Department of Mathematics OSU
Qualifying Examination
Fall 2013

PART II: COMPLEX ANALYSIS and LINEAR ALGEBRA

- Do any two of the problems in Part CX, *use the correspondingly marked blue book* and indicate on the selection sheet with your identification number those problems that you wish graded.

  Similarly, do the same for Part LA.

- Your solutions should contain all mathematical details. Please write them up as clearly as possible.

- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.

- On problems with multiple parts, individual parts may be weighted differently in grading.

- You have three hours to complete Part II.

- When you are done with the examination, place examination blue books and selection sheets back into the envelope in which the test materials came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

PART CX : COMPLEX ANALYSIS QUALIFYING EXAM

1. (a) For which values of real numbers α, β, γ, δ, ϵ, ζ is the function $f(z) = u(x, y) + iv(x, y) = \alpha x^3 + \beta x^2 + \gamma y^2 + i(\delta xy + \epsilon y^2 + \zeta x^3)$ analytic?

   (b) Let $a_j \in \mathbb{C}$ with $|a_j| = 1$, for $j = 1, 2, \ldots, n$. Prove that there is $z \in \mathbb{C}$ with $|z| = 1$ such that the product $P = |z - a_1||z - a_2| \cdots |z - a_n|$ of the distances from z to a_j, for $j = 1, 2, \ldots, n$, is strictly greater than 1.

2. (a) Let $p(z)$ be a polynomial. Find all entire functions $f(z)$ satisfying $|f(z)| \leq |p(z)|$ for all $z \in \mathbb{C}$.

   (b) Let $f(z)$ be analytic in Ω, where Ω contains the closed unit disk $\overline{D}_1 = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$. Suppose that $f(z)$ maps $\overline{D}_1$ into the open unit disk $D_1 = \{z \in \mathbb{C} \,|\, |z| < 1\}$.

      i. Show that for every integer $n \geq 1$, the equation $f(z) - z^n = 0$ has exactly n solutions (counting multiplicities) in D_1.
      ii. Then prove that $f(z)$ has a *unique* fixed point in D_1.

3. (a) Let $\Omega \subset \mathbb{C}$ be open, and assume that the points 1, -1 lie in the same component of $\mathbb{C} \setminus \Omega$. Show that there exists an analytic $f(z)$ on Ω such that $f(z)^2 = \dfrac{1+z}{1-z}$ for all $z \in \Omega$.

   (b) Evaluate the integral
   $$\int_{|z|=2} \frac{dz}{\sin^2 z \, \cos z},$$
   where the circle is traversed in the counterclockwise direction.

Exam continues on next page ...

PART LA: LINEAR ALGEBRA QUALIFYING EXAM

1. (a) Give an example of a normal linear operator T on a finite dimensional vector space V which has no (nonzero) eigenvector.

   (b) Let V be a real, finite-dimensional inner product space of dimension $n > 0$. Let T be a self-adjoint linear operator on V. Prove that T has a (nonzero) eigenvector.

2. Let T be a linear operator on a finite-dimensional vector space V, and let W_1 and W_2 be T-invariant subspaces of V such that $V = W_1 \oplus W_2$.

   If $m_1(x)$ and $m_2(x)$ are the minimal polynomials of $T|_{W_1}$ and $T|_{W_2}$, respectively, is $m_1(x)m_2(x)$ the minimal polynomial of T? Either prove this, or conjecture an alternate minimal polynomial in terms of m_1 and m_2 and prove your conjecture.

3. Let A be a real $n \times n$ matrix. We call A a *difference of two squares* if there exist real $n \times n$ matrices B and C such that both $BC = CB = 0$ and $A = B^2 - C^2$.

   (a) Show that if A is diagonal, then A is a difference of two squares.

   (b) Show that if A is a real symmetric matrix (not necessarily diagonal), then A is a difference of two squares.

   (c) Let A be a difference of two squares, with B, C, as above. Show that if B has a nonzero real eigenvalue, then A has a positive real eigenvalue.

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