Department of Mathematics OSU
Qualifying Examination
Fall 2008

PART I : Real Analysis

- Do any of the four problems in Part I. Indicate on the sheet with your identification number the four which you wish graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete Part I.
- On problems with multiple parts, individual parts may be weighted differently in grading.

1. Suppose that the real valued function f is Lebesgue integrable on $[0,\infty)$. Use techniques from Lebesgue theory to prove

$$\int_0^\infty f(x)\,dx \;=\; \lim_{b\to\infty}\int_0^b f(x)\,dx\ .$$

Here, b is a continuous variable, not a discrete sequence $\{b_n\}$.

Note. In a calculus class, the given relation is a definition of an improper integral. In the present problem, the integrals are Lebesgue integrals, and the relation is a theorem instead of a definition.

2. Prove that if the real valued function $f \in L^1(\mathbf{R})$, then

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\cos nx\,dx \;=\; 0\,.$$

(In other words, prove the Riemann-Lebesgue lemma.)

Hint. You may use, without proof, the fact that the set of all step functions with compact support and finitely many steps is dense in $L^1(\mathbf{R})$.

3. Suppose that f is a bounded, real valued function on the closed interval $[a,b]$.

(a) Define the Lebesgue integral of f on $[a,b]$. Your answer should contain a definition of whether the integral exists.

(b) Modify your answer to part (a) so as to yield a definition of Riemann integral of f on $[a,b]$. (You should need to change only a few words.)

(c) Give an example of a function f on the interval $[0,1]$ for which the Lebesgue integral exists but the Riemann integral does not exist. Prove that your example has the required properties.

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4. Suppose that f is a real valued function on $\mathbf{R}$ and $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of continuous functions converging pointwise to f. Prove that there exists a nonempty open subset U of $\mathbf{R}$ and a real number M such that $|f_n(x)| \leq M$ for all $x \in U$ and all $n \geq 1$.

Hint. For every positive integer M, let $E_M = \{x \in \mathbf{R} : |f_n(x)| \leq M \text{ for all } n \geq 1\}$.

5.

a.) State the Contraction Mapping Theorem.

b.) Let X be a complete metric space. Suppose that a map $A : X \to X$ is such that there is a natural number n for which the nth power of A, $A^n = \underbrace{A \circ \cdots \circ A}_{n}$, is a contraction. Prove that A has a unique fixed point.

6. Let H be a Hilbert space with inner product denoted by $\langle x, y \rangle$. Suppose that $f, f_n \in H$, $n = 1, 2, \ldots$, are such that for every $g \in H$ one has $\langle f_n, g \rangle \to \langle f, g \rangle$ as $n \to \infty$.

a.) Show that if H is finite dimensional this implies that $f_n \to f$, that is, $\lim_{n\to\infty} \| f - f_n \| = 0$.

b.) Show that the conclusion of part (a) need not hold if H is infinite dimensional.