OSU Department of Mathematics
Qualifying Examination
Fall 2021

Linear Algebra

Instructions:

- Do **any three** of the four problems.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have **four** hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use the problem selection sheet to indicate your identification number and the three problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection sheet on top and any scratch-work on the bottom.
  3. Submit the exam: place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.

Exam continues on next page ...

Common notation:

- For a linear operator T on a vector space V and a T-invariant subspace W, we denote by $T|_W$ the restriction of T to W.
- lcm stands for the *least common multiple*.
- $\mathbb{R}[x]$ denotes the space of all polynomials with real coefficients.

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Problems:

1. (10 pts) Let n be a positive integer, and let $A = (a_{ij})_{i,j=1}^{n}$ be an $n \times n$ matrix with

$$a_{ij} = \begin{cases} 2 & \text{if } j = i, \\ -1 & \text{if } j = i \pm 1, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{That is} \quad A = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & \dots & -1 & 2 \end{bmatrix}.$$

Prove that every eigenvalue of A is a positive real number.

2. (10 pts) Find the Jordan Canonical Form and an associated Jordan Canonical Basis for the following complex matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ -1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix}.$$

3. **a**. (6 pts) Let T be a linear operator on a vector space V over field F, and suppose $W_1, \dots, W_k$ are T-invariant subspaces of V such that $V = W_1 + \cdots + W_k$. Let $m_i(x)$ be the minimal polynomial of $T|_{W_i}$ for each $1 \le i \le k$. Prove that the minimal polynomial of T is $\operatorname{lcm}(m_1, \dots, m_k)$.

   b. (4 pts) For $\alpha \in \mathbb{C}$ set $T : \mathbb{C}^3 \to \mathbb{C}^3$ be defined by $T(x, y, z) = (\alpha x + y, -x + y, 2z)$. Let $W_1 = \{(x, y, 0) \mid x, y \in \mathbb{C}\}$ and $W_2 = \{(0, 0, z) \mid z \in \mathbb{C}\}$. Use (a) to determine which $\alpha \in \mathbb{C}$ make T diagonalizable.

4. (10 pts) Use linear algebra to find $f(x) \in \mathcal{P}_2(\mathbb{R}) = \{f \in \mathbb{R}[x] \mid \deg(f) \le 2\}$ such that $f'(0) = 0$ and $f(x)$ minimizes

$$\int_0^1 (2x - f(x))^2 dx.$$