**Department of Mathematics OSU**
**Qualifying Examination**
**Fall 2010**

## PART II: COMPLEX ANALYSIS and LINEAR ALGEBRA

- Do any of the two problems in Part CA and any two problems in Part LA . Indicate on the sheet with your identification number those problems that you wish graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete Part II.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place test and examination book back into the envelope. If you use extra examination books, be sure to place your code number on them.

## PART : COMPLEX ANALYSIS QUALIFYING EXAM

1. a.) Let $f(z) = u(x,y) + \mathrm{i}v(x,y)$ be analytic in an open, connected domain $\Omega \subset \mathbb{C}$, and suppose that $v(x,y) = \exp(-y)(y\cos x + x\sin x)$. Find $f(z)$.

b.) Let $f(z)$, $g(z)$ be analytic in an open, connected domain $\Omega$ containing 0, and suppose that

$$f(z+w) = f(z)f(w) - g(z)g(w), \quad g(z+w) = g(z)f(w) + g(w)f(z),$$

whenever $z$, $w$, and $z+w$ are all contained in $\Omega$. Suppose further that $f(0) = 1$, $f'(0) = 0$. Find all such pairs $f(z)$ and $g(z)$.

2. a.) Let $\mathbf{c}$ be a circle in $\mathbb{C}$, and let $w_1, w_2 \in \mathbb{C} \setminus \mathbf{c}$ be two points contained in the complement of $\mathbf{c}$. Show that there is a fractional linear transformation $f(z)$ satisfying $f(\mathbf{c}) \subset \mathbf{c}$ and $f(w_1) = w_2$.

b.) Let $f(z)$ be analytic in the open unit disk $|z| < 1$ and continuous in $|z| \leq 1$, and suppose that $|f(\mathrm{e}^{\mathrm{i}t})| = 1$ for all $t$. Suppose furthermore that $f(z)$ has exactly one zero of order $k$ in $|z| < 1$. Find all such $f(z)$.

3. a.) (i.) Show that there is an analytic function defined in $\Omega = \{z \in \mathbb{C} \,|\, |z| > 4\}$ whose derivative is $f(z) = z/((z-1)(z-2)(z-3))$.

(ii.) Can you find an analytic function in $\Omega$ with the derivative $g(z) = z^2/((z-1)(z-2)(z-3))$?

b.) Use the calculus of residues to evaluate

$$\int_0^\infty \frac{x^{1/2}\,dx}{1+x^2}.$$

**Exam continues on next page ...**

**PART: LINEAR ALGEBRA QUALIFYING EXAM**

1. Consider the real vector space of real $2\times 2$ matrices with its usual scalar multiplication and addition. Let $\mathcal{V}$ be the subspace spanned by the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \ C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix},$$

and let $T : \mathcal{V} \to \mathbb{R}$ be the function that takes each $M \in V$ to its trace.

a.) Show that $T$ is a linear transformation.

b.) Choose explicit ordered bases $\mathcal{B}$ of $\mathcal{V}$ and $\mathcal{C}$ of $\mathbb{R}$ and give the matrix representation of $T$ with respect to these bases.

c.) Let $\mathcal{W}$ be the kernel of $T$. Give an explicit basis for $\mathcal{W}$.

d.) Again let $\mathcal{W}$ denote the kernel of $T$. Give an explicit basis for the quotient vector space $\mathcal{V}/\mathcal{W}$.

**Exam continues on next page ...**

2. Let $V$ be a finite dimensional real vector space and $b : V \times V \to \mathbb{R}$ a bilinear form — that is, $b(u, v)$ gives a linear transformation in each variable when the other variable is held fixed. Assume both that (1) $b$ is antisymmetric, i.e. $b(u, v) = -b(v, u)$ for all $u, v \in V$, and that (2) $b$ is *non-degenerate*, i.e. for every non-zero $v \in V$ there exists some $w \in V$ such that $b(v, w) \neq 0$.

For any non-empty subset $S$ in $V$ we define the $b$-annihilator of $S$, denoted $S^{\perp}$, by

$$S^{\perp} = \{v \in V : b(u, v) = 0 \text{ for all } u \in S\}.$$

a.) Prove that $S^{\perp}$ is a subspace of $V$.

b.) Let $v_1 \neq 0$ in $V$. Show that there is $w_1 \in V$ such that $b(v_1, w_1) = 1$. Prove that $\{v_1, w_1\}$ is a linearly independent set, and that if $S_1 = \text{Span}\{v_1, w_1\}$, then $V$ is the direct sum of $S_1$ and $S_1^{\perp}$.

c.) Prove that the restriction of $b$ to $S_1^{\perp}$ is non-degenerate.

d.) Prove that the dimension of $V$ is even.

3. For each non-negative integer $n$, let $\mathcal{P}_n$ denote the real vector space of the polynomials in one variable of degree less than or equal to $n$. (We include the zero polynomial in this set). For $f \in \mathcal{P}_n$, let $f'$ denote its derivative.

Give a Jordan canonical basis for the linear operator

$$\begin{aligned} T : \mathcal{P}_n &\to \mathcal{P}_n \\ f &\mapsto f + f' \,. \end{aligned}$$