Department of Mathematics OSU
Qualifying Examination
Spring 2017

Complex Analysis and Linear Algebra

- Do any two of the three problems in Part CA, *use the corresponding marked blue book* and indicate on the selection sheet with your indentification code those problems which you want to have graded. Similarly, do any two of the three problems in Part LA in the *corresponding marked blue book* and mark those which you want to have graded on the selection sheet.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete Part II.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheets back into the envelope in which the test materials came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them and mark whether they are for *complex analysis* or *linear algebra*.

DO NOT WRITE YOUR NAME ANYWHERE – USE ONLY YOUR TEST ID CODE

Part CA: Complex Analysis

1. Let $a > 1$ be real. Evaluate the integral $\displaystyle\int_0^\pi \frac{\cos\theta}{a+\cos\theta}\,d\theta$ by residue methods.

2. Prove the fundamental theorem of algebra using Rouché's theorem.

3. Let f be an entire function such that for some positive integer k and positive constant A and all $0 < r < \infty$ it holds that

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leq Ar^{2k}.$$

   Prove that f is a constant multiple of the function z^k.

Exam continues on next page ...

Part LA: Linear Algebra

1. (a) Let T be a linear operator on a finite dimensional real inner product space, and let $m(x) \in \mathbb{R}[x]$ be the minimal polynomial of T. Prove that T has a matrix representation that is upper-triangular if and only if $m(x)$ splits into linear factors in $\mathbb{R}[x]$. You may use named theorems in your proof.

   (b) Let A be the real 3×3 matrix given by

$$\begin{bmatrix} b & b+1 & 1 \\ b-1 & b & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Determine for which $b \in \mathbb{R}$ the matrix A is upper-triangularizable.

2. Let V be a finite dimensional complex vector space, and let T, U be normal operators on V such that $TU = UT$.

   (a) Prove that there exists an orthonormal basis β of V such that $[T]_\beta$ and $[U]_\beta$ are both diagonal.

   (b) Prove that TU is a normal operator.

3. Let V be the real vector space of polynomial functions of (total) degree at most 2 in two real variables x and y. Let T be the linear operator on V defined by

$$T\left(f(x, y)\right) = \frac{\partial}{\partial x} f(x, y) + \frac{\partial}{\partial y} f(x, y).$$

   Does T have a Jordan canonical form? If so, determine it.