

OSU Department of Mathematics
Qualifying Examination
Spring 2021

Real Analysis

Instructions:

- Do any **four of the six problems**.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
 1. Use a separate sheet of paper to clearly indicate your identification number and the four problems which you wish to be graded.
 2. Arrange your solutions according to the problem order with the problem selection page on top and any scratch-work on the bottom.
 3. Submit the exam:
 - **For the in-person exam:** place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.
 - **For the on-line exam:**
 - * scan your exam in the order arranged as above, starting with the selection page and ending with the scratch-work, as a single pdf file (using e.g. CamScan phone app);
 - * check that your scan is legible and contains all the necessary pages to be graded;
 - * email the file directly to Nichole Sullivan (Nikki.Sullivan@oregonstate.edu);
 - * wait online until it is confirmed that your submission was received.

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Common Notation:

- $C^k(I)$ is the set of all functions on an interval I that have continuous derivatives up to and including order k .
- $\|a_n\|_p$ is the p -norm of a sequence (a_n) : $\|a_n\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}$, for $1 \leq p < \infty$, and $\|a_n\|_{\infty} = \sup_{n \in \mathbb{N}} |a_n|$, $p = \infty$.
- l_p – the normed space of all p -summable sequences: $l_p = \{(a_n) : \|a_n\|_p < \infty\}$, $1 \leq p \leq \infty$.

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Problems:

1. (a) (3pt) Let $\beta \leq 0$, and suppose that $0 \leq x < y \leq 1$. Explain why:

$$\int_x^y (1-s)^\beta ds \leq \int_{1-(y-x)}^1 (1-s)^\beta ds .$$

- (b) (7pt) Let $\{f_n\}$ be a sequence of $C^2([0, 1])$ functions satisfying $|f_n(0)| \leq M$ and $|f'_n(0)| \leq M$, for some $M > 0$. Suppose that there exists some $\alpha \in (1, 2)$ such that:

$$|f''_n(x)| < \frac{1}{(1-x)^\alpha}, \text{ for all } x \in (0, 1).$$

Prove that the sequence (f_n) has a subsequence which converges uniformly on $[0, 1]$.

2. Assume (M, d) is a compact metric space, and let $f : M \rightarrow M$ satisfy $d(f(x), f(y)) = d(x, y)$ for all $x, y \in M$.

- (a) (7pt) Show that f is onto.

Hint: try to create a sequence with no convergent subsequences.

- (b) (3pt) Show that the conclusion in (a) may fail if M is not compact.

3. Let (a_n) be a sequence in \mathbb{R} , $a_n \rightarrow a$. Suppose (b_n) is a bounded sequence in \mathbb{R} such that $\exists k \in \mathbb{N}$ for which the sequence $c_n = b_n - a_n b_{n+k}$ is convergent to c .

- (a) (6pt) Prove that if $a = 1$, then $c = 0$.

- (b) (4pt) Prove that if $0 \leq a < 1$, then (b_n) is convergent. (Note: in fact, (b_n) must converge also for $a > 1$, but you don't have to show this.)

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4. (10pt) Let $K \subset \mathbb{R}$ be a compact set, and let (f_n) be an equicontinuous sequence of functions on K . Show that if (f_n) converges pointwise to some function f on K , then (f_n) converges uniformly to f on K .

5. Let $(a_n), (\beta_n), (\gamma_n) \in l_\infty$ – fixed sequences. For a sequence (b_n) define

$$T(b_n) = a_n + \beta_n b_{n+1} + \gamma_n b_{n+2}, \quad n \in \mathbb{N}.$$

- (a) (3pt) Prove that T is a well-defined continuous mapping from l_∞ to l_∞ .
- (b) (3pt) Prove that if $\| |\beta_n| + |\gamma_n| \|_\infty < 1$, then T has a fixed point. (Here $(|\beta_n|), (|\gamma_n|)$ are the sequences of the absolute values.)
- (c) (4pt) Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $|\beta| + |\gamma| < |\alpha|$. Prove that for any $(a_n) \in l_\infty$ there exists $(b_n) \in l_\infty$ such that for all $n \in \mathbb{N}$

$$a_n = \alpha b_n + \beta b_{n+1} + \gamma b_{n+2}.$$

Also show that the result fails if the assumption $|\beta| + |\gamma| < |\alpha|$ is removed.

6. For each $n \in \mathbb{N}$, let $f_n : [0, 1] \rightarrow [0, \infty)$ be a continuous function, and assume that the sequence $\{f_n\}$ satisfies the property that for all $x \in [0, 1]$ and for all $n \in \mathbb{N}$, $f_n(x) \geq f_{n+1}(x)$. Assume $f_n(x) \rightarrow f(x)$ for each $x \in [0, 1]$.

- (a) (3pt) Construct an example showing that the sequence (f_n) satisfying the conditions above is not necessarily uniformly convergent to f on $[0, 1]$.
- (b) (7pt) Set $M = \sup_{x \in [0, 1]} f(x)$. Prove that there exists $t \in [0, 1]$ such that $f(t) = M$.