Department of Mathematics OSU
Qualifying Examination
Spring 2017

Real Analysis

- Do any four of the six problems. Indicate on the sheet with your identification number which four you wish to have graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this exam.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheet back into the envelope in which the test materials came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

DO NOT WRITE YOUR NAME ANYWHERE – USE ONLY YOUR TEST ID CODE

Standard Notations and Conventions: $\mathbb{R}$ denotes the reals, $\mathbb{C}$ denotes the complex numbers, m_n is Lebesgue measure on $\mathbb{R}^n$ (the subscript will be omitted when $n = 1$), for any Hausdorff space M, $\mathcal{B}(M)$ denotes the sigma-algebra of Borel sets (the sigma-algebra generated by the open sets), set complementation is denoted with ^{c}, i.e. A^c will denote the complement of A with respect to the ambient space, but set difference notation $A \setminus B$ will be used for relative complement. For a measurable set E, $1 \leq p < \infty$, $L^p(E)$ denotes the set of (equivalence classes of) p-th power integrable functions with norm $\|f\|_p = \left(\int_E |f|^p \, dm\right)^{1/p}$ and for $p = \infty$ denotes the set of (equivalence classes of) essentially bounded measurable functions with essential supremum as norm. Unless otherwise noted, you may assume that functions in L^p are real valued. In integration problems where several variables occur, integrals will be written with the name of the variable of integration in parentheses, as in $\int f(x,t) \, dm(x)$.

1. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set of positive Lebesgue measure, i.e. $m(E) > 0$. Show that

$$E - E \equiv \{x - y \ : x, y \in E\}$$

contains an interval.

Possible hint: You may assume (i.e. proof not required) that there is an interval I such that $m(I) > 0$ and $m\Big(E \cap I\Big) \geq \frac{3}{4} m(I)$.

2. Prove that if $f(x,y) = ye^{-(1+x^2)y^2}$, then $\int_0^\infty \int_0^\infty f(x,y) \, dm(x) \, dm(y)$ exists and

$$\int_0^\infty \int_0^\infty f(x,y) \, dm(x) \, dm(y) = \int_0^\infty \int_0^\infty f(x,y) \, dm(y) \, dm(x).$$

Use the above equality to give an alternative proof of the formula

$$\int_0^\infty e^{-x^2} \, dm(x) = \frac{\sqrt{\pi}}{2}.$$

3. Let f belong to $L^1((0,\infty))$ and define, for $t > 0$,

$$g(t) = \int_0^\infty e^{-tx} f(x) \, dm(x).$$

Prove, justifying each step, that g is differentiable and that $g'(t) = -\int_0^\infty e^{-tx} x f(x) \, dm(x)$.

The exam continues on the next page

4. Let μ be a non-trivial measure on $\mathcal{B}(\mathbb{R})$ which is finite on compact sets and which is translation invariant: that is, $\mu(E) = \mu(E+c)$ for every real number c and every Borel set E, where $E+c = \{y \in \mathbb{R} : y = x+c \text{ for some } x \in E\}$ is the translate of E by c. Prove that μ is a constant multiple of Lebesgue measure m: that is, there is a constant K such that $\mu(E) = Km(E)$ for all Borel sets E. [One possible path is to show the claim holds true for intervals $[a, b)$ with rational endpoints, that the sets of m measure zero have μ measure zero, and to then extend to bounded measurable sets by coverings. However, you may use any method you choose.]

5. (a) Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$, and let $F(x) = \|x\|$ for all $x \in \mathbb{R}^n$. Prove that the function F is continuous on $\mathbb{R}^n$ with respect to the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, which is defined by $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$ for all $x \in \mathbb{R}^n$. Your proof should not use any assumptions about the norm $\|\cdot\|$, other than t he fact that it satisfies the definition of a norm on a vector space.

   (b) Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. Prove that $\|\cdot\|$ and the Euclidean norm $\|\cdot\|_2$ are *equivalent*, in the sense that there exist positive constants C_1 and C_2 such that $C_1\|x\|_2 \le \|x\| \le C_2\|x\|_2$ for all $x \in \mathbb{R}^n$.

   (c) Let $X = L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then $\|f\|_1 < \infty$ and $\|f\|_2 < \infty$ for all $f \in X$. Prove that the L^1 norm and the L^2 norm are *not* equivalent on X.

6. For this problem, let f be a mapping defined on a neighborhood of 0 in $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $f(0) = 0$. Denote by $\bar{B}_r$ the closed ball of radius r centered at the origin. Assume that there exists a $\delta > 0$ such that $f(x) - x$ is Lipschitz continuous on $\bar{B}_\delta$ with Lipschitz constant less than 1/2. That is,

$$\|f(x) - x - (f(y) - y)\| \le c\|x - y\|$$

   for some $c < 1/2$ and all x, y in $\bar{B}_\delta$.

   (a) Show that f is injective (1-1) on $\bar{B}_\delta$.

   (b) Prove that the image under f of $\bar{B}_\delta$ contains $\bar{B}_{\delta/2}$. [Hint: Show that for $y \in \bar{B}_{\delta/2}$, the map $g_y(x) = y + x - f(x)$ takes $\bar{B}_\delta$ to itself and has a fixed point.]

   (c) Conclude that f has a continuous inverse on a domain containing $\bar{B}_{\delta/2}$