects all nonspacelike curves which start on I^+ to the future of S, then $M_{\rm B} > 0$ to the future of S. (In fact, one suspects $M_{\rm B} \geq \left(\frac{\rm SA}{16\pi}\right)^{1/2}$, where SA is the area of the apparent horizon.) See Figure 9.

Conjecture 3: If, in the (maximal) conformally completed manifold, there exists an acausal hypersurface S with boundary at I^+, which intersects all nonspacelike curves which start on I^+ to the future of S, then $M_{\rm B} \geq 0$ to the future of S. See Figure 10.

We will ignore the fact that the C−metrics are not *AEFANSI*; we expect that, so long as the other hypotheses are satisfied, the conjectures should still be applicable to the C−metrics.[28,29] We will consider each of these conjectures in

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[26] I am grateful to Pong Soo Jang for summarizing the state of the art for Bondi mass conjectures, and to Richard Schoen for discussions of ADM mass conjectures.

[27] Asymptotically Minkowskian is probably sufficient.

[28] Throughout this chapter we will ignore the flat C−metrics: Although the results as stated are also applicable to these C−metrics, the $r = 0$ "singularity" there is only a coordinate singularity, and could thus be removed.

[29] However, we must now **assume** that all apparent horizons lie behind or on an event horizon; i.e. cannot be seen from I^+. Although one expects this to be true if the physical interpretation is correct, the standard proof [16] doesn't work here. See Footnote 22. (For the definition of an appar-

turn, and show that the result that the Bondi mass is negative in a neighborhood of i^o on I^+ does **not** lead to any contradictions. We showed in Chapter 3 that the C−metrics satisfy the dominant energy condition. However, it is easy to see that the C−metrics have **no** global, regular Cauchy surfaces: One can never predict the future of the bullet holes from data given in their past. For the same reason, the hypersurface required in Conjecture 2 cannot exist (since at least one bullet hole would lie in its future), while that required in Conjecture 3 must intersect I^+ to the future of both bullet holes. Thus, $M_{\rm B} < 0$ in a neighborhood of i^o in I^+ is not in any way inconsistent with these conjectures. Furthermore, for nodal C−metrics it is essential to place the nodes at $\vartheta = 0$ in order to discuss Conjecture 3. As is shown in the Appendix, when this is done for vacuum C−metrics the Bondi mass on crosssections in the future of the bullet holes is, in fact, positive. Although we have not checked this explicitly for charged C−metrics, we expect to have no difficulty showing that, as in the vacuum case, $M_{\rm B}$ approaches zero in the limit towards I^+; which implies $M_{\rm B} \geq 0$ to the future of the bullet holes.

We now turn to the ADM mass $M_{\rm ADM}$, and the recent results of Schoen and Yau [25,26].[30] Let $(\hat{M}, \hat{g}_{ab})$ be asymptotically flat at spatial infinity and satisfy the dominant energy condition.

ent horizon see [16].)

[30]These results are presented here as conjectures, rather than as theorems, because they are expressed in a different language than the theorems proved in [25,26]. The exact correspondence has not been verified in detail.

Conjecture 4: If there exists a regular, geodesically complete, spacelike hypersurface diffeomorphic to $\mathbf{R}^3$, then $M_{\mathrm{ADM}} \geq 0$, with equality holding if and only if $\hat{M}$ is (a submanifold of) Minkowski space.

Conjecture 5: If there exists a regular, geodesically complete, spacelike hypersurface with (disconnected) boundary at i^o and at an apparent horizon, which is diffeomorphic to $\mathbf{R}^3$ minus a ball, then $M_{\mathrm{ADM}} \geq 0$, with equality holding as above.[31] See Figure 11.

However, there are **no** geodesically complete spacelike hypersurfaces for the C−metrics. (This is stronger than the statement above that there are no global Cauchy surfaces.) This is clear for those C−metrics in which the $r = 0$ singularity is timelike (deg $F(y) = 4$), but us also true for the vacuum C−metrics, since there are timelike curves from I^+ to I^- with $\vartheta = \vartheta_o$. Thus, Conjecture 4 does not apply. The same argument shows that Conjecture 5 does not apply (to the vacuum C−metrics) when one notices that the above timelike curves with $\vartheta = \vartheta_o$ must lie entirely outside the horizon $y = y_3$, whereas one expects that all apparent horizons should lie inside this horizon.

In conclusion, the only Conjecture whose hypotheses can be satisfied is Conjecture 3, and, if the nodes are placed so that Conjecture 3 is applicable, we find that the result ($M_{\mathrm{B}} \geq 0$) does, in fact, hold.

Now that we have shown that $M_{\mathrm{ADM}} = 0$ does not lead to any contradictions, we present an **intuitive** explanation of what is going on:[32] The physical

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[31] One can generalize this to allow for the possibility of several apparent horizons and/or several asymptotic regions. The result proved in [25,26] is in fact stronger than the statement here; the "apparent horizon" requirement can be weakened considerably.

[32] This point of view is implicitly contained in [4].

interpretation of the C−metrics [4] provides no explanation of the acceleration of the particles. One thus interprets [4,13] the 2−dimensional sheet of nodal singularities at $\vartheta = \vartheta_o$ as a strut keeping the particles apart. Alternatively, placing the nodes at $\vartheta = 0$, one interprets the two resulting sheets of singularities as strings holding the particles apart. In either case, one interprets $M_{\rm ADM} = 0$ as representing a positive contribution from the particles and a negative contribution from the potential energy of the strut or strings.

Supporting this point of view, Ernst [6,7, see also 8] has shown that the introduction of a gravitational or electric field to provide a physical explanation for the acceleration can be done so as to remove the nodal problems.

7.2 Test Particle Orbits

The intuitive explanation of zero total mass as presented in the last section for nodal C−metrics does not work for the regular C−metrics. In fact, there is a simple argument which seems to indicate that the mass of the two particles described by the regular C−metrics is negative: From Conjecture 3 of the last section, we expect the Bondi mass to be positive in the future of the bullet holes on I^+,[33] and we know the Bondi mass is negative to the past of the bullet holes on I^+. Thus, it seems that the particles must carry away negative mass in order to explain the jump from negative to positive Bondi mass.[34] Note that this argument makes no sense at all for nodal C−metrics, since one cannot then discuss the Bondi mass to the future **and** to the past of the bullet holes.

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[33] This has only been explicitly verified for the vacuum C−metrics.

[34] An alternative explanation is that the large amount of radiation hitting I^+ near the bullet holes carries negative energy. Since the radiation elsewhere appears to carry positive energy, we will assume that this is also the case in a neighborhood of each bullet hole.

We will not examine the case $m = 0$ in detail. These C−metrics do have a well-behaved limit to the corresponding Reissner-Nordström spacetimes, and we expect that any nonstandard properties of these C−metrics will either carry over to the Reissner-Nordström spacetimes or be negligible for small A.

We start our discussion of the case $|e| = m > \frac{1}{4A}$ by emphasizing that these C−metrics **DO NOT** have a Reissner-Nordström limit: As $A \to 0$ (with e and m held constant), we eventually reach $|e| = m = \frac{1}{4A}$, after which we must either redefine ϑ (and thus get a Reissner-Nordström limit at the cost of introducing nodal singularities) or have our metric no longer remain Lorentzian. We will now show that these C−metrics probably do not have any bound test particle orbits, although (most of) those C−metrics with a Reissner-Nordström limit do have bound orbits.

We can write the C−metrics using coordinates $(\tau, r, \vartheta, \varphi)$, where $\tau = A^{-1}t$, as

$$\hat{g}_{ab}dx^a dx^b = -A^2 r^2 F(y) d\tau^2 + \frac{dr^2}{A^2 r^2 F(y)} + \frac{2\rho(\vartheta) dr d\vartheta}{AF(y)} \tag{7.1}$$

$$+ r^2\left(1 + \frac{\rho(\vartheta)^2}{F(y)}\right) d\vartheta^2 + r^2 \kappa^2 \rho(\vartheta)^2 d\varphi^2$$

where y is now regarded as a function of r and ϑ.

The timelike geodesics of the C−metrics can now be characterized as follows:

$$\lambda^m \nabla_m \lambda^a = 0$$

$$\lambda^a = \dot{\tau}(\partial_\tau)^a + \dot{r}(\partial_r)^a + \dot{\vartheta}(\partial_\vartheta)^a + \dot{\varphi}(\partial_\varphi)^a$$

$$\dot{\varphi} = \frac{J_z}{r^2\kappa^2\rho(\vartheta)^2} \tag{7.2}$$

$$\dot{\tau} = \frac{E}{A^2 r^2 F(y)}$$

$$\pm\dot{\vartheta} = r^{-2}\left[J^2 - \frac{J_z^2}{\kappa^2\rho(\vartheta)^2}\right]^{1/2}$$

$$\pm\dot{r} = -A\rho(\vartheta)\left[J^2 - \frac{J_z^2}{\kappa^2\rho(\vartheta)^2}\right]^{1/2}$$

$$\pm\left[E^2 - A^2 r^2 F(y) M^2 - A^2 F(y) J^2\right]^{1/2}$$

where J_z and E are the constants of the motion corresponding to the two Killing vectors (which can be interpreted as the z−component of angular momentum and the "energy" of the test particle in the rest frame of the particle at $r = 0$, respectively, and

$$-M^2 := -A^2 R^2 F(y)\dot{\tau}^2 + \frac{\dot{r}^2}{A^2 r^2 F(y)} + \frac{2\rho(\vartheta)}{AF(y)}\dot{r}\dot{\vartheta} \tag{7.3}$$

$$+r^2\left(1 + \frac{\rho(\vartheta)^2}{F(y)}\right)\dot{\vartheta}^2 + r^2\kappa^2\rho(\vartheta)^2\dot{\varphi}^2$$

is the mass of the test particle, and

$$J^2 := r^4\dot{\vartheta}^2 + r^4\kappa^2\rho(\vartheta)^2\dot{\varphi}^2. \tag{7.4}$$

J^2 is not in general a constant of the motion, but satisfies

$$(J^2)^{\cdot} = 2A\rho(\vartheta)M^2r^3\dot{\vartheta}. \tag{7.5}$$

We will restrict ourselves to the case $\rho(\vartheta) \neq 0 \neq F(y)$, $M > 0$, $\dot{\vartheta} = 0$. One can then interpret J as the total angular momentum of the test particle; it is associated with the conformal Killing tensor $\hat{K}^{ab}$. The last condition ensures that J^2 is in fact a constant of the motion, and that we can find **all** such geodesics explicitly. At the end of this section, when we discuss the Newtonian analog, we will give a plausibility argument that there should be bound geodesics if and only if there are bound geodesics with $\dot{\vartheta} = 0$. The remaining conditions ensure that we are not at a Killing horizon, that we are not at a pole of the $\{(y, t)$=constant$\}$ 2−spheres (where we certainly don't expect any bound geodesics), and that our test particle has some (nonzero) mass.

However, **not** all values of E, J, J_z, and M yield solutions of the second-order geodesic equations.[35] One finds that $\dot{\vartheta} = 0$ forces

$$J^2\rho'(\vartheta) \equiv -A\rho(\vartheta)^2r^3M^2 \tag{7.6}$$

in order for $\ddot{\vartheta} \equiv 0$ to be satisfied. Note that this immediately yields $J^2 \neq 0$ and $\dot{r} = 0$,[36] which in turn forces

$$J^2F'(y) = Ar^3\left(2F(y) - \frac{F'(y)}{Ar}\right)M^2 \tag{7.7}$$

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[35]I am indebted to Charles Misner for pointing out the existence of spurious solutions for $\dot{r} = 0$ and/or $\dot{\vartheta} = 0$. Eqs. (7.6) and (7.7) can be derived from Eq. (7.2) by **careful** differentiation.

[36]For $A = 0$ this condition correctly reduces to $\vartheta = \frac{\pi}{2}$ or $J = 0$, and we cannot conclude anything about $\dot{r}$.

in order for $\ddot{r} \equiv 0$ to be satisfied. Combining these two equations we obtain

$$F'(y)\,(\rho(\vartheta)^2 - \frac{\rho'(\vartheta)}{Ar}) = -2\rho'(\vartheta)F(y)\,. \tag{7.8}$$

Also, the defining equation for M^2 is now

$$E^2 = A^2F(y)(r^2M^2 + J^2)\,. \tag{7.9}$$

Eq. (7.9) implies that $F(y) > 0$, while Eq. (7.6) implies that $\rho'(\vartheta) < 0$;[37] from Eq. (7.8) we now conclude that $F'(y) > 0$. We rewrite Eq. (7.8) as a polynomial in x and y using $\rho(\vartheta)^2 = G(x)$ and $2\rho'(\vartheta) = G'(x)$:

$$B(x, y) := F'(y)G(x) + F(y)G'(x) - ½(x + y)F'(y)G'(x)\,. \tag{7.10}$$

We have thus shown that $B(x, y) = 0$ is a necessary condition for the existence of bound test particle orbits. However, we now show that for regular C−metrics with $|e| = m > \frac{1}{4A}$ we cannot have $B(x, y) = 0$ subject to the above constraints, namely $G(x) > 0$, $G'(x) < 0$, $F(y) > 0$, and $F'(y) > 0$.

The graphs of $G(x)$ and $F(y)$ for these C−metrics are given in Figure 12. Note that the conditions on $G(x)$ and $G'(x)$ restrict us to the regions $-\frac{1}{mA} < x < -\frac{1}{2mA}$ or $0 < x < \frac{-1+\sqrt{1+4mA}}{2mA}$, while those on $F(y)$ and $F'(y)$ force $y > \frac{1+\sqrt{1+4mA}}{2mA}$. We will treat $B(x, y)$ as a fourth order polynomial in x for fixed y. Note that

$$B(-\frac{1}{mA}, y) = B(0, y) = F'(y) > 0$$

$$0 < B(-\frac{1}{2mA}, y) = G(-\frac{1}{2mA})F'(y) < F'(y)$$

[37]We interpret this physically as a dragging effect due to the accelerated coordinate system. One would normally have expected $\rho'(\vartheta) = 0$, which corresponds to the equatorial plane in the limit as $A \to 0$. Compare the discussion of the Newtonian analog at the end of this Section.

$$B(-y, y) = 0 \tag{7.11}$$

$$-y < \frac{-1 - \sqrt{1 + 4mA}}{2mA} < -\frac{1}{mA}$$

$$\lim_{x \to \pm\infty} B(x, y) = +\infty$$

See Figure 13. These equations guarantee that $B(x, y)$ is monotonically increasing for $x > 0$, and thus that $B(x, y) \neq 0$ for $0 < x < \frac{-1 + \sqrt{1 + 4mA}}{2ma}$. However, since $G'(-\frac{1}{2mA}) = 0$, $G''(-\frac{1}{2mA}) = 1$, and $G'''(-\frac{1}{2mA}) = 0$, we have

$$\partial_x{}^2 B(-\frac{1}{2mA}, y) \equiv G'''(-\frac{1}{2mA})\left[F(y) - ½(-\frac{1}{2mA} + y)F''(y)\right] = 0 \tag{7.12}$$

and thus $x = -\frac{1}{2mA}$ is a point of inflection of the graph of $B(x, y)$. But since $x = -\frac{1}{2mA}$ lies to the right of the local maximum of $B(x, y)$, it is clear that we must have $\partial_x B(-\frac{1}{2mA}, y) < 0$ and therefore $B(x, y) > 0$ for $-\frac{1}{mA} < x < -\frac{1}{2mA}$. (To be slightly more rigorous, $\partial_x{}^3 B(-\frac{1}{2mA}, y) > 0$, and thus $\partial_x B(x, y)$ has a local minimum at $x = -\frac{1}{2mA}$.) In any case, we see that $B(x, y) \neq 0$ subject to the above constraints on x and y, and we conclude that the regular C−metrics with $|e| = m > \frac{1}{4A}$ have **no** bound timelike geodesics with $\dot{\vartheta} = 0$.

Note that if $G'(x) = 0$ and $F'(y) = 0$ then $B(x, y) = 0$ and Eqs. (7.6) and (7.7) are satisfied (for $M = 0$). For vacuum C−metrics and for C−metrics with $|e| = m < \frac{1}{4A}$,[38] we can choose (x_o, y_o) such that $G'(x_o) = 0 = F'(y_o)$,

[38]Or, more generally, for any C−metric of type v. These, together with the vacuum C−metrics, comprise all the C−metrics with a Reissner-Nordström limit except $|e| > m$ and $m = e = 0$.

$G(x_o) > 0$, $F(y_o) > 0$, and $G''(x_o) \neq 0 \neq F''(y_o)$. We can choose E and J so that Eq. (7.9) is satisfied, thus obtaining a bound photon orbit. However, since

$$\partial_x B(x_o, y_o) = G''(x_o)F(y_o) \neq 0 \tag{7.13}$$

$$\partial_y B(x_o, y_o) = F''(y_o)G(x_o) \neq 0$$

there exists a neighborhood W of x_o on which we can find $y(x)$ so that $B(x, y(x)) = 0$, with $y(x_o) = y_o$ and $F(y(x)) > 0$. Furthermore, using $\frac{dy}{dx} = -\frac{\partial_x B(x, y)}{\partial_y B(x, y)}$ we find that choosing $x \in W$ such that $G'(x) < 0$ forces $F'(y(x)) > 0$! We can now use Eq. (7.6) to define $\frac{M}{J}$ and Eq. (7.9) to define $\frac{E}{J}$, thus obtaining a bound timelike geodesic (Eq. (7.7) is automatically satisfied since $B(x, y(x)) \equiv 0$), and strongly suggesting that each particle at $r = 0$ has positive mass.

Let us now consider the Newtonian version of the situation. We consider a single point mass undergoing constant acceleration. In the rest frame of the particle we can replace the acceleration by a constant gravitational field. Assuming that our mass is positive, there will be a surface on which the component of the field of the point mass in the direction of the constant field exactly cancels out the constant field, resulting in the existence of bound orbits (see Figure 14).[39] Note that these orbits occur only for $\frac{\pi}{2} < \vartheta < \pi$; the orbits are dragged along behind the source of the acceleration. Furthermore, there are bound orbits with $\dot{\vartheta} = 0 = \dot{r}$. Finally, note that there are no bound orbits if the source has negative mass. We take this as a strong indication that a relativistic accelerating particle

[39]Note that the word "bound" is perhaps not quite correct since the constant acceleration will drag the test particle with it to infinity (in the relativistic case: to $\imath$). I am grateful to Bill Cordwell for suggesting this argument, and for providing Figure 14.

has positive mass if and only if there are bound orbits in the sense discussed here.

Note that even though the separation of the two particles at $r = 0$ is on the order of m at closest approach, they are hidden from each other by the accelerated motion, and thus have **no** effect on each other or on the test particle orbits just considered. Furthermore, the close approach makes it impossible to approximate the mass of either particle by applying Kepler's laws to these test particle orbits.

We were unable to show there are **no** bound test particle orbits for $|e| = m > \frac{1}{4A}$ because we could not solve the general geodesic equations. However, we interpret our failure to find any bound test particle orbits with $\dot{\vartheta} = 0$ as strongly supporting the conclusion that the C−metrics with $|e| = m > \frac{1}{4A}$ describe particles with negative mass.

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Appendix: Bondi Mass Calculation for Vacuum C−Metrics[40]

We follow Geroch [17] in defining the Bondi Mass in a divergence−free conformal frame on a crosssection S of I as:

$$M_{\mathrm{B}}(S,\alpha) := \frac{1}{8\pi}\int_S (Y_\alpha)^m \varepsilon_{mab}\, dS^{ab} \tag{A.1}$$

where

$$(Y_\alpha)^a := ¼\ \alpha \underline{K}^{am} l_m + (\alpha D_m l_n + l_m D_n \alpha) q^{np} N_{pq} q^{q[m} n^{a]}$$

and where q^{ab} is any "inverse" of q_{ab} (i.e. $q_{am}q^{mn}q_{nb} = q_{ab}$), l_m is any 1-form satisfying $l_m n^m = 1$, αn^a is a BMS time translation, N_{pq} is the news tensor, D_m is the derivative operator intrinsic to I, and $\underline{K}^{ab}$ is constructed from the Weyl tensor. Note that this depends both on the choice of crosssection and on the choice of time translation. We proceed to work out each of these terms, but first clarify the general approach. We use the same notation as [17] throughout. The final integration will be technically difficult, because of the necessity of integrating in both the u and $\tilde{u}$ charts. However, since everything so far is invariant under the transformation $(u,\psi) \to (\tilde{u},\tilde{\psi})$, we will take advantage of this symmetry to reduce the integral to an integral over **part** of the crosssection S, which we can then integrate in **one** chart. We proceed as follows: Anywhere we have a choice we choose all tensors so that they are invariant under the above transformation; we refer to this invariance as **functional symmetry.** This is the case e.g. with the selection of l_m and the choice of crosssection. Thus, the resulting integrand will also have this symmetry, i.e. will have **exactly** the same functional form in both charts. Furthermore, since the integrand is conformally invariant, we can even work in two **different** conformal frames, one for each chart. In

[40]This calculation was done jointly with Michael Streubel, to whom I am deeply indebted.

fact, this appears to be necessary: There does not seem to be a **global** divergence-free frame!

We introduce some notation. Let h be the function determined implicitly by Eq. (4.22); i.e. $\psi = h(\tilde{\psi})$, and of course also $\tilde{\psi} = h(\psi)$. Define the function σ via

$$\sigma(\vartheta) := \int \frac{d\vartheta}{\kappa\rho(\vartheta)} \tag{A.2}$$

Thus, from Eq. (4.22), $\sigma(h(\psi)) = -\sigma(\psi)$, and $h^2(\psi) = \psi$. Define ψ_o as the **unique** point where $h(\psi_o) = \psi_o$. This is also the unique point where $\sigma(\psi_o) \equiv 0$. Assuming that the integrand dM is independent of φ and functionally symmetric (i.e. $dM = f(\psi)d\psi \equiv -f(\tilde{\psi})d\tilde{\psi}$, where the factor of –1 comes from the opposite orientations of ψ and $\tilde{\psi}$) we can write symbolically

$$\begin{aligned} M &= \frac{1}{8\pi}\int dM = \frac{1}{8\pi}\int_{\psi=0}^{\vartheta_o} f(\psi)d\psi \\ &= \tfrac{1}{4}\int_{\psi=0}^{\psi_o} f(\psi)d\psi - \tfrac{1}{4}\int_{\tilde{\psi}=\psi_o}^{0} f(\tilde{\psi})d\tilde{\psi} \\ &= \tfrac{1}{2}\int_{\psi=0}^{\psi_o} f(\psi)d\psi \end{aligned} \tag{A.3}$$

and we can evaluate this integral without going into the $\tilde{u}$ chart! We now proceed to determine $f(\psi)$, making sure that the above assumptions are satisfied.

We do the entire calculation in the divergence-free conformal frame (u, ψ, φ) first introduced in Section 4.1. We have (dropping the primes)

$$q_{ab}dx^a dx^b = d\psi^2 + \kappa^2\rho(\psi)^2 d\varphi^2 \tag{A.4}$$

$$n^a = \frac{\rho(\vartheta)}{\rho(\psi)}(\bar{\partial}_u)^a$$

where $\vartheta = \vartheta(u, \psi)$ satisfies $\dfrac{d\vartheta}{\rho(\vartheta)} = \dfrac{d\psi}{\rho(\psi)} + A du$, i.e. $\sigma(\vartheta) \equiv \sigma(\psi) + \dfrac{Au}{\kappa}$. We first settle the terms where we have some choice. We choose

$$q_{ab} = (\bar{\partial}_\psi)^a (\bar{\partial}_\psi)^b + \frac{1}{\kappa^2 \rho(\psi)^2} (\bar{\partial}_\varphi)^a (\bar{\partial}_\varphi)^b \ . \tag{A.5}$$

When we express q^{ab} in the $(\tilde{u}, \tilde{\psi}, \varphi)$ divergence-free conformal frame to obtain $\tilde{q}^{ab}$, it is clear that $\tilde{q}^{ab}$ is functionally the same as q^{ab}. The choice of l_m is not quite so obvious. In an attempt to keep everything independent of φ we try

$$l_m dx^m = \frac{\rho(\psi)}{\rho(\vartheta)} \left(du + \frac{b}{\rho(\psi)} \right) d\psi \ ,$$

where b is a constant. Since $du = d\tilde{u} + \dfrac{2}{A\rho(\tilde{\psi})} d\tilde{\psi}$, we have

$$\tilde{l}_m dx^m = \frac{\rho(\tilde{\psi})}{\rho(\vartheta)} \left(d\tilde{u} + \frac{2}{A\rho(\tilde{\psi})} d\tilde{\psi} - \frac{b}{\rho(\tilde{\psi})} d\tilde{\psi} \right),$$

where we have, of course, changed conformal frames. Functional symmetry forces $\dfrac{2}{A} - b \equiv b$, and thus

$$l_m dx^m = \frac{\rho(\psi)}{\rho(\vartheta)} \left(du + \frac{1}{A\rho(\psi)} d\psi \right). \tag{A.6}$$

We next turn to the translations αn^a, which are the solutions to

$$D_a D_b \alpha + ½ \alpha \rho_{ab} = f_\alpha q_{ab} \tag{A.7}$$

with $L_n \alpha = 0$, i.e. $\bar{\partial}_u \alpha = 0$ and where f_α is an arbitrary function. The derivative operator D_a is determined from ∇_a via

$$D_a \underline{k}_b := \underleftarrow{\nabla_a k_b} \tag{A.8}$$

where the arrow denotes the pullback of to I, and k_b is **any** 1−form on M whose

pullback to I is $\underline{k}_b$. Note that this definition only makes sense in a divergence-free conformal frame. Eq. (A.8) enables us to determine the connection $\gamma^c{}_{ab}$ of D_a in terms of the connection $\Gamma^c{}_{ab}$ of ∇_a. After a lengthy but straightforward calculation, one obtains

$$\gamma^c{}_{ab}dx^a dx^b = -A\rho'(\vartheta)(\bar{\partial}_u)^c du^2 - \kappa^2\rho(\psi)\rho'(\psi)(\bar{\partial}_\psi)^c d\varphi^2 \tag{A.9}$$

$$+2\,\frac{\rho'(\psi)-\rho'(\vartheta)}{\rho(\psi)}\,(\bar{\partial}_u)^c du d\psi + 2\,\frac{\rho'(\psi)}{\rho(\psi)}\,(\bar{\partial}_\varphi)^c d\varphi d\psi\ .$$

The tensor ρ_{ab} is the **unique** symmetric tensor satisfying

$$\rho_{ab}n^b = 0;\ \rho_{ab}q^{ab} = -\frac{2\rho''(\psi)}{\rho(\psi)};\ D_{[a}\rho_{b]c} = 0 \tag{A.10}$$

where the right side of the second of these equations is just the scalar curvature of q_{ab}. Direct calculation yields

$$\rho_{ab}dx^a dx^b = \left(\frac{\rho'(\psi)^2-\kappa^{-2}}{\rho(\psi)^2} - \frac{2\rho''(\psi)}{\rho(\psi)}\right)d\psi^2 + \kappa^2\left(\kappa^{-2}-\rho'(\psi)^2\right)d\varphi^2. \tag{A.11}$$

One can now substitute Eq. (A.11) into Eq. (A.7) to obtain

$$\alpha,_{\varphi\varphi} + \kappa^2\rho(\psi)\rho'(\psi)\alpha,_\psi + \tfrac{1}{2}\alpha\left(1-\kappa^2\rho'(\psi)^2\right) = f_\alpha\kappa^2\rho(\psi)^2$$

$$\alpha,_{\psi\varphi} = \frac{\rho'(\psi)}{\rho(\psi)}\,\alpha,_\varphi \tag{A.12}$$

$$\alpha,_{\psi\psi} + \tfrac{1}{2}\alpha\left(\frac{\rho'(\psi)^2-\kappa^{-2}}{\rho(\psi)^2} - 2\,\frac{\rho''(\psi)}{\rho(\psi)}\right) = f_\alpha\ .$$

Unfortunately, we have been unable to solve these equations directly, so we resort to the following trick. Define Θ via $\dfrac{d\Theta}{\sin\Theta} := \dfrac{d\psi}{\kappa\rho(\psi)}$, which implies that

$$q_{ab}dx^a dx^b = \frac{\kappa^2\rho(\psi)^2}{\sin^2\Theta}\,(d\Theta^2 + \sin^2\Theta d\varphi^2)\ . \tag{A.13}$$

But the right side of this equation is conformally the standard 2−sphere metric, for which we know that the general translation is given by $\alpha' n^a$ with

$$\alpha' = a + b\,\sin\Theta\,\cos\varphi + c\,\sin\Theta\,\sin\varphi + d\,\cos\Theta\,. \tag{A.14}$$

Since the translations behave very simply under conformal transformations ($\alpha' = \omega\alpha$), we can obtain the solutions to Eq. (A.12) from Eq. (A.13) merely by multiplying by the conformal factor $\omega := \dfrac{\kappa\rho(\psi)}{\sin\Theta}$, and using $\ln\tan\dfrac{\Theta}{2} \equiv \sigma(\psi)$, i.e. $\sin\Theta \equiv \dfrac{1}{\cosh\sigma(\psi)}$. The general translation is thus given by αn^a, where

$$\alpha = a\,\kappa\rho(\psi)\cosh\sigma(\psi) + b\,\kappa\rho(\psi)\cos\varphi \tag{A.15}$$

$$+\,c\,\kappa\rho(\psi)\sin\varphi + d\,\kappa\rho(\psi)\sinh\sigma(\psi)$$

which we write as $\alpha = a\alpha_0 + b\alpha_1 + c\alpha_2 + d\alpha_3$, and one can check directly that this is the general solution to Eq. (A.12)!

The news tensor is defined by $N_{ab} := S_{ab} - \rho_{ab}$, where $S_a{}^b$ is the **unique** tensor satisfying

$$S_a{}^b n^a = \lambda n^b;\; S_{ab}q^{ab} = -\frac{2\rho''(\psi)}{\rho(\psi)}; \tag{A.16}$$

$$R_{abc}{}^d = q_{c[a}S_{b]}{}^d + S_{c[a}\delta_{b]}{}^d$$

where λ is an arbitrary constant, and $S_{ab} \equiv q_{ab}S_a{}^c \equiv S_{(ab)}$. We first need to determine $R_{abc}{}^d$, the Riemann tensor of q_{ab}. Direct calculation yields

$$R_{abc}{}^d = -\frac{2\rho''(\psi)}{\rho(\psi)}(\bar\partial_\varphi)^d D_{[a}\psi D_{b]}\varphi D_c\psi + 2\kappa^2\rho(\psi)\rho''(\psi)(\bar\partial_\psi)^d D_{[a}\psi D_{b]}\varphi D_c\varphi$$

$$+\frac{2}{\rho(\psi)^2}\Big(\rho'(\psi)\rho'(\vartheta) - \rho'(\vartheta)^2 + \rho(\vartheta)\rho''(\vartheta) - \rho(\psi)\rho''(\psi)\Big)(\bar\partial_u)^d D_{[a}\psi D_{b]}uD_c\psi$$

$$+2\kappa^2\rho'(\psi)\big(\rho'(\psi) - \rho'(\vartheta)\big)(\bar\partial_u)^d D_{[a}uD_{b]}\varphi D_c\varphi \tag{A.17}$$

and

$$S_{ab}dx^a dx^b = \frac{1}{\rho(\psi)^2}(-\rho'(\vartheta)^2 + \rho'(\psi)^2 + \rho(\vartheta)\rho''(\vartheta) - 2\rho(\psi)\rho''(\psi))d\psi^2$$

$$+\kappa^2\left(\rho'(\vartheta)^2 - \rho'(\psi)^2 - \rho(\vartheta)\rho''(\vartheta)\right)d\varphi^2 \tag{A.18}$$

and thus

$$N_{ab} = \frac{N(\vartheta)}{\rho(\psi)^2}\left(d\psi^2 - \kappa^2\rho(\psi)^2 d\varphi^2\right) \tag{A.19}$$

where $N(\vartheta) := \rho(\vartheta)\rho''(\vartheta) - \rho'(\vartheta)^2 + \kappa^{-2}$. Note that N_{ab} is, of course, tracefree.

We now turn our attention to $\underline{K}^{ab}$, defined by[41]

$$\underline{K}^{ab} := \varepsilon^{amn}\varepsilon^{bpq}K_{mnpq} \tag{A.20}$$

where ε^{abc} is the usual alternative tensor, defined up to sign by

$$\varepsilon^{abc}\varepsilon^{mnp}q_{bn}q_{cp} \equiv 2n^a n^b. \tag{A.21}$$

We choose

$$\varepsilon^{abc} = +\frac{6\rho(\vartheta)}{\kappa\rho(\psi)^2}(\bar{\partial}_u)^{[a}(\bar{\partial}_\psi)^b(\bar{\partial}_\varphi)^{c]} \tag{A.22}$$

and thus, from $\varepsilon^{abc}\varepsilon_{abc} \equiv 3!$,

$$\varepsilon_{abc} = \frac{6\kappa\rho(\psi)^2}{\rho(\vartheta)}D_{[a}uD_b\psi D_{c]}\varphi\ . \tag{A.23}$$

We thus have

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[41] Note that due to differences in convention $K_{ab} = -\frac{1}{4}\, q_{am}q_{bn}\underline{K}^{mn}$, where K_{ab} was determined in Eq. (4.10). Furthermore, note that Eq. (4.10) was obtained in a different conformal frame than the one we are using here.

$$\underline{K}^{ab} = \frac{m\rho(\vartheta)^5}{\kappa^2\rho(\psi)^5}\Big[8\kappa^2(\bar{\partial}_u)^a(\bar{\partial}_u)^b + 12A^2\kappa^2\rho(\psi)^2(\bar{\partial}_\psi)^a(\bar{\partial}_\psi)^b \tag{A.24}$$

$$-12A^2(\bar{\partial}_\varphi)^a(\bar{\partial}_\varphi)^b - 24A\kappa^2\rho(\psi)(\bar{\partial}_u)^{(a}(\bar{\partial}_\psi)^{b)}\Big]$$

which agrees with the previous calculation in Chapter 4.[41]

We now finally have all of the pieces necessary to calculate $(Y_\alpha)^a$; it only remains to choose the appropriate translation. For the moment setting $\alpha = \alpha_0$, we obtain

$$(Y_{\alpha_0})^a = -\frac{\kappa m\rho(\vartheta)^4}{\rho(\psi)^3}\cosh\sigma(\psi)(\bar{\partial}_u)^a$$

$$+\frac{N(\vartheta)}{2A\rho(\psi)^3}(\sinh\sigma(\psi) - \kappa\rho'(\vartheta)\cosh\sigma(\psi))(\bar{\partial}_u)^a \tag{A.25}$$

$$-\frac{N(\vartheta)}{2\rho(\psi)^2}\sinh\sigma(\psi)(\bar{\partial}_\psi)^a$$

The last step is to determine dS^{ab}. We will **assume** that our crosssections are independent of φ. Thus, a crosssection is given by

$$u = g(\psi) \text{ and/or } \tilde{u} = \tilde{g}(\tilde{\psi}) \tag{A.26}$$

wherever these expressions are defined. Functional symmetry forces $g \equiv \tilde{g}$. Thus,

$$dS^{ab} = \frac{\partial x^{[a}}{\partial\psi}\frac{\partial x^{b]}}{\partial\varphi}\,d\psi\,d\varphi \tag{A.27}$$

$$\equiv \left(g'(\psi)\delta_u{}^{[a}\delta_\varphi{}^{b]} + \delta_\psi{}^{[a}\delta_\varphi{}^{b]}\right)d\psi\,d\varphi\,.$$

Thus,

$$(Y_{\alpha_0})^m\varepsilon_{mab}dS^{ab} \equiv -\frac{\kappa^2}{\rho(\psi)}\left(m\rho(\vartheta)^3 + \frac{\rho'(\vartheta)N(\vartheta)}{2A\rho(\vartheta)}\right)\cosh\sigma(\psi)d\psi\,d\varphi \tag{A.28}$$

$$+\frac{\kappa N(\vartheta)}{2\rho(\vartheta)}\left(\frac{1}{A\rho(\psi)} + g'(\psi)\right)\sinh\sigma(\psi)d\psi\,d\varphi\,.$$

Using $\cosh\sigma(\tilde{\psi}) \equiv +\cosh\sigma(\psi)$, $\sinh\sigma(\tilde{\psi}) \equiv -\sinh\sigma(\psi)$, and $u + \frac{\kappa}{A}\sigma(\psi) \equiv \tilde{u} + \frac{\kappa}{A}\sigma(\tilde{\psi})$, it is straightforward to verify that the desired criteria have been met; that dM is independent of φ and functionally symmetric[42]

Putting it all together, we obtain

$$M_0(S) = -\tfrac{1}{2}\int_0^{\psi_o} \frac{\kappa^2\cosh\sigma(\psi)}{\rho(\psi)}\left[m\rho(\vartheta)^3 + \frac{N(\vartheta)\rho'(\vartheta)}{2A\rho(\vartheta)}\right]d\psi \qquad \text{(A.29)}$$

$$+\tfrac{1}{2}\int_0^{\psi_o} \frac{\kappa N(\vartheta)\sinh\sigma(\psi)}{2\rho(\vartheta)}\left[\frac{1}{A\rho(\psi)} + g'(\psi)\right]d\psi$$

where the subscript on M is to remind us that we chose $\alpha = \alpha_0$. Repeating the calculation using $\alpha = \alpha_3$ yields precisely the same result, but with $\sinh\sigma(\psi)$ and $\cosh\sigma(\psi)$ interchanged:

$$M_3(S) = -\tfrac{1}{2}\int_0^{\psi_o} \frac{\kappa^2\sinh\sigma(\psi)}{\rho(\psi)}\left[m\rho(\vartheta)^3 + \frac{N(\vartheta)\rho'(\vartheta)}{2A\rho(\vartheta)}\right]d\psi \qquad \text{(A.30)}$$

$$+\tfrac{1}{2}\int_0^{\psi_o} \frac{\kappa N(\vartheta)\cosh\sigma(\psi)}{2\rho(\vartheta)}\left[\frac{1}{A\rho(\psi)} + g'(\psi)\right]d\psi\ .$$

Furthermore, since $\int_0^{2\pi}\sin\varphi\, d\varphi \equiv 0 \equiv \int_0^{2\pi}\cos\varphi\, d\varphi$, it is clear that choosing $\alpha = \alpha_1$ or α_2 yields zero, i.e.

$$M_1(S) = M_2(S) = 0\ . \qquad \text{(A.31)}$$

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[42]Note that we did not need to use $g = \tilde{g}$ explicitly. Implicitly it is needed, however, in order to guarantee that the "conjugate" points ($u = U, \psi = \Psi, \varphi = \Phi$) and ($\tilde{u} = U, \tilde{\psi} = \Psi, \varphi = \Phi$) are both on the same crosssection!

As a check on these calculations, one verifies by direct calculation that, as expected, the energy flux on I is the square of the news, i.e.

$$D_a(Y_\alpha)^a = -\tfrac{1}{4}\,\alpha N_{mn} q^{np} N_{pq} q^{qm}\,; \tag{A.32a}$$

in particular

$$D_a(Y_{\alpha_0})^a = -\frac{\kappa\cosh\sigma(\psi)}{2\rho(\psi)^3}\,N(\vartheta)^2\,. \tag{A.32b}$$

The Bondi 4−momentum is a 1−form $(P_{\mathrm{B}})_a$ which acts on translations v^a so that

$$(P_{\mathrm{B}})_a v^a = \frac{1}{8\pi}\int (Y_\alpha)^m \varepsilon_{mab}\, dS^{ab} \tag{A.33}$$

where α is the BMS translation naturally associated with v^a. We have thus shown

$$(P_{\mathrm{B}})_a \equiv M_{\alpha_0}\delta_a{}^0 + M_{\alpha_3}\delta_a{}^3\,. \tag{A.34}$$

We now finally turn to the tricky topic of choosing crosssections. We will construct our crosssections so that they are independent of φ; the crosssections are thus given by Eq. (A.26), with $g \equiv \tilde{g}$. A C^∞ crosssection is therefore given by any C^∞ $g(\psi)$ defined for $\psi\varepsilon[0, \vartheta_o)$, with $u + \tilde{u} \equiv \dfrac{2\kappa}{A}\,\sigma(\vartheta)$ forcing

$$\tilde{g}(\tilde{\psi}) \equiv g(h(\psi)) \equiv g(\psi) + \frac{2\kappa}{A}\,\sigma(\psi) \tag{A.35}$$

where defined, i.e. for $\psi \neq 0, \vartheta_o$.[43] A possible choice of $g(\psi)$ is given by

$$g(\psi) = \frac{\kappa}{A}\ln\left|\frac{\lambda}{1+e^{2\sigma(\psi)}}\right| \tag{A.36}$$

[43]Note that even **without** the assumption of functional symmetry, since the relationship between u and $\tilde{u}$ breaks down as $A \to 0$ (and thus so does the attempt to extend the original manifold), there are **no** C^∞ crosssections which are well behaved in the limit as $A \to 0$!

for λ =constant; the crosssections are parameterized by $\lambda > 0$. It is easy to check that $g(\psi)$ satisfies Eq. (A.36) and is C^∞, but some motivation for this choice of $g(\psi)$ can be given. Functional symmetry becomes obvious if we rewrite the defining equation for the crosssection as

$$e^{A\kappa^{-1}u} + e^{A\kappa^{-1}\tilde{u}} = \lambda > 0 \tag{A.37}$$

where we have used $\sigma(\psi) \equiv -\frac{A}{\kappa}\, u + \sigma(\vartheta)$ and $u + \tilde{u} \equiv \frac{2\kappa}{A}\, \sigma(\vartheta)$. Furthermore, the exponentials are necessary because $\lim_{\psi \to \vartheta_o} u \equiv -\infty \equiv \lim_{\psi \to 0} \tilde{u}$. We thus obtain

$$g'(\psi) = \frac{-e^{\sigma(\psi)}}{A\rho(\psi)\cosh\sigma(\psi)}\ . \tag{A.38}$$

Substituting this in Eq. (A.29), we obtain

$$M_0(\lambda) = -\frac{\kappa^2 m}{2}\int_0^{\psi_o} \frac{\rho(\vartheta)^3 \cosh\sigma(\psi)}{\rho(\psi)}\, d\psi \tag{A.39}$$

$$-\frac{\kappa}{4A}\int_0^{\psi_o} \frac{N(\vartheta)}{\rho(\vartheta)\rho(\psi)}\,(\kappa\rho'(\vartheta)+1)\cosh\sigma(\psi)d\psi$$

$$+\frac{\kappa}{4A}\int_0^{\psi_o} \frac{N(\vartheta)}{\rho(\vartheta)\rho(\psi)}\cdot\frac{1}{\cosh\sigma(\psi)}\, d\psi\ .$$

Although this appears to be independent of λ, both ϑ and u are implicit functions of λ (and ψ).

Although we see no way to evaluate this integral explicitly, we can investigate its behavior as $\lambda \to 0$. This gives us the limit as bout u and $\tilde{u}$ approach $-\infty$ on **each** generator where they are defined. This is thus precisely the limit of the Bondi mass to the infinite past on I^+, and we expect this limit to be the ADM mass provided the news tensor falls off sufficiently fast [23].

For the vacuum C−metrics we have [13]

$$e^{2\kappa^{-1}y^*} \equiv \frac{y-y_2}{(y-y_1)^{n_1}(y_3-y)^{n_3}}$$

or, equivalently,

$$e^{2\kappa^{-1}x^*} \equiv \frac{x-x_2}{(x_1-x)^{n_1}(x-x_3)^{n_3}}$$

where

$$0 < n_1 = \frac{(y_3-y_2)}{(y_3-y_1)}; \qquad 0 < n_3 = \frac{(y_2-y_1)}{(y_3-y_1)} .$$

We thus have

$$\frac{x-x_2}{(x_1-x)^{n_1}(x-x_3)^{n_3}} \equiv e^{2\sigma(\vartheta)} \equiv \frac{\lambda^2}{4\cosh^2\sigma(\psi)} . \tag{A.40}$$

Assuming $\psi = 0, \vartheta_o$, we can thus expand $x - x_2$ in powers of λ^2 as follows:

$$x - x_2 = s[\lambda^2 + p\lambda^4 + 0(\lambda^6)] \tag{A.41}$$

where

$$s := \frac{(x_1-x_2)^{n_1}(x_2-x_3)}{4\cosh^2\sigma(\psi)}$$

and

$$p := \frac{(x_1-x_2)^{n_1-1}(x_2-x_3)^{n_3-1}(x_1+x_3-2x_2)}{4\cosh^2\sigma(\psi)} .$$

Since $x \to x_2$ as $\lambda \to 0$ (i.e. $\vartheta = 0$ at i^o) we can use this expression in order to expand $G(x)$ and its derivatives in powers of λ^2, and use the result to expand

$\rho(\vartheta)$ and $N(\vartheta)$ in powers of λ. After some messy algebra one obtains

$$\rho(\vartheta) = \sqrt{\frac{2s}{\kappa}}\left(1 + \frac{\lambda^2}{4}[2p - \kappa s(1 + 6mAx_2)]\right)\lambda + O(\lambda^5) \qquad \text{(A.42a)}$$

and

$$N(\vartheta) = -\left(\frac{3ma(x_1 - x_2)^{2n_1}(x_2 - x_3)^{2n_2}}{8\kappa \cosh^4 \sigma(\psi)}\right)\lambda^4 + O(\lambda^6)\,. \qquad \text{(A.42b)}$$

Furthermore, the exact coefficients here are not particularly important; we have shown that each term in the integrand of Eq. (A.39) goes to zero as $O(\lambda^3)$. We thus conclude that

$$\lim_{\lambda\to 0} M_0(\lambda) = 0\,. \qquad \text{(A.43)}$$

Note that Eq. (A.42b) implies immediately that the falloff condition on the news in [23] is satisfied. Furthermore, repeating this calculation for $\alpha = \alpha_3$ does not change anything except the details; we can thus conclude that $\lim_{\lambda\to 0} M_3(\lambda) = 0$, and thus

$$\lim_{\lambda\to 0} (P_{\mathrm{B}})^a(\lambda) = 0 \qquad \text{(A.44)}$$

i.e. the limit to the infinite past (on I^+) of the Bondi 4−momentum is zero. This guarantees that the ADM−mass is zero (so long as it is defined and the falloff condition on the news in [23] is satisfied), and that the Bondi mass is negative for **any** choice of BMS time translation (e.g. $\alpha = \alpha_0$).

Note that if we had placed the nodes at $\vartheta = 0$ instead of at ϑ_o (thus obtaining a neighborhood of i^+ instead of a neighborhood of i^o) we can modify the preceding argument very slightly to show that Conjecture 3 of Chapter 7 is satisfied.

The crosssections given by

$$g'(\psi) = -\frac{\kappa}{A}\ln\left|\frac{\lambda'}{1+e^{-2\sigma(\psi)}}\right| \tag{A.44a}$$

or, equivalently,

$$e^{-A\kappa^{-1}u} + e^{-A\kappa^{-1}\tilde{u}} = \lambda' \tag{A.44b}$$

can be thought of as the boundary of a hypersurface satisfying the hypotheses of Conjecture 3. However, a completely analogous argument to the one given above shows that

$$\lim_{\lambda'\to 0}(P_{\mathrm{B}})^a(\lambda') = 0\,, \tag{A.45}$$

and this is now the limit to $i+$. Thus, **if** a regular neighborhood of i^+ exists in I, then the Bondi mass is positive there.

We now discuss the sense in which this procedure can be said to have a "Schwarzschild" limit. We would like to simply take the limit as $A \to 0$ in the integrand, but we must be very careful. First of all, a nice Schwarzschild cross-section is given by u =constant, which is **not** a functionally symmetric crosssection at all! (In fact, it's not even a regular C−metric crosssection!) But we claim that taking an appropriate limit as $A \to 0$ of M on a u =constant crosssection should still yield m. However, as can be easily verified, taking $\lim_{A\to 0}(Y_{\alpha_0})^a$ and then integrating leads to a divergent result. What went wrong? The problem is that it's not clear when "lim" and "$\int$" commute. In fact, since the l_a we chose diverges as $A \to 0$, this result is not so surprising. Thus, in order to get a Schwarzschild limit which makes sense, we will carefully arrange things so that each term of the integrand has a well-defined limit as $A \to 0$. It turns out, of course, that the integrand itself will reduce to the correct integrand for the

Schwarzschild metric. We thus repeat the previous calculation substituting $\bar{l}_m dx^m := \dfrac{\rho(\psi)}{\rho(\vartheta)}\, du$ for l_m and using $\alpha = \alpha_0$, since these have the correct limit. (αn^a reduces to the BMS time translation associated with the timelike Killing vector.) We obtain

$$M_0(S) = \frac{\kappa^2 m}{2} \int_{\psi=0}^{\vartheta_o} \frac{\rho(\vartheta)^3}{\rho(\psi)} \cosh\sigma(\psi) d\psi \tag{A.46}$$

$$+ \frac{3\kappa^2 m}{4} \int_{\psi=0}^{\vartheta_o} \rho(\vartheta)^3 \cosh\sigma(\psi) g'(\psi) d\psi$$

$$+ \frac{\kappa}{8} \int_{\psi=0}^{\vartheta_o} \frac{N(\vartheta)}{A\rho(\vartheta)} \Big(\kappa\rho'(\vartheta)\cosh\sigma(\psi) + \sinh\sigma(\psi) \Big) g'(\psi) d\psi \ .$$

This expression is, of course, entirely equivalent to the one previously given (Eq. (A.29)). However, this expression is **not** functionally symmetric; in order to carry out the integration, we would first have to transform the integrand to the $(\tilde{u}, \tilde{\psi}, \varphi)$ chart in a neighborhood of $\psi = \vartheta_o$. But for our present purpose this is irrelevant. We simply set $g'(\psi) = 0$ (u =constant), take the limit of the integrand as $A \to 0$, and then integrate, using the following relations, valid in the limit $A = 0$:

$$\rho(\vartheta) = \sin\vartheta;\ \rho(\vartheta)\cosh\sigma(\vartheta) = 1;\ \rho(\vartheta)\sinh\sigma(\vartheta) = -\cos(\vartheta); \tag{A.47}$$

$$\vartheta_o = \pi;\ \psi = \vartheta;\ \kappa = 1;\ \lim_{A\to 0} \frac{N(\vartheta)}{A} = -2m(2+\cos\vartheta)(1-\cos\vartheta)^2 \ .$$

We obtain

$$M_S \equiv \frac{m}{2} \int_0^{\pi} \sin\vartheta d\vartheta \equiv m \tag{A.48}$$

as desired. We cannot emphasize too strongly, however, that this limit has nothing whatsoever to do with the Bondi mass of the C−metrics; the crosssection used here is **not** a C−metric crosssection.