Department of Mathematics
Qualifying Examination
Fall Term 2000

Part I: Real Analysis

Do any four of the problems in Part I. Your solutions should include all essential mathematical details; please write them up as clearly as possible. You have three hours to complete Part I of the examination.

1. Let $f : \mathbf{R} \to \mathbf{R}$ be a nonnegative, Lebesgue integrable function.

   (a) Define functions $f_n : \mathbf{R} \to \mathbf{R}$ by

   $$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n; \\ n & \text{otherwise} \end{cases}$$

   Suppose that E is any measurable set in $\mathbf{R}$. Show that

   $$\lim_{n\to\infty} \int_E f_n(x)\,dx = \int_E f(x)\,dx.$$

   Note: If you use any major theorems in your solution, clearly state the hypotheses and conclusion of the theorems, and indicate how your use of the theorem is justified.

   (b) Let $F : \mathbf{R} \to \mathbf{R}$ be given by

   $$F(x) = \int_{[-\infty,x]} f$$

   Show that F is continuous. (Do not assume that f is bounded.)

2. Show that every convergent sequence of measurable functions on a set of finite measure is almost uniformly convergent. That is, show the following:

   Let E be a Lebesgue measurable subset of R with finite measure. Let $\mu(A)$ represent the Lebesgue measure of A. Let (f_n) be a sequence of measurable functions defined on E. Suppose that f is a real valued function such that $f_n(x) \to f(x)$ almost everywhere on E. Then given $\epsilon > 0$, and $\delta > 0$, show that there is a measurable set $A \subset E$ with $\mu(A) < \delta$ and an integer N such that for all x **not in** A, and for all $n \geq N$

$$|f_n(x) - f(x)| < \epsilon$$

Note: State clearly any properties of measurable sets that you use.

3. Consider the relation on $I := [0,1]$ defined by $x \sim y$ if and only if $x - y \in \mathbf{Q}$.

   (a) Show that this relation is an equivalence relation.

   (b) By an application of the axiom of choice, form a set $\mathcal{S}$ of distinct equivalence class representatives, one for each class of the relation. Prove that $\mathcal{S}$ is **not** Lebesgue integrable.

4. (a) Prove that the (improper) Riemann integral

   $$\int_0^\infty \frac{\sin x}{x}\, dx$$

   exists.

   (b) Prove that the (improper) Riemann integral

   $$\int_0^\infty \left|\frac{\sin x}{x}\right| dx$$

   diverges.

5. Recall that a metric space X is called separable if it has a countable (or finite) dense subset. Show that if a metric space X is separable then every subset Y of X is also separable.

6. (a) Let I denote the unit interval, $[0,1]$. Consider a function f from I to itself. Suppose the the graph

   $$\Gamma_f = \{\, (x, f(x)) \,|\, x \in I \,\}$$

   is a closed subset of $I \times I$. Prove that f is continuous.

   (b) Give an example of a discontinuous function $f : \mathbf{R} \to \mathbf{R}$ whose graph is closed.

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Part II: Complex Analysis and Linear Algebra

Do any two problems in Part CA and any two problems in Part LA. Your solutions should include all essential mathematical details; please write them up as clearly as possible. You have three hours to complete Part II of the examination.

Part CA

1. Let $\mathcal{C}$ be the boundary of the square of vertices $\pm 2 \pm 2i$, oriented counterclockwise. Let $\alpha = 1 + i$. Evaluate

$$\oint_{\mathcal{C}} \frac{z^3}{(z-\alpha)^2}\, dz$$

2. Suppose $f(z)$ is a non-constant entire function. Prove, without appealing to Picard's theorem, that there exists a $z_0 \in \mathbf{C}$ such that $f(z_0)$ is a positive real number.

3. Suppose $f(z) = \frac{az+b}{cz+d}$ be a fractional-linear transformation of the complex plane. Here a, b, c, and d are complex numbers and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.

   (a) Show that $f(z)$ can be written as the composition of maps of the following three types: (i) $f_1(z) = z + z_0$, for some $z_0 \in \mathbf{C}$, (ii) $f_2(z) = \alpha z$ for some $\alpha \in \mathbf{C}$ and (iii) $f_3(z) = 1/z$.

   (b) Show that if A is a line and B is a circle then $f(A)$ is either a line or a circle and $f(B)$ is either a line or a circle.

Continued on back

Part LA

1. Let A be a 4×4 matrix with entries in $\mathbf{C}$ such that $\operatorname{rank}(A) = 1$. Show that either A is diagonalizable (over $\mathbf{C}$) or $A^2 = 0$, but not both.

2. Consider the complex numbers $\mathbf{C}$ as a vector space over the reals $\mathbf{R}$; note that $\mathcal{B} = \{1, i\}$ is a basis for this vector space. For each $\alpha \in \mathbf{C}$, let

$$l_\alpha : \mathbf{C} \to \mathbf{C}$$

be defined by $l_\alpha(z) = \alpha z$ for all $z \in \mathbf{C}$.

   (a) Given $\alpha \in \mathbf{C}$, find the matrix M_α representing the linear operator l_α with respect to the basis $\mathcal{B} = \{1, i\}$; that is, $M_\alpha = [l_\alpha]_{\mathcal{B}}$.

   (b) Determine the exact set of $\alpha \in \mathbf{C}$ such that M_α is diagonalizable over $\mathbf{R}$.

   (c) For which $\alpha \in \mathbf{C}$ is the characteristic polynomial of l_α equal to its minimal polynomial?

   (d) Let $\rho : \mathbf{C} \to M_2(\mathbf{R})$ by sending α to M_α. Show that ρ is an injective $\mathbf{R}$-linear map.

3. (a) Prove that if V is a finite dimensional inner product space over a field F and $\phi : V \to F$ is a linear functional then there exists a unique vector v_0 in V such that for any $v \in V$, $\phi(v) = \langle v, v_0 \rangle$.

   (b) Consider the real vector space of polynomials with real coefficients, with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx.$$

   Fix $x_0 \in \mathbf{R}$ and let L be the linear functional given by $L(f) = f(x_0)$. Show that there is no polynomial $p(x)$ such that for all polynomials f, $L(f) = \langle f, p \rangle$.