MTH 112: Elementary Functions

Section 8.3: Vectors

- Learn basic concepts about vectors.
- Learn representations of vectors.
- Find the magnitude, direction angle, and components of a vector.
- Perform operations on vectors.
- Learn to apply the dot product.
- Use vector to calculate work.
Vectors

A Vector quantity involves both magnitude and direction. Magnitude can be interpreted as size or length.

A vector quantity can be represented by a directed line segment called a vector.

- Two vectors are equal if they have the same magnitude and direction.
Two ways to represent a vector

- A vector is usually represented symbolically by a letter printed in boldface type.

- A second way to denote a vector is to use two points. If the initial point of a vector \( \mathbf{v} \) is \( P \) and its terminal point is \( Q \), then \( \mathbf{v} = \overrightarrow{PQ} \).
Representation of vectors

If we place the initial point of vector $\mathbf{v}$ at the origin, then its terminal point can be used to determine $\mathbf{v}$. To distinguish the point $(a_1,a_2)$ from the vector $\mathbf{v}$, we use the notation

$$\mathbf{v} = \langle a_1, a_2 \rangle$$
The **horizontal component** of \( \mathbf{v} \) is \( a_1 \) and the **vertical component** of \( \mathbf{v} \) is \( a_2 \).
A vector with its initial point at the origin in the rectangular coordinate system is called a position vector. The figure shows the position vector \( \mathbf{v} = \langle a_1, a_2 \rangle \)
The positive angle $\theta$ between the x-axis and the position vector is called the **direction angle** for the vector. $\theta$ is the direction angle for vector $\mathbf{v}$.

If $\mathbf{v} = \langle a_1, a_2 \rangle$ then the direction angle satisfies

$$\tan(\theta) = \frac{a_2}{a_1}, \text{ where } a_1 \neq 0.$$
Magnitude of a vector

If \( \mathbf{v} = < a_1, a_2 > \) then the magnitude (or length) of \( \mathbf{v} \) is given by

\[
\| \mathbf{v} \| = \sqrt{a_1^2 + a_2^2}
\]

If \( \| \mathbf{v} \| = 1 \) then \( \mathbf{v} \) is the unit vector.
Magnitude of a vector

Example 1

Find the magnitude and direction angle $\theta$ for $\mathbf{v} = \langle -9, 40 \rangle$.

$\parallel \mathbf{v} \parallel = 41$

$\theta = 102.68^\circ$
Magnitude of a vector

**Example 2**

Find the magnitude and direction angle $\theta$ for $\mathbf{v} = < -28, -45 >$.

\[
\text{magnitude: } \sqrt{(-28)^2 + (-45)^2} = 53 \\
\text{direction angle: } \theta = \tan^{-1}\left(\frac{-45}{-28}\right) + \pi \approx 4.15579 \text{ rad} \approx 238.1092 \degree
\]
Horizontal and vertical components

The horizontal and vertical components for a vector \( \mathbf{v} = \langle a_1, a_2 \rangle \) having direction angle \( \theta \) are given by

\[
a_1 = \| \mathbf{v} \| \cos(\theta) \quad \text{and} \quad a_2 = \| \mathbf{v} \| \sin(\theta)
\]

That is,

\[
\mathbf{v} = \langle a_1, a_2 \rangle = \langle \| \mathbf{v} \| \cos(\theta), \| \mathbf{v} \| \sin(\theta) \rangle.
\]
Example

Vector \( \mathbf{w} \) has magnitude 12.5 and direction angle 53.6°. Find the horizontal and vertical components. Round to the nearest tenth.
Vector addition

If \( \mathbf{a} = \langle a_1, a_2 \rangle \) and \( \mathbf{b} = \langle b_1, b_2 \rangle \) then the sum of \( \mathbf{a} \) and \( \mathbf{b} \) is given by

\[
\mathbf{a} + \mathbf{b} = \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle.
\]

Here \( c = \mathbf{a} + \mathbf{b} \).

Notice \( \mathbf{a} + \mathbf{b} \) is the diagonal of a parallelogram.
Vector addition. Graphing $\mathbf{u} + \mathbf{v}$

\[ \mathbf{u} = \langle 2, 2 \rangle \]
\[ \mathbf{v} = \langle -4, -1 \rangle \]
\[ \mathbf{u} + \mathbf{v} = \langle -2, 1 \rangle \]
\[ = \langle 2 + (-4), 2 + (-1) \rangle \]
Vector subtraction

If \( \mathbf{a} = \langle a_1, a_2 \rangle \) and \( \mathbf{b} = \langle b_1, b_2 \rangle \) then the difference of \( \mathbf{a} \) and \( \mathbf{b} \) is given by

\[
\mathbf{a} - \mathbf{b} = \langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle = \langle a_1 - b_1, a_2 - b_2 \rangle.
\]
Example 3

Let $u = \langle 3, 4 \rangle$ and $v = \langle 5, -6 \rangle$. Find $u - v$.

\[
\begin{align*}
\mathbf{u} &= \langle 3, 4 \rangle \\
\mathbf{v} &= \langle 5, -6 \rangle \\
\mathbf{u} - \mathbf{v} &= \langle 3 - 5, 4 + 6 \rangle \\
&= \langle -2, 10 \rangle
\end{align*}
\]
Scalar multiplication

If \( \mathbf{v} = \langle v_1, v_2 \rangle \) and \( k \) is a real number, then the scalar product \( k\mathbf{v} \) is given by

\[
k\mathbf{v} = k\langle v_1, v_2 \rangle = \langle kv_1, kv_2 \rangle
\]
Example

Find each of the following expressions graphically and symbolically if \( a = \langle 12, 5 \rangle \) and \( b = \langle 4, 7 \rangle \).

1. \(-2b\)
2. \(a + 2b\)
Vector notation using $i$ and $j$

A second type of vector notation involves the vectors $i = \langle 1, 0 \rangle$ and $j = \langle 0, 1 \rangle$. A vector $\mathbf{a} = \langle a_1, a_2 \rangle$ can be expressed as

$$\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j}.$$ 

For example, $\langle 3, -4 \rangle$ and $3\mathbf{i} - 4\mathbf{j}$ represent the same vector.
Dot product

Let \( \mathbf{a} = \langle a_1, a_2 \rangle \) and \( \mathbf{b} = \langle b_1, b_2 \rangle \). The dot product of \( \mathbf{a} \) and \( \mathbf{b} \), denoted \( \mathbf{a} \cdot \mathbf{b} \), is a real number given by

\[
\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2
\]

**Example 4**

Calculate \( \mathbf{a} \cdot \mathbf{b} \) if:

1. \( \mathbf{a} = \langle 4, 3 \rangle \) and \( \mathbf{b} = \langle 1, 2 \rangle \)
2. \( \mathbf{a} = 2\mathbf{i} + 5\mathbf{j} \), \( \mathbf{b} = 3\mathbf{i} + 2\mathbf{j} \)
Angle between two vectors

If \( \mathbf{a} \) and \( \mathbf{b} \) are nonzero vectors, then the angle between \( \mathbf{a} \) and \( \mathbf{b} \) is given by

\[
\theta = \cos^{-1}\left( \frac{\mathbf{a} \cdot \mathbf{b}}{\| \mathbf{a} \| \| \mathbf{b} \|} \right)
\]

Vectors \( \mathbf{a} \) and \( \mathbf{b} \) are perpendicular if and only if \( \mathbf{a} \cdot \mathbf{b} = 0 \).
Example

Let \( \mathbf{a} = \langle 4, -5 \rangle \) and \( \mathbf{b} = \langle 2, -2 \rangle \)

1. Find \( \mathbf{a} \cdot \mathbf{b} \)
2. Approximate the angle \( \theta \) between \( \mathbf{a} \) and \( \mathbf{b} \) to the nearest tenth of a degree.
3. State if vectors \( \mathbf{a} \) and \( \mathbf{b} \) are perpendicular, parallel, or neither. If \( \mathbf{a} \) and \( \mathbf{b} \) are parallel, state whether they point in the same direction or in opposite directions.
Applications to vectors

**Example 5**

Suppose that vector $\mathbf{a}$ represents a force of 80 pounds pulling on a water-ski towrope and $\mathbf{b}$ represents a force of 60 pounds pulling on a second towrope. The resultant force $\mathbf{c} = \mathbf{a} + \mathbf{b}$ is given by the diagonal of the parallelogram. Vector $\mathbf{c}$ represents the net force exerted by the two water skiers. Find the magnitude of the resultant force on the ski boat if the angle between the towropes is $25^\circ$. 
Example 6: Work

If a constant force $F$ is applied to an object that moves along a vector $D$, then the work $W$ done is

$$W = F \cdot D$$
Example

Find the work done when a force $\mathbf{F} = \langle 3, 2 \rangle$ moves an object from point $P = (2, 1)$ to point $Q = (3, 1)$, where force is measured in pounds and distance in feet.