Department of Mathematics OSU
Qualifying Examination
Fall 2016

Complex Analysis and Linear Algebra

- Do any two of the three problems in Part CA, *use the corresponding marked blue book* and indicate on the selection sheet with your indentification code those problems which you want to have graded. Similarly, do any two of the three problems in Part LA in the *corresponding marked blue book* and mark those which you want to have graded on the selection sheet.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this exam.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheets back into the envelope in which the test materials came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them and mark whether they are for *complex analysis* or *linear algebra*.

DO NOT WRITE YOUR NAME ANYWHERE – USE ONLY YOUR TEST ID CODE

Part CA: Complex Analysis

1. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0, n \geq 1$. Prove that either $p(z) \equiv z^n$ or else there exists a point ζ on the unit circle such that $|p(\zeta)| > 1$.

2. Determine the number of zeros of the polynomial $2z^5 - 15z^2 - z + 1$ in the annulus $\frac{1}{3} \leq |z| \leq 2$.

3. Let f be a holomorphic function on the disk $D_R = \{|z| < R\}$ of radius R, such that the real part of f is bounded above by some real number c (note that c can be negative). For any positive radius r, define $M_r(f)$ to be the maximum of $|f(z)|$ over all z in the closed disk $\overline{D}_r$ of radius r.

   (a) Prove that if $f(0) = 0$, then for any r satisfying $0 < r < R$,

   $$M_r(f) \leq \frac{2rc}{R-r}.$$

   (b) Use part (a) to prove that when $f(0) \neq 0$,

   $$M_r(f) \leq \frac{2rc}{R-r} + \frac{(R+r)|f(0)|}{R-r}.$$

Exam continues on next page ...

Part LA: Linear Algebra

1. Let A be a 2×2 real matrix with distinct real eigenvalues. For a 2×2 real matrix B, let $C(B)$ denote the 4×4 matrix whose block form is

$$\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}$$

(where 0 denotes the two-by-two zero matrix). Prove that the subset of $M_{2\times 2}(\mathbb{R})$ (the vector space of real 2×2 matrices) consisting of all B for which the corresponding $C(B)$ is diagonalizable forms a two dimensional subspace of $M_{2\times 2}$. [Hint: What must be the minimal polynomial of $C(B)$ if it is diagonalizable?]

2. Let a and b be real numbers. Prove that there are two orthogonal unit vectors u and v in $\mathbb{R}^3$ such that $u = (u_1, u_2, a)$ and $v = (v_1, v_2, b)$ if and only if $a^2 + b^2 \leq 1$.

3. Let A be any $n \times n$ complex matrix. Prove that A can be written as $A = B + N$ where B is diagonalizable, N is nilpotent (some power is the zero matrix) and the matrices B and N commute.