Section 8.3: Vectors

- Learn basic concepts about vectors.
- Learn representations of vectors.
- Find the magnitude, direction angle, and components of a vector.
- Perform operations on vectors.
- Learn to apply the dot product.
- Use vector to calculate work.

A Vector quantity involves both magnitude and direction. Magnitude can be interpreted as size or length.

A vector quantity can be represented by a directed line segment called a vector.

- Two vectors are equal if they have the same magnitude and direction.
Two ways to represent a vector

- A vector is usually represented symbolically by a letter printed in boldface type.

- A second way to denote a vector is to use two points. If the initial point of a vector \( \mathbf{v} \) is \( P \) and its terminal point is \( Q \), then \( \mathbf{v} = \mathbf{PQ} \)

Representation of vectors

If we place the initial point of vector \( \mathbf{v} \) at the origin, then its terminal point can be used to determine \( \mathbf{v} \). To distinguish the point \((a_1, a_2)\) from the vector \( \mathbf{v} \), we use the notation
The horizontal component of \( \mathbf{v} \) is \( a_1 \) and the vertical component of \( \mathbf{v} \) is \( a_2 \).

A vector with its initial point at the origin in the rectangular coordinate system is called a position vector. The figure shows the position vector \( \mathbf{v} = \langle a_1, a_2 \rangle \).
The positive angle $\theta$ between the x-axis and the position vector is called the direction angle for the vector. $\theta$ is the direction angle for vector $\mathbf{v}$.

If $\mathbf{v} = \langle a_1, a_2 \rangle$ then the direction angle satisfies

$$\tan(\theta) = \frac{a_2}{a_1}, \text{ where } a_1 \neq 0.$$
Magnitude of a vector

**Example 1**

Find the magnitude and direction angle θ for $v = \langle -9, 40 \rangle$.

\[
\begin{align*}
\text{magnitude:} & \quad \|v\| = \sqrt{(-9)^2 + 40^2} = 41 \\
\text{direction angle:} & \quad \theta = \tan^{-1}\left(\frac{40}{-9}\right) + \pi \\
& \approx -77.3196^\circ \\
& \approx 102.68^\circ
\end{align*}
\]

**Example 2**

Find the magnitude and direction angle θ for $v = \langle -28, -45 \rangle$.

\[
\begin{align*}
\text{magnitude:} & \quad \|v\| = \sqrt{(-28)^2 + (-45)^2} = 53 \\
\text{direction angle:} & \quad \theta = \tan^{-1}\left(\frac{-45}{-28}\right) + \pi \\
& \approx 238.1092^\circ \\
& \approx 238.11^\circ
\end{align*}
\]
Horizontal and vertical components

**Horizontal and vertical components**

The **horizontal and vertical components** for a vector \( \mathbf{v} = <a_1, a_2> \) having direction angle \( \theta \) are given by

\[
\begin{align*}
    a_1 &= \|\mathbf{v}\| \cos(\theta) \quad \text{and} \quad a_2 = \|\mathbf{v}\| \sin(\theta),
\end{align*}
\]

That is,

\[
\mathbf{v} = <a_1, a_2> = (\|\mathbf{v}\| \cos(\theta), \|\mathbf{v}\| \sin(\theta)).
\]

**Example**

Vector \( \mathbf{w} \) has magnitude 12.5 and direction angle 53.6°. Find the horizontal and vertical components. Round to the nearest tenth.
Vector addition

If \( \mathbf{a} = \langle a_1, a_2 \rangle \) and \( \mathbf{b} = \langle b_1, b_2 \rangle \) then the sum of \( \mathbf{a} \) and \( \mathbf{b} \) is given by

\[
\mathbf{a} + \mathbf{b} = \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle.
\]

Here \( \mathbf{c} = \mathbf{a} + \mathbf{b} \).
Notice \( \mathbf{a} + \mathbf{b} \) is the diagonal of a parallelogram.

Vector addition. Graphing \( \mathbf{u} + \mathbf{v} \)

\[
\mathbf{u} = \langle 2, 2 \rangle
\]

\[
\mathbf{v} = \langle -4, -1 \rangle
\]

\[
\mathbf{u} + \mathbf{v} = \langle -2, 1 \rangle
\]

\[
= \langle 2 + (-4), 2 + (-1) \rangle
\]
Vector subtraction

If \( \mathbf{a} = \langle a_1, a_2 \rangle \) and \( \mathbf{b} = \langle b_1, b_2 \rangle \) then the difference of \( \mathbf{a} \) and \( \mathbf{b} \) is given by

\[
\mathbf{a} - \mathbf{b} = \langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle = \langle a_1 - b_1, a_2 - b_2 \rangle.
\]

Example 3
Let \( \mathbf{u} = \langle 3, 4 \rangle \) and \( \mathbf{v} = \langle 5, -6 \rangle \). Find \( \mathbf{u} - \mathbf{v} \).

\[
\mathbf{u} = \langle 3, 4 \rangle
\]

\[
\mathbf{v} = \langle 5, -6 \rangle
\]

\[
\mathbf{u} - \mathbf{v} = \langle 3 - 5, 4 - (-6) \rangle = \langle -2, 10 \rangle
\]
Scalar multiplication

If \( \mathbf{v} = \langle v_1, v_2 \rangle \) and \( k \) is a real number, then the scalar product \( k\mathbf{v} \) is given by

\[
k\mathbf{v} = k\langle v_1, v_2 \rangle = \langle kv_1, kv_2 \rangle
\]
Vector notation using $i$ and $j$

A second type of vector notation involves the vectors $i = \langle 1, 0 \rangle$ and $j = \langle 0, 1 \rangle$. A vector $\mathbf{a} = \langle a_1, a_2 \rangle$ can be expressed as

$$\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j}.$$ 

For example, $\langle 3, -4 \rangle$ and $3\mathbf{i} - 4\mathbf{j}$ represent the same vector.

Dot product

Let $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$. The dot product of $\mathbf{a}$ and $\mathbf{b}$, denoted $\mathbf{a} \cdot \mathbf{b}$, is a real number given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$

**Example 4**

Calculate $\mathbf{a} \cdot \mathbf{b}$ if:

1. $\mathbf{a} = \langle 4, 3 \rangle$ and $\mathbf{b} = \langle 1, 2 \rangle$
2. $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j}$
Angle between two vectors

If \( \mathbf{a} \) and \( \mathbf{b} \) are nonzero vectors, then the angle between \( \mathbf{a} \) and \( \mathbf{b} \) is given by

\[
\theta = \cos^{-1}\left( \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \right)
\]

Vectors \( \mathbf{a} \) and \( \mathbf{b} \) are perpendicular if and only if \( \mathbf{a} \cdot \mathbf{b} = 0 \).

---

Example

Let \( \mathbf{a} = \langle 4, -5 \rangle \) and \( \mathbf{b} = \langle 2, -2 \rangle \)

1. Find \( \mathbf{a} \cdot \mathbf{b} \)
2. Approximate the angle \( \theta \) between \( \mathbf{a} \) and \( \mathbf{b} \) to the nearest tenth of a degree.
3. State if vectors \( \mathbf{a} \) and \( \mathbf{b} \) are perpendicular, parallel, or neither. If \( \mathbf{a} \) and \( \mathbf{b} \) are parallel, state whether they point in the same direction or in opposite directions.
Applications to vectors

**Example 5**

Suppose that vector $a$ represents a force of 80 pounds pulling on a water-ski towrope and $b$ represents a force of 60 pounds pulling on a second towrope. The resultant force $c = a + b$ is given by the diagonal of the parallelogram. Vector $c$ represents the net force exerted by the two water skiers. Find the magnitude of the resultant force on the ski boat if the angle between the towropes is $25^\circ$.

**Work**

**Example 6: Work**

If a constant force $F$ is applied to an object that moves along a vector $D$, then the work $W$ done is

$$W = F \cdot D$$
Example

Find the work done when a force $\mathbf{F} = \langle 3, 2 \rangle$ moves an object from point $P = (2, 1)$ to point $Q = (3, 1)$, where force is measured in pounds and distance in feet.