OSU Department of Mathematics
Qualifying Examination
Fall 2021

Real Analysis

Instructions:

- Do **any three** of the four problems.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have **four** hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use the problem selection sheet to indicate your identification number and the three problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection sheet on top and any scratch-work on the bottom.
  3. Submit the exam: place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.

Exam continues on next page ...

Common notation:

- $\| \cdot \|_\infty$ denotes the L^∞ norm (supremum norm).
- $C[0, 1]$ denotes the space of all continuous functions on $[0, 1]$.
- $C^1[0, 1]$ denotes the space of all continuously differentiable functions on $[0, 1]$.
- For a transformation T from a space S to itself and a positive integer n, transformation T^n is the nth iteration of T.

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Problems:

1. Let $X = \{f \in C^1[0,1] : f(0) = 0\}$ be the space of all functions $f : [0,1] \to \mathbb{R}$ such that f and f' are continuous on $[0,1]$ and $f(0) = 0$. (Interpret the values of f' at the endpoints as one-sided derivatives.) For each $f \in X$, let

$$\|f\| = \max\{|f'(x)| : x \in [0,1]\}.$$

Note that $\|f\|$ is defined in terms of the *derivative* of f.

   a. (2 pts) Show that if the definition of X is changed by deleting the condition $f(0) = 0$, then $\|\cdot\|$ is not a norm.

   b. (4 pts) Show that if a sequence $(f_n)_{n=1}^{\infty}$ converges in X (i.e., with respect to the norm $\|\cdot\|$), then the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly on $[0,1]$.

   c. (4 pts) Show that the space X is complete, with respect to the norm $\|\cdot\|$ defined above.

2. Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous functions that map $[0,1]$ into $\mathbb{R}$. Also let D be a countable dense subset of $[0,1]$, and assume that the sequence $(f_n(x))_{n=1}^{\infty}$ converges, for all $x \in D$. That is, the sequence of functions $(f_n)_{n=1}^{\infty}$ converges pointwise on the dense subset D.

   a. (3 pts) Use an example to show that the sequence $(f_n)_{n=1}^{\infty}$ need not converge everywhere on $[0,1]$.

   b. (7 pts) Prove that if the sequence $(f_n)_{n=1}^{\infty}$ converges *uniformly* on the dense subset D, then the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly on $[0,1]$.

3. Consider space $C[0,1]$ equipped with $\|\cdot\|_\infty$ norm (i.e., $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$) and a transformation

$$T : C[0,1] \to C[0,1] \quad \text{defined by} \quad Tf(x) = \int_0^x f(y)\,dy, \qquad x \in [0,1].$$

   a. (3 pts) Show that T is not a contraction and that T has a unique fixed point.

   b. (5 pts) Show that T^2 is a contraction.

   c. (2 pts) Is T^3 a contraction? Justify your answer.

4. (10 pts) Consider a sequence $(f_n)_{n=1}^{\infty}$ of real valued functions in $C^1[0,1]$ such that

$$|f_n(x)| \le -\frac{\ln x}{x^2} \quad \text{and} \quad |f_n'(x)| \le -\ln x \qquad \text{for all } x \in (0,1] \text{ and all } n.$$

Prove that the sequence $(f_n)_{n=1}^{\infty}$ has a uniformly converging subsequence.

Hint: The triangle inequality yields $|f_n(x)| \le |f_n(y)| + |f_n(y) - f_n(x)|$.